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A Formal Linearization of Nonlinear Systems based on the Discrete-Fourier Transform

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ABSTRACT

The problem regarding nonlinear systems has come to occupy an important position. In order to solve a nonlinear problem we have methods of linearization which are developed through linear approximation to adapt linear system theories. In this paper we present a formal linearization of nonlinear systems based on the discrete-Fourier transform (D.F.T.).

1. INTRODUCTION

In recent years research about linearization has been done. And many results have been published but their methods are not practical. Because they have some problems like low accuracy of approximation in wide region. Here we present a formal linearization of nonlinear systems based on the D.F.T. The excellent characteristics of this linearization are having high accuracy of approximation and simple transformation using D.F.T. for any nonlinear systems.

Composing an augmented vector space we can transform nonlinear system into formal linear system on the function space. Here we introduce the Trigonometric functions for composing, so as to be made use of D.F.T. with simplicity and high accuracy. Through the formal linearization we can adapt the linear system theory to the given nonlinear system and get the solution by the inversion.

Concretely the outline of this method in a scalar case is as follows. Let $\dot{x}(t)=f(x(t))$ be a given nonlinear system and let f be a real-valued function defined on R and x be a state variable. A formal linearization function introduced here is $\phi(y(t))=\tilde{\varphi}(y(t))-\tilde{\phi}(y(\infty))$ where

$$\tilde{\phi}(y)=[\sin y,\cos y,\sin 2y,\cos 2v,\cdots,\sin(n-1)y,\cos(n-1)y]^T$$
$$=[\tilde{\phi}_1,\tilde{\phi}_2,\tilde{\phi}_3,\tilde{\phi}_4,\cdots,\tilde{\phi}_{2(n-1)}]^T.$$

$\dot{\tilde{\phi}}(y)$ is expanded by D.F.T. so that the linear system $\dot{\phi}(y)=A\phi(y)$ is obtained. The inversion is carried out by using $\tilde{\phi}_1$ and $\tilde{\phi}_2$.

This paper also propose examples of adapting this method to scalar and vector systems. Numerical examples show satisfactory results. As an application of this method we propose an observer for a nonlinear system too.

2. A FORMAL LINEARIZATION OF A SCALAR SYSTEM

We consider a scalar system. Assume that a nonlinear system is given as

$$\Sigma_1 : \dot{x}(t)=f(x(t)) \quad (\cdot=d/dt) \qquad (2.1)$$
$$x(0)=x_0\in[0,\ell]\subset R$$

where x is a state variable defined on $[0,\ell]$, R is the set of all real-valued, f is a nonlinear square integrable function with the first continuous derivative.

We here define a formal linearization function

$$\phi(y(t))=\tilde{\phi}(y(t))-\tilde{\phi}(y(\infty)) \tag{2.2}$$

where

$$\tilde{\phi}(y)=[\sin y,\cos y,\sin 2y,\cos 2y,\cdots,\sin(n-1)y,\cos(n-1)y]^T$$
$$=[\tilde{\phi}_1,\tilde{\phi}_2,\tilde{\phi}_3,\tilde{\phi}_4,\cdots,\tilde{\phi}_{2(n-1)-1},\tilde{\phi}_{2(n-1)}]^T. \tag{2.3}$$

The $\phi(y)$ is called as the nth-order linearization function.

We transform the system Σ_1 into a linear system :

$$\Sigma_2 \ : \ \dot{\phi}(y)=A\phi(y) \tag{2.4}$$
$$\phi(y(0))=\phi(y_0),y(0)=\frac{2\pi}{\ell}x_0$$

as follows.

To expand $\tilde{\phi}_K(y)$ in Fourier series on $[0,2\pi]$, we introduce a new variable :

$$y(t)=\frac{2\pi}{\ell}x(t), \quad y(t)\in[0,\ 2\pi]. \tag{2.5}$$

By this equation, Eq.(2.1) is transformed as

$$\Sigma_1' \ : \ \dot{y}(t)=g(y(t)). \quad (g(y)=\frac{2\pi}{\ell}f(\frac{\ell}{2\pi}y)) \tag{2.6}$$

From Eqs.(2.3) and (2.6),

$$\dot{\tilde{\phi}}_{2r-1}(y)=\frac{d}{dt}\sin ry=(\frac{d}{dt}\sin ry)\dot{y}$$
$$=r(\cos ry)g(y)=G_{2r-1}(y). \tag{2.7}$$

$$\dot{\tilde{\phi}}_{2r}(y)=\frac{d}{dt}\cos ry=(\frac{d}{dt}\cos ry)\dot{y}=-r(\sin ry)g(y)=G_{2r}(y). \tag{2.8}$$

Expanding each G_r $(r=1,2,\cdots,2(n-1))$ by D.F.T., we have

$$\dot{\tilde{\phi}}_r(y)=\sum_{k=1}^{n-1}(\alpha_{r\ 2k-1}\sin ky+\alpha_{r\ 2k}\cos ky)+\frac{\alpha_{r\ 0}}{2} \tag{2.9}$$

where

$$\alpha_{r\ 2k-1}=\frac{2}{N}\sum_{i=0}^{N-1}G_r(y_i)\sin ky_i$$

$$\alpha_{r\ 2k}=\frac{2}{N}\sum_{i=0}^{N-1}G_r(y_i)\cos ky_i, \quad y_i=\frac{2\pi}{N}i \ , \ N=2n-1.$$

From Eq.(2.3), the nth-order linear system with respect to $\tilde{\phi}$ is obtained as

$$\dot{\tilde{\phi}}(y)=A\tilde{\phi}(y)+b \tag{2.10}$$

where

$$A=\begin{pmatrix} \alpha_{1\ 1} & \alpha_{1\ 2} & \cdots & \alpha_{1\ N-1} \\ \alpha_{2\ 1} & \alpha_{2\ 2} & \cdots & \alpha_{2\ N-1} \\ \vdots & \vdots & & \vdots \\ \alpha_{N-1\ 1} & \alpha_{N-1\ 2} & \cdots & \alpha_{N-1\ N-1} \end{pmatrix}$$

$$b=[\frac{\alpha_{1\ 0}}{2}\ \ \frac{\alpha_{2\ 0}}{2}\ \cdots\ \frac{\alpha_{N-1\ 0}}{2}]^T.$$

Another approximate equation without the constant term is obtained with respect to $\phi(y)$ as

$$\dot{\phi}(y)=\dot{\tilde{\phi}}(y)-\dot{\tilde{\phi}}(y(\infty))=(A\tilde{\phi}(y)+b) \tag{2.11}$$
$$-(A\tilde{\phi}(y(\infty))+b)=A(\tilde{\phi}(y)-\tilde{\phi}(y(\infty))=A\phi(y).$$

Thus the linear system Eq.(2.4) is obtained.

The inverse transformation is as follows. From Eq.(2.4), ϕ is derived and then $\tilde{\phi}$ is obtained from Eq.(2.2). y(t) is evaluated by $\tilde{\phi}_1$ and $\tilde{\phi}_2$ as

$$y=\cos^{-1}\tilde{\phi}_2 \qquad (0\leq\tilde{\phi}_1)$$
$$=2\pi-\cos^{-1}\tilde{\phi}_2 \qquad (\tilde{\phi}_1<0). \tag{2.12}$$

Then the solution of the nonlinear system x(t) is acquired from Eq.(2.5).

In the next, we are going to deal with a vector system.

3. A FORMAL LINEARIZATION OF A VECTOR SYSTEM

For the sake of simplicity we are going to deal only with a system with two variables. However, from what follows it will be obvious that this restriction is merely simplicity and that all considerations can be similarly expanded in the case of systems with more variables.

The nonlinear system is given as

$$\Sigma_3 \ : \ \dot{x}(t)=f(x(t)) \quad (\ \cdot=d/dt) \tag{3.1}$$

where

$$x(t)=[x_1(t),x_2(t)]^T$$
$$\in[\ell_{11},\ell_{12}]\times[\ell_{21},\ell_{22}]\subset R^2$$
$$f(x)=[f_1(x),f_2(x)]^T\in C^1\cap L^2.$$

We here define the nth-order linearization function as

$$\phi(y(t))=\tilde{\phi}(y(t))-\tilde{\phi}(y(\infty)) \tag{3.2}$$

where

$$\tilde{\phi}(y)=[\cos y_2,\sin y_2,\cos 2y_2,\sin 2y_2,\cdots,$$
$$\cos(n-1)y_2,\sin(n-1)y_2,$$
$$\cos y_1,\cos y_2\cos y_1,\sin y_2\cos y_1,\cdots,$$
$$\cos(n-1)y_2\cos y_1,\sin(n-1)y_2\cos y_1,$$
$$\sin y_1,\cos y_2\sin y_1,\sin y_2\sin y_1,\cdots,$$
$$\cos(n-1)y_2\sin y_1,\sin(n-1)y_2\sin y_1,\cdots$$
$$\cdots,\sin(n-1)y_1,\cos y_2\sin(n-1)y_1,\sin y_2\sin(n-1)y_1,$$
$$\cdots,\sin(n-1)y_2\sin(n-1)y_1]^T$$
$$=[\tilde{\phi}_1\ \ \tilde{\phi}_2\ \ \tilde{\phi}_3\ \ \tilde{\phi}_4\ \cdots\ \tilde{\phi}_{(2n-1)^2-1}]^T.$$

Next we introduce a new variable :

$$y_1(t)=(\frac{m_1-x_1(t)}{p_1}+1)\pi,\ y_2(t)=(\frac{m_2-x_2(t)}{p_2}+1)\pi \quad (3.3)$$

where

$$D_y=[0,2\pi]\times[0,2\pi]$$

$$m_1=\frac{\ell_{11}+\ell_{12}}{2} \qquad m_2=\frac{\ell_{21}+\ell_{22}}{2}$$

$$p_1=\frac{\ell_{11}-\ell_{12}}{2} \qquad p_1=\frac{\ell_{21}-\ell_{22}}{2}.$$

By these equations Eq.(3.1) is exchanged with

$$\Sigma_3' : \dot{y}(t)=g(y(t)),\ g=[g_1,g_2]^T \quad (3.4)$$

which is similar to Eq.(2.6). From Eqs.(3.2) and (3.4),

$$\dot{\tilde{\phi}}_K(y)=\frac{\partial\tilde{\phi}_K}{\partial y_1}\dot{y}_1+\frac{\partial\tilde{\phi}_K}{\partial y_2}\dot{y}_2=\frac{\partial\tilde{\phi}_K}{\partial y_1}g_1+\frac{\partial\tilde{\phi}_K}{\partial y_2}g_2$$
$$=F_K(y_1,y_2). \quad (3.5)$$

Expanding each $F_K(K=1,2,\cdots,(2n-1)^2-1)$ with respect to y_1 and y_2 by D.F.T., we have

$$\dot{\tilde{\phi}}_K(y)=\frac{1}{2}(\frac{a_{0\ 0}^K}{2}+\sum_{i=1}^{n-1}[a_{0\ i}^K\cos iy_2+b_{0\ i}^K\sin iy_2] \quad (3.6)$$

$$+\sum_{r=1}^{n-1}\{(\frac{a_{r\ 0}^K}{2}+\sum_{i=1}^{n-1}[a_{r\ i}^K\cos iy_2+b_{r\ i}^K\sin iy_2])\cos ry_1$$

$$+(\frac{c_{r\ 0}^K}{2}+\sum_{i=1}^{n-1}[c_{r\ i}^K\cos iy_2+d_{r\ i}^K\sin iy_2])\sin ry_1\}$$

where

$$a_{r\ i}^K=\frac{2}{N}\sum_{j=0}^{N-1}[\frac{2}{N}\sum_{m=0}^{N-1}F_K(y_{1m},y_{2j})\cos ry_{1m}]\cos iy_{2j}$$

$$b_{r\ i}^K=\frac{2}{N}\sum_{j=0}^{N-1}[\frac{2}{N}\sum_{m=0}^{N-1}F_K(y_{1m},y_{2j})\cos ry_{1m}]\sin iy_{2j}$$

$$c_{r\ i}^K=\frac{2}{N}\sum_{j=0}^{N-1}[\frac{2}{N}\sum_{m=0}^{N-1}F_K(y_{1m},y_{2j})\sin ry_{1m}]\cos iy_{2j}$$

$$d_{r\ i}^K=\frac{2}{N}\sum_{j=0}^{N-1}[\frac{2}{N}\sum_{m=0}^{N-1}F_K(y_{1m},y_{2j})\sin ry_{1m}]\sin iy_{2j}$$

$$y_{1m}=\frac{2\pi}{N}m,\quad y_{2j}=\frac{2\pi}{N}j,\quad N=2n-1.$$

From Eq.(3.2), the nth-order linear system with respect to $\tilde{\phi}$ is obtained as

$$\dot{\tilde{\phi}}(y)=A\tilde{\phi}(y)+b. \quad (3.7)$$

From Eq.(3.2), we have the linear system without the constant term with respect to ϕ as

$$\dot{\phi}(y)=\dot{\tilde{\phi}}(y)-\dot{\tilde{\phi}}(y(\infty))=A\phi(y). \quad (3.8)$$

Thus the nonlinear system (Eq.(3.1)) is transformed into linear system (Eq.(3.8)). The inverse transformation is carried out in a similar way in the case of a scalar system. y_1 is obtained from $\tilde{\phi}_N$ and $\tilde{\phi}_{2N}$, y_2 is from $\tilde{\phi}_1$ and $\tilde{\phi}_2$. Then we can get solutions x_1 and x_2 from Eq.(3.3).

4. OBSERVER

In this section we compose a nonlinear observer as an application of the linearization. Here we consider a scalar system, for a vector system is straightforward.

Assume that a nonlinear system with measurement is given as the same as Eq.(2.1) : Dynamic equation is

$$\dot{x}(t)=f(x(t)),\quad x(0)\in[0,\ell]. \quad (4.1)$$

Measurement equation is

$$Z(t)=h(x(t)) \quad (4.2)$$

where Z is real-valued measurement datum, and h is a nonlinear function which satisfies with

$$\sum_{i=0}^{N-1}h(\frac{\ell}{N}i)=0.$$

By the way of section 2 f(x) is expanded by D.F.T. so that the linear equation of (2.4) is derived as

$$\dot{\phi}(y)=A\phi(y). \quad (4.3)$$

In a similar way, h(x) is also expanded by D.F.T. so that we have

$$Z(y)=B\phi(y) \quad (4.4)$$

where

$$B=[\beta_1,\cdots,\beta_K,\cdots,\beta_{N-1}]^T$$

$$\beta_K=\frac{2}{N}\sum_{i=0}^{N-1}\frac{2\pi}{\ell}h(\frac{\ell}{2\pi}y_i)\phi_K(y),\quad y_i=\frac{2\pi}{N}i.$$

The linear observer theory(6) is applied to the linear system Eqs.(4.3) and (4.4). Identity observer, for example, is

$$\dot{\hat{\phi}}(t)=A\hat{\phi}(t)+K(Z(t)-B\hat{\phi}(t)). \quad (4.5)$$

K is appropriately chosen so that all eigenvalues of the matrix (A-KB) have negative real parts. The solution of this observer is carried out as shown at the end of section 2.

5. NUMERICAL EXAMPLES

We are going to illustrate the use of this method. Two examples are shown in this section. One is of the scalar system and the other is of the vector system.

5.1 SCALAR SYSTEM

Given a nonlinear scalar system as

$$\Sigma_1 : \dot{x}(t)=-x(t)+x^2(t) \qquad (5.1)$$

$$x(0)=x_0 \in [0, \ell] \subset R, (x_0=0.8, \ell=0.9).$$

From Eq.(2.11), the linear system respect to ϕ is obtained. For the purpose of comparison, we solve the given nonlinear equation (5.1) and the linear equation (2.11). In this case let the order of ϕ be parameter. The coefficients of Eq.(2.11) is automatically evaluated as n is given. Fig.1 shows the trajectories of the results by computer. $\hat{x}(t)$ is of the linearized system where n=4, 6, 11 and 21. x(t) is the true value from the original equation (5.1). Fig.2 is the integration of the square error :

$$J(t)=\int_0^t (x(\tau)-\hat{x}(\tau))^2 d\tau. \qquad (5.2)$$

5.2 VECTOR SYSTEM

We show the example of an electric power system for a nonlinear system with two variables :

$$M\ddot{\delta}+D\dot{\delta}+P_{em}\sin\delta=P_{in}. \qquad (5.3)$$

The expressions for x_1 and x_2 in terms of state variables are given by

$$\dot{x}_1(t)=\delta(t)-\delta(\infty) \qquad (5.4)$$

$$\dot{x}_2(t)=\dot{\delta}(t).$$

Eq.(5.3) is written by x_1 and x_2 as

$$\begin{cases} \dot{x}_1(t)=x_2(t) \\ \dot{x}_2(t)=a_1\sin(x_1(t)+\delta(\infty))+a_2x_2+a_0 \end{cases} \qquad (5.5)$$

$$x(t) \in [\ell_{11}, \ell_{12}] \times [\ell_{21}, \ell_{22}]$$

$$=[-0.41,0.1] \times [-0.8,1.0] \subset R^2$$

$$x_1(0)=\delta(0)-\delta(\infty)=0.6-\delta(\infty)$$

$$x_2(0)=\dot{\delta}(0)=0.2$$

where

$M=0.0265$, $D=0.005$, $P_{em}=1.0$, $P_{in}=0.8$,

$a_0=P_{in}/M$, $a_1=-P_{em}/M$, $a_2=-D/M$,

$\delta(\infty)=\sin^{-1}(P_{in}/P_{em})$.

From Eq.(3.8), the linear system respect to ϕ is obtained. Figs.3 and 4 show the trajectories of $\hat{x}_1(t)$ and $\hat{x}_2(t)$ when n is parameter (n=2,3,4). Figs.5 and 6 show the integrations of the square error :

$$J_1(t)=\int_0^t (x_1(\tau)-\hat{x}_1(\tau))^2 d\tau \qquad (5.6)$$

$$J_2(t)=\int_0^t (x_2(\tau)-\hat{x}_2(\tau))^2 d\tau \qquad (5.7)$$

6. CONCLUSION

Nonlinear systems are transformed into linear systems formally. Numerical examples show that the accuracy of this method improved as n increases.

A nonlinear observer also proposed as an application of this method.

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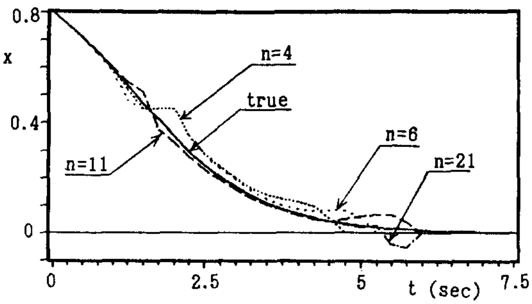

Fig.1 $x(t)$ and $\hat{x}(t)$

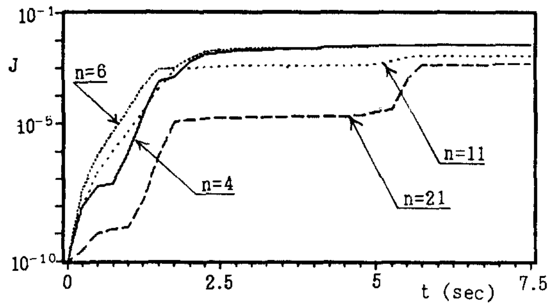

Fig.2 $J(t)$

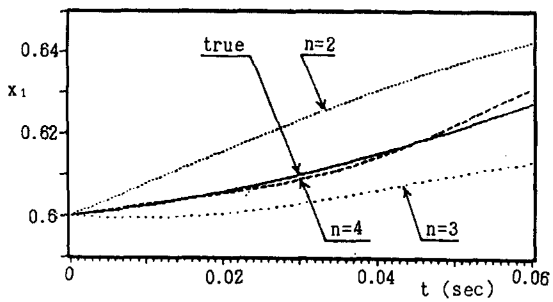

Fig.3 $x_1(t)$ and $\hat{x}_1(t)$

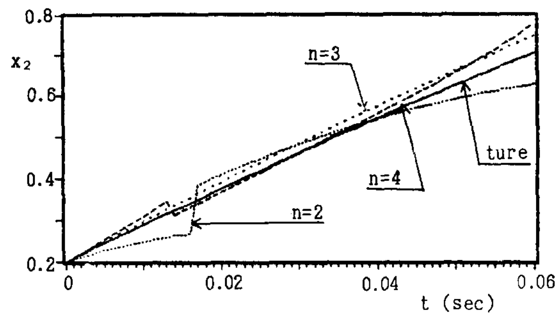

Fig.4 $x_2(t)$ and $\hat{x}_2(t)$

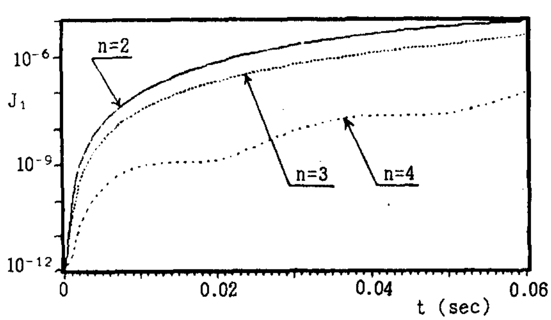

Fig.5 $J_1(t)$

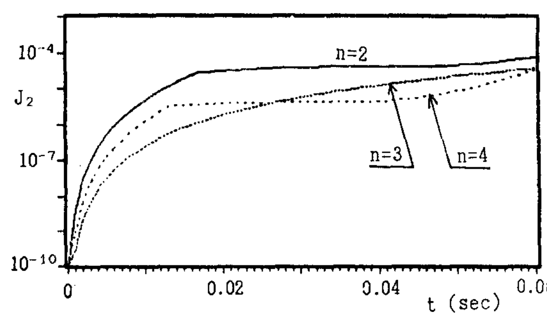

Fig.6 $J_2(t)$