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On the Linear Quadratic Regulator for Descriptor Systems

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Abstract

This paper deals with the linear quadratic optimal regulator problem for descriptor systems without performing a preliminary transformation for a descriptor system. We derive a generalized Riccati differential equation (GRDE) based on the two-point boundary value problem for a Hamiltonian equation. We then obtain an optimal feedback control and the optimal cost in terms of the solution of GRE. A simple example is included.

I Introduction

The linear quadratic (LQ) optimal regulator theory, although very useful, often does not provide a physical meaning to the control law, since the state variables do not necessarily correspond to physical variables of a system of interest, and since the parameters of a state-space model are usually some functions of the independent parameters of a physical system. It is claimed (Lewis [7], Luenberger [8]) that the descriptor systems,or singular systems, have a great capacity for the system modeling in that they can preserve the structure of physical systems and can include nondynamic constraints and impulsive elements. Hence the system theory including the LQ regulator for the descriptor system has a potential applicability to a wide class of systems.

The basic theory for the descriptor systems is well developed for the last decades (Cobb [5], Lewis [7], Luenberger [8]). Moreover, Cobb [4] has considered an optimal LQ regulator problem for the descriptor systems by transforming the descriptor system into a state-space system with a reduced dimension to eliminate impulsive modes, and Lewis [6] has also dealt with an LQ problem based on a Hamiltonian formulation and derived a generalized Riccati differential equation, whose solution does not always exist. Moreover, Bender and Laub [2] have developed fairly complete results for an LQ regulator problem for descriptor systems by employing a singular value decomposition (SVD) coordinate system and a Hamiltonian formulation. In fact, they have determined the extended control vector in terms of the state and costate vectors, and derived four types of Riccati differential equations with reduced dimensions. The discrete-time results are also developed by Bender and Laub [3].

In this paper, we consider an LQ optimal regulator problem for a continuous-time descriptor system along the line of Bender and Laub [2], but without using an SVD coordinate system or a preliminary transformation for a descriptor system. For the finite-horizon problem, we derive a generalized Riccati differential equation that has a dynamic part and nondynamic constraints and gives an optimal feedback control and the optimal cost. It is shown that the generalized Riccati differential equation (GRDE) has always a solution under mild conditions. Hence our results are complementary to earlier results (Cobb [4], Lewis [6], Bender and Laub [2]).

II Problem Formulation

In this section, the LQ regulator problem is formulated, after reviewing the basic results for descriptor systems based on [5], [7].

We consider the continuous-time descriptor system

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{2.1}$$

$$Ex(0-) = Ex_0 \tag{2.2}$$

where $E \in \mathbf{R}^{n\times n}$, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $x(t)$ is the n-dimensional descriptor vector, and $u(t)$ is the m-dimensional control vector. The matrix E is, in general, a singular matrix with $\text{rank}E = r \leq n$.

We assume that the descriptor system (2.1),(2.2) is *regular*, namely, $\det(sE - A) \not\equiv 0$. If (E, A) is regular, then a solution $x(t)$ of the system (2.1), (2.2) is uniquely determined by the inverse Laplace transform of

$$X(s) = (sE - A)^{-1}Ex_0 + (sE - A)^{-1}BU(s) \tag{2.3}$$

The finite eigenvalues of the characteristic equation

$$\det(sE - A) = 0 \tag{2.4}$$

are called *finite dynamic modes.* If all the finite dynamic modes of (2.4) are stable, then we say that (E, A) is *stable.* Furthermore we define two modes for the infinite eigenvalues of (2.4) which are the zero eigenvalues of $\det(E-\lambda A)=0$. The infinite eigenvalues corresponding to the generalized eigenvectors v that satisfy

$$Ev = 0 \tag{2.5}$$

are the ***nondynamic modes***. Suppose v^1 satisfies $Ev^1 = 0$. The infinite eigenvalues corresponding to the generalized principal vectors v^k that satisfy

$$Ev^k = Av^{k-1} \qquad (k \geq 2) \tag{2.6}$$

are the ***impulsive modes***.

Let the characteristic equation of (2.4) be q-dimensional, i.e.

$$\text{degree } \det(sE - A) = q \leq r. \tag{2.7}$$

It is well known that if $(E,\ A)$ is regular, then the descriptor system (2.1) has q finite dynamic modes, $(n-r)$ nondynamic modes, and $(r-q)$ impulsive modes. Moreover, it follows that if the system has impulsive modes, then $x(t)$, the inverse Laplace transform of (2.3), may include δ function, its derivatives, and the derivatives of the input $u(t)$.

It is shown that the descriptor system

$$E\dot{x}(t) = Ax(t) \tag{2.8}$$

is regular, and has no impulsive mode if and only if

$$\text{Im}E + \text{Im}A(\ker E) = \mathbf{R}^n \tag{2.9}$$

If the system (2.8) has no impulsive mode, then we say that $(E,\ A)$ is *impulse-free*.

If there exists a descriptor variable feedback $u(t) = Kx(t)$ such that all impulsive modes can be transformed into finite dynamic modes, then the descriptor system is *impulse controllable*, or *controllable at* ∞. It follows that the descriptor system (2.1) is impulse controllable if and only if $\text{Im}E + \text{Im}A(\ker E) + \text{Im}B = \mathbf{R}^n$. Also, if there exists a descriptor variable feedback $u(t) = Kx(t)$ such that all finite dynamic modes can be stabilized, then the descriptor system is ***finite dynamics stabilizable***. The descriptor system (2.1) is finite dynamics stabilizable if and only if $\text{rank}\,[\, sE - A \quad B\,] = n$ for all s, $\text{Re}\,[s] \geq 0$.

If the feedback system $E\dot{x} = (A+BK)x$ is (a) regular, (b) impulse-free, and (c) stable, then we say that $(E, A+BK)$ is ***admissible***, and the feedback gain K is called an ***admissible feedback gain***.

Now we consider the problem of computing a control $u(t)$ to minimize the cost function

$$J(Ex_0,u,T) = \frac{1}{2}x^T(T)E^TPEx(T) + \frac{1}{2}\int_0^T [\, x^T(t)\ u^T(t)\,] \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \tag{2.10}$$

where $P \in \mathbf{R}^{n\times n}$, $Q \in \mathbf{R}^{n\times n}$, $R \in \mathbf{R}^{m\times m}$ and $S \in \mathbf{R}^{n\times m}$. The weighting matrix P is symmetric and non-negative definite, R is symmetric and positive definite, and

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} C^T \\ D^T \end{bmatrix} [\, C \ D\,] \tag{2.11}$$

is symmetric and non-negative definite, where $C \in \mathbf{R}^{p\times n}$ and $D \in \mathbf{R}^{p\times m}$. If $y(t)$ is the p-dimensional output vector

$$y(t) = Cx(t) + Du(t) \tag{2.12}$$

then the cost function J can also be expressed as

$$J(Ex_0,u,T) = \frac{1}{2}x^T(T)E^TPEx(T) + \frac{1}{2}\int_0^T y^T(t)y(t)\,dt \tag{2.13}$$

In this paper, we assume the following for the regulator problem.

Assumption 1 : The descriptor system (2.1),(2.2) is regular, impulse controllable, and finite dynamics stabilizable, namely

$$\det(sE - A) \not\equiv 0 \tag{2.14}$$

$$\text{Im}E + \text{Im}A(\ker E) + \text{Im}B = \mathbf{R}^n \tag{2.15}$$

$$\text{rank}\,[\, sE - A \quad B\,] = n,\ \forall s,\ \text{Re}\,[s] \geq 0 \tag{2.16}$$

III LQ Regulator Problem

In this section we derive a generalized Riccati differential equation that characterizes an optimal feedback gain and the optimal cost.

A. Necessary conditions

We can derive necessary conditions for minimization of the cost function (2.10) based on the caluculus of variations [1], [2], [6].

Lemma 1 : Assume that $x(t)$ and $u(t)$ satisfy (2.1),(2.2). Further assume that $x(t)$, $E\dot{x}(t)$, and $u(t)$ contain no impulse in the interval $(0,\ T)$. Then necessary conditions for the cost function J to be minimized are

$$E\dot{x}(t) = \frac{\partial H(t)}{\partial \gamma(t)},\ E^T\dot{\gamma}(t) = -\frac{\partial H(t)}{\partial x(t)},\ \frac{\partial H(t)}{\partial u(t)} = 0 \tag{3.1}$$

$$E^T\gamma(T) = E^TPEx(T) \tag{3.2}$$

where

$$H(t) := \frac{1}{2}[\, x^T(t)\ u^T(t)\,] \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + \gamma^T(t)[\, Ax(t) + Bu(t)\,] \tag{3.3}$$

Proof : See [1], [2]. □

Combining the necessary conditions (3.1),(3.2) and the system equations (2.1),(2.2), we obtain the two-point boundary value problem (TPBVP)

$$\begin{bmatrix} E & 0 & 0 \\ 0 & E^T & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{\gamma}(t) \\ \dot{u}(t) \end{bmatrix} = \begin{bmatrix} A & 0 & B \\ -Q & -A^T & -S \\ S^T & B^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ \gamma(t) \\ u(t) \end{bmatrix} \tag{3.4}$$

with

$$Ex(0-) = Ex_0, \quad E^T\gamma(T) = E^T PEx(T) \tag{3.5}$$

Before solving this problem, we shall assume the following by regarding the equations (3.4), (3.5) as a descriptor system.

Assumption 2 : The descriptor system (3.4),(3.5) is regular and impulse-free, i.e.

$$\mathrm{Im}\tilde{E} + \mathrm{Im}\tilde{A}(\ker\tilde{E}) = \mathbf{R}^{2n+m} \tag{3.6}$$

where

$$\tilde{E} := \begin{bmatrix} E & 0 & 0 \\ 0 & E^T & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{A} := \begin{bmatrix} A & 0 & B \\ -Q & -A^T & -S \\ S^T & B^T & R \end{bmatrix} \tag{3.7}$$

Hence $\det(s\tilde{E} - \tilde{A}) \not\equiv 0$.

Under Assumption 2, solutions $x(t)$, $\gamma(t)$, and $u(t)$ of the TPBVP (3.4), (3.5) are impulse-free and smooth. Therefore $E\dot{x}(t)$ is also impulse-free.

Now, under Assumptions 1 and 2, we solve the TPBVP (3.4),(3.5) by using the state transition matrix [1],[2]. Since it is assumed that R is invertible, the bottom row of (3.4) yields

$$u(t) = -R^{-1}\{ S^T x(t) + B^T\gamma(t) \} \tag{3.8}$$

Using this, we can rewrite (3.4) as

$$\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{\gamma}(t) \end{bmatrix} = \begin{bmatrix} A - BR^{-1}S^T & -BR^{-1}B^T \\ -Q + SR^{-1}S^T & -A^T + SR^{-1}B^T \end{bmatrix} \begin{bmatrix} x(t) \\ \gamma(t) \end{bmatrix} \tag{3.9}$$

Under Assumption 2, we see that (3.9) is also regular and impulse-free. Therefore we can transform the pencil associated with (3.9) into a Weierstrass standard form [7]. In fact, there exist $2n \times 2n$ non-singular matrices $\tilde{M}$ and $\tilde{N}$ such that

$$\tilde{M} \begin{bmatrix} sE - A + BR^{-1}S^T & BR^{-1}B^T \\ Q - SR^{-1}S^T & sE^T + A^T - SR^{-1}B^T \end{bmatrix} \tilde{N} = \begin{bmatrix} sI - \tilde{F} & 0 \\ 0 & I \end{bmatrix} \tag{3.10}$$

where $\tilde{F}$ is a $2r \times 2r$ matrix. Substituting (3.10) into the Laplace transform of (3.9), we have

$$\begin{bmatrix} X(s) \\ \Gamma(s) \end{bmatrix} = \tilde{N} \begin{bmatrix} (sI - \tilde{F})^{-1} & 0 \\ 0 & I \end{bmatrix} \tilde{M} \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} \begin{bmatrix} x(0-) \\ \gamma(0-) \end{bmatrix} \tag{3.11}$$

Take the inverse Laplace transform of this to get

$$\begin{bmatrix} x(t) \\ \gamma(t) \end{bmatrix} = \tilde{N} \begin{bmatrix} e^{\tilde{F}t} & 0 \\ 0 & I \end{bmatrix} \tilde{M} \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} \begin{bmatrix} x(0-) \\ \gamma(0-) \end{bmatrix} \tag{3.12}$$

Next we define the backward-time state transition matrix $\Omega(t,T)$ for (3.9) as

$$\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} \begin{bmatrix} x(t) \\ \gamma(t) \end{bmatrix} = \begin{bmatrix} \Omega_{11}(t,T) & \Omega_{12}(t,T) \\ \Omega_{21}(t,T) & \Omega_{22}(t,T) \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} \begin{bmatrix} x(T) \\ \gamma(T) \end{bmatrix}, \quad \forall t \le T \tag{3.13}$$

where $\Omega_{ij}(t,T)$ are $n \times n$ submatrices formed by partitioning $\Omega(t,T)$. Substitute $E^T\gamma(T) = E^T PEx(T)$ from (3.5) into (3.13) to get

$$Ex(t) = [\, \Omega_{11}(t,T) + \Omega_{12}(t,T)E^T P \,]\, Ex(T) \tag{3.14}$$

$$E^T\gamma(t) = [\, \Omega_{21}(t,T) + \Omega_{22}(t,T)E^T P \,]\, Ex(T) \tag{3.15}$$

where $t \le T$. Moreover we define the state transition matrix $\Psi(t,T)$ relating $Ex(t)$ to $Ex(T)$:

$$Ex(T) = \Psi(t,T)\, Ex(t). \tag{3.16}$$

Combining (3.15) and (3.16), we get

$$E^T\gamma(t) = [\, \Omega_{21}(t,T) + \Omega_{22}(t,T)E^T P \,]\, \Psi(t,T)\, Ex(t) \tag{3.17}$$

Thus it follows from (3.17) with $t = 0-$ and (3.12) that

$$\begin{bmatrix} x(t) \\ \gamma(t) \end{bmatrix} = \begin{bmatrix} \Lambda_{11}(t) & \Lambda_{12}(t) \\ \Lambda_{21}(t) & \Lambda_{22}(t) \end{bmatrix} \begin{bmatrix} I \\ Y(0-) \end{bmatrix} Ex(0-) \tag{3.18}$$

where $\Lambda_{ij}(t)$ and $Y(t)$ are $n \times n$ time-varying matrices such that

$$\begin{bmatrix} \Lambda_{11}(t) & \Lambda_{12}(t) \\ \Lambda_{21}(t) & \Lambda_{22}(t) \end{bmatrix} := \tilde{N} \begin{bmatrix} e^{\tilde{F}t} & 0 \\ 0 & I \end{bmatrix} \tilde{M} \tag{3.19}$$

$$Y(t) := [\, \Omega_{21}(t,T) + \Omega_{22}(t,T)E^T P \,]\, \Psi(t,T) \tag{3.20}$$

By the regularity of Assumption 1, the matrix relating $Ex(0-)$ to $x(t)$ must be non-singular, so that we can rewrite (3.18) as

$$Ex(0-) = [\, \Lambda_{11}(t) + \Lambda_{12}(t)Y(0-) \,]^{-1} x(t) \tag{3.21}$$

$$\gamma(t) = [\, \Lambda_{21}(t) + \Lambda_{22}(t)Y(0-) \,]\, Ex(0-) \tag{3.22}$$

Hence we obtain the following.

Lemma 2 : Define an $n \times n$ time-varying matrix $X(t)$ by

$$X(t) := [\, \Lambda_{21}(t) + \Lambda_{22}(t)\, Y(0-) \,][\, \Lambda_{11}(t) + \Lambda_{12}(t)\, Y(0-) \,]^{-1} \tag{3.23}$$

Then

$$\gamma(t) = X(t)x(t) \tag{3.24}$$

where $(x(t), \gamma(t))$ is a solution of the TPBVP (3.4), (3.5).

B. Optimal Regulator

By solving the TPBVP (3.4),(3.5), we can obtain the main results for the finite-horizon problem. The following assumption is made for the descriptor system (3.4),(3.5).

Assumption 3 : No finite dynamic mode of the descriptor system (3.4),(3.5) lies on the imaginary axis, namely,

$$\mathrm{rank} \begin{bmatrix} A - \lambda E & B \\ Q & S \\ S^T & R \end{bmatrix} = n + m, \quad \forall \lambda, \ \mathrm{Re}[\lambda] = 0 \tag{3.25}$$

Theorem 1 : Let $T < \infty$ for (2.10). Under Assumptions 1, 2 and 3, there exists a solution $(X(t),\ Y(t))$ of the simultaneous differential equations

$$E^T\dot{X}(t)+Y(t)A+A^TX(t)+Q - \{\,Y(t)B+S\,\}\,R^{-1}\,\{\,S^T+B^TX(t)\,\}=0 \tag{3.26}$$

$$Y(t)E = E^TX(t) \tag{3.27}$$

with the final condition

$$E^TX(T) = E^TPE \tag{3.28}$$

where $X(t)$ and $Y(t)$ are $n\times n$ time-varying matrices. Such $X(t)$ gives rise to an optimal feedback gain

$$K(t) := -R^{-1}\,\{\,S^T+B^TX(t)\,\} \tag{3.29}$$

Moreover the minimum value of the cost function J is given by

$$J_{min} = \frac{1}{2}x_0^TE^TX(0)x_0. \tag{3.30}$$

The simultaneous differential equations (3.26),(3.27) are called GRDE in the sequel. Before proving this theorem, we shall state two lemmas.

Lemma 3 : Under Assumptions 1, 2 and 3, there exists a solution $X_o \in \mathbf{R}^{n\times n}$ of the algebraic simultaneous equations

$$X_o^TA+A^TX_o+Q-(X_o^TB+S)R^{-1}(S^T+B^TX_o)=0 \tag{3.31}$$

$$X_o^TE = E^TX_o \tag{3.32}$$

such that $(E,\ A+BK_o)$ is admissible, where

$$K_o := -R^{-1}(S^T+B^TX_o). \tag{3.33}$$

Proof : A proof is given in [9]. □

Lemma 4 : Under Assumptions 1, 2 and 3, there exists a solution $Z_o(t)$ of the simultaneous differential equations

$$E^T\dot{Z}_o(t)+Z_o^T(t)A+A^TZ_o(t)+Q - \{\,Z_o^T(t)B+S\,\}\,R^{-1}\,\{\,S^T+B^TZ_o(t)\,\}=0 \tag{3.34}$$

$$Z_o^T(t)E = E^TZ_o(t) \tag{3.35}$$

with the final condition

$$E^TZ_o(T) = E^TPE \tag{3.36}$$

where $Z_o(t)$ is an $n\times n$ time-varying matrix.

Proof : A proof is given in [9]. □

Now we shall give a proof of Theorem 1.

(*Necessity*) From Lemma 2 there exists a matrix X such that $\gamma = Xx$, where $(x(t),\ \gamma(t))$ is a solution of the TPBVP (3.4),(3.5). And defining Y by (3.20), from (3.17) we get

$$E^T\gamma = YEx \tag{3.37}$$

Substituting $\gamma = Xx$ into this, we have

$$(YE-E^TX)\,x = 0 \tag{3.38}$$

Since this holds for arbitrary x, (3.27) must hold.

Substitute $\gamma = Xx$ into (3.9) to get

$$E\dot{x} = \{A-BR^{-1}(S^T+B^TX)\}\,x \tag{3.39}$$

$$E^T\dot{X}x+E^TX\dot{x} = (-Q+SR^{-1}S^T-A^TX+SR^{-1}B^TX)x \tag{3.40}$$

Arranging (3.27), (3.39) and (3.40), we have

$$\{E^T\dot{X}+YA+A^TX+Q-(YB+S)R^{-1}(S^T+B^TX)\}\,x=0. \tag{3.41}$$

Since this holds for arbitrary x, (3.26) must hold. The final condition (3.28) is derived by substituting $\gamma = Xx$ into (3.5). Hence X and Y defined respectively by (3.23) and (3.20) must satisfy (3.26),(3.27) and (3.28).

For X, the control law

$$u(t) = -R^{-1}\,\{\,S^T+B^TX(t)\,\}\,x(t) \tag{3.42}$$

is derived from (3.8) and (3.24), so that a feedback gain is given by (3.29). Furthermore

$$\begin{aligned}
&\frac{d}{dt}\left(x^TE^TXx\right)\\
&=\dot{x}^TE^TXx+x^TE^T\dot{X}x+x^TE^TX\dot{x}\\
&=(u^TB^T+x^TA^T)Xx\\
&\quad+x^T\{-YA-A^TX-Q+(YB+S)R^{-1}(S^T+B^TX)\}\,x\\
&\quad+x^TY(Ax+Bu)\\
&=-[\,x^T\ u^T\,]\begin{bmatrix} Q & S\\ S^T & R\end{bmatrix}\begin{bmatrix} x\\ u\end{bmatrix}\\
&\quad+\{\,x^T(YB+S)R^{-1}+u^T\,\}\,R\,\{\,u+R^{-1}(S^T+B^TX)x\,\}
\end{aligned} \tag{3.43}$$

Applying the control law (3.42), the value of the cost function (2.10) is equal to

$$\begin{aligned}
J &= \frac{1}{2}x^T(T)E^TPEx(T)+\frac{1}{2}\int_0^T -\frac{d}{dt}\left(x^TE^TXx\right)dt\\
&= \frac{1}{2}x_0^TE^TX(0)x_0
\end{aligned} \tag{3.44}$$

(*Sufficiency*) From Lemma 4 there exists a solution $Z_o(t)$ of the simultaneous differential equations (3.34), (3.35) and (3.36). For $Z_o(t)$, similarly to (3.43), we have

$$\begin{aligned}
&\frac{d}{dt}\left(x^TE^TZ_ox\right)\\
&=-[\,x^T\ u^T\,]\begin{bmatrix} Q & S\\ S^T & R\end{bmatrix}\begin{bmatrix} x\\ u\end{bmatrix}\\
&\quad+\{x^T(Z_o^TB+S)R^{-1}+u^T\}\,R\,\{u+R^{-1}(S^T+B^TZ_o)x\}
\end{aligned} \tag{3.45}$$

so that

$$\begin{aligned}
&J(Ex_0,u,T)\\
&=\frac{1}{2}x^T(T)E^TPEx(T)+\frac{1}{2}\int_0^T -\frac{d}{dt}\left(x^TE^TZ_ox\right)\,dt\\
&\quad+\frac{1}{2}\int_0^T\{x^T(Z_o^TB+S)R^{-1}+u^T\}\\
&\qquad\times R\,\{u+R^{-1}(S^T+B^TZ_o)x\}\,dt\\
&\geq \frac{1}{2}x_0^TE^TZ_o(0)x_0
\end{aligned} \tag{3.46}$$

Thus the control law

$$u_o(t) := -R^{-1}\,\{\,S^T+B^TZ_o(t)\,\}\,x(t) \tag{3.47}$$

minimizes the cost function J.

It may be noted that $E^T\gamma(0)$ is uniquely determined from Ex_0 by (3.17), where $(x(t),\ \gamma(t))$ is a solution of the TPBVP (3.4),(3.5). Both the control law (3.42) and (3.47) satisfy the TPBVP (3.4),(3.5), so that

$$E^T\gamma(0) = E^T X(0)x_0 = E^T Z_o(0)x_0 \tag{3.48}$$

This implies that

$$J_{min} = \frac{1}{2}x_0^T E^T X(0)x_0 = \frac{1}{2}x_0^T E^T Z_o(0)x_0 \tag{3.49}$$

Therefore (3.29) is an optimal feedback gain. □

As an extension of the Riccati equation for state-space systems, Lewis [7] has derived a GRDE

$$\begin{aligned} E^T\dot{X}(t)E + &E^T X(t)A + A^T X(t)E + Q \\ &- \{\ E^T X(t)B + S\ \}R^{-1}\{\ S^T + B^T X(t)E\ \} = 0, \end{aligned} \tag{3.50}$$

which may have no solution [2]. Assuming the invertibility of $\bar{R}$, generalized R-matrix, Bender and Laub [2] have derived four types of Riccati differential equations with reduced dimension, all of which have the same solution. But, since these Riccati equations are based on an SVD coordinate system, they are rather complicated. It is to be noted that the present generalized Riccati equation of Theorem 1 consists of a dynamic equation and some nondynamic constraints, and has always a solution under mild conditions, since it is derived under the stronger assumption that R is positive definite. Of course, if $E = I$, then the simultaneous differential equations (3.26),(3.27) are equivalent to the Riccati equation for a state-space model.

C. Example

Consider the system ([2], [4])

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t) \tag{3.51}$$

$$x_1(0-) = x_{10} \tag{3.52}$$

with a nondynamic mode ∞ and an impulsive mode ∞/∞. Set the weighting matrices as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \geq 0, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$S = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R = 1. \tag{3.53}$$

From (3.27), we have

$$X(t) = \begin{bmatrix} X_{11}(t) & 0 \\ X_{21}(t) & X_{22}(t) \end{bmatrix}, Y(t) = \begin{bmatrix} Y_{11}(t) & Y_{12}(t) \\ 0 & Y_{22}(t) \end{bmatrix}$$

$$X_{11} = Y_{11} \tag{3.54}$$

Thus, the GRDE of (3.26) gives

$$\begin{aligned} \dot{X}_{11}(t) + X_{21}(t) + Y_{12}(t) + 1 - X_{21}(t)Y_{12}(t) &= 0 \\ X_{22}(t) + Y_{11}(t) - X_{22}(t)Y_{12}(t) &= 0 \\ X_{11}(t) + Y_{22}(t) - X_{21}(t)Y_{22}(t) &= 0 \\ 1 - X_{22}(t)Y_{22}(t) &= 0 \end{aligned} \tag{3.55}$$

From the last equation of (3.55), we simply take $X_{22}(t) = a(\neq 0)$ and $Y_{22}(t) = 1/a$. Thus we have a Riccati equation

$$\dot{X}_{11}(t) + 2 - X_{11}^2(t) = 0, \quad X_{11}(T) = P_{11}. \tag{3.56}$$

Since $P_{11} \geq 0$, $X_{11}(t)$ exists for $t \leq T$, so that

$$X(t) = \begin{bmatrix} X_{11}(t) & 0 \\ 1 + aX_{11}(t) & a \end{bmatrix}, Y(t) = \begin{bmatrix} X_{11}(t) & 1 + \frac{1}{a}X_{11}(t) \\ 0 & \frac{1}{a} \end{bmatrix} \tag{3.57}$$

Hence an optimal feedback gain is given by

$$K(t) = -\begin{bmatrix} 1 + aX_{11}(t) & a \end{bmatrix}, \quad a \neq 0 \tag{3.58}$$

and the minimum value of the cost function J is given by

$$J_{min} = \frac{1}{2}X_{11}(0)x_{10}^2. \tag{3.59}$$

For the stationary case with $T \to \infty$, the gain $K(t)$ converges to

$$K = -\begin{bmatrix} 1 + a\sqrt{2} & a \end{bmatrix}, \quad a \neq 0. \tag{3.60}$$

If we take $a = \frac{\sqrt{2}}{3}$, then $K = (-\frac{1}{3}\ \ \frac{\sqrt{2}}{3})$, the minimum norm solution of Bender and Laub [2]; if $a = \frac{\sqrt{2}}{4}$, then we get $K = (-\frac{1}{2}\ \ \frac{\sqrt{2}}{4})$, the solution computed by Cobb [4].

IV Conclusions

This paper has considered finite-horizon LQ regulator problems for a continuous-time descriptor system without using an SVD coordinate system or a preliminary transformation. We have derived a GRDE that yields an optimal solution.

The present approach can be extended to the infinite-horizon LQ regulator problem, and all the optimal feedback gains are parametrized by the solution of the generalized eigenvalue problem associated with a Hamiltonian equation and a free parameter [9].

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