Equivalent Classes of Decouplable and Controllable Linear Systems

In-Joong Ha† Sung-Joon Lee‡

†Department of Control and Instrumentation Engineering,
Automation and Systems Research Institute, Seoul National University
‡Department of Electronics Engineering, Seoul National University,
San 56-1 Shinrim-Dong, Kwanak-Ku, Seoul 151-742, Korea

ABSTRACT

The problem we consider in this paper is more demanding than the problem of input-output linearization with state equivalence recently solved by Cheng, Isidori, Respondek, and Tarn. We request that the MIMO nonlinear system, for which the problem of input-output linearization with state-equivalence is solvable, can be decoupled. In exchange for further requirement like this, our problem produces more usable and informative results than the problem of input-output linearization with state-equivalence. We present the necessary and sufficient conditions for our problem to be solvable. We characterize each of the nonlinear systems satisfying these conditions by a set of parameters which are invariant under the group action of state feedback and transformation. Using this set of parameters, we can determine directly the unique one, among the canonical forms of decouplable and controllable linear systems, to which a nonlinear system can be transformed via appropriate state feedback and transformation. Finally, we present the necessary and sufficient conditions for our problem to be solvable with internal stability, that is, for stable decoupling.

I. INTRODUCTION

In the last decade, the problem of exact linearization of nonlinear systems has been extensively studied. The so-called, state-space linearization problem, which is to find how a nonlinear system of dimension n with no output can be transformed into a linear system of dimension n with no output via state feedback and transformation, was solved by Su [22] and Dayawansa *et al.* [5] for the case of single-input systems and by Jakubczyk and Respondek [16] and Hunt *et al.* [11] for the case of multiple-input systems. On the other hand, the input-output linearization problem is to find how a nonlinear system with the output can be transformed so as to have the input-output dynamic characteristics of a linear system. Isidori and Ruberti [12] utilized the Volterra series expansions of input-output maps to solve this input-output linearization problem.

The state-space linearization technique is an efficient method of stabilizing a nonlinear system via state feedback because the well-developed linear system theory can be applied directly to the linearized form of the nonlinear system. However, the resulting closed-loop system, in general, does not have the dynamic characteristics of a linear system. Moreover, most of practical systems have the output to be controlled in a desired fasion. The input-output linearization technique resolves this issue and enables nonlinear systems to have the input-output dynamic characteristics of linear systems. However, it considers only the input-output dynamic behavior of the nonlinear system and can leave some part of the hidden dynamics (often called the zero dynamics [3], [14]) unstable in exchange for giving the linear input-output dynamic behavior. Hence, it cannot guarantee internal stability of the resulting closed-loop system.

Therefore, it is more desirable in practical applications if we can transform a nonlinear system of dimension n with the output into a linear system of dimension n with the output via appropriate state feedback and transformation. If it is possible, the closed-loop system will have the input-output dynamic characteristics of the linear system. Moreover, the internal stability of the closed-loop system can be readily guaranteed. This approach is often called input-output linearization with state equivalence. The necessary and sufficient conditions for linearization with state equivalence have been found by Lee *et al.* [17] for the case of SISO (single-input single-output) systems and by Cheng *et al.* [4] for the case of MIMO (multi-input multi-output) systems.

The problem we consider in this paper is more demanding than the problem of input-output linearization with state equivalence. We require that the MIMO nonlinear system, for which the problem of input-output linearization with state equivalence is solvable, can be decoupled. If this is possible, the well-developed results in decoupling of linear systems (for instance, [6], [7],[10], [23]) can be directly applied to the MIMO nonlinear system. Our problem has not been solved before. Besides this fact, there are other important reasons why we attempt to solve this problem. First, decoupling is known to be one of the most efficient control methods for MIMO systems. Second, all of the well-known results on decoupling of nonlinear systems (for instance, [8], [9], [14], [15], [18], [20], [21]) can provide the input-output dynamic characteristics of decoupled linear systems but cannot handle efficiently the issue of internal stability except for some special cases [3], [13]. Third, the necessary and sufficient conditions for our problem to be solvable are more easily verified than those for the input-output linearization with state equivalence. Fourth, we can characterize the invariant properties such as stable decouplability without solving a set of partial differential equations,

Just as the Brunovsky canonical forms of controllable linear systems [2] play important roles in the state-space linearization problem, the canonical forms of decouplable and controllable linear systems presented recently in [10] do in our problem. Unfortunately, the canonical forms of general MIMO linear systems have not been known until now. For these reasons, our problem should produce more informative and usable results than the problem of input-output linearization with state-equivalence. In fact, our results are quite informative and, successfully, extend some of the well-known results in decoupling of linear systems to a special class of nonlinear systems.

The presentation of our results is organized in the following way. In Section II we introduce some notations and definitions used frequently in our development and describe the prior works which are closly related to ours. In Section III, we give the necessary and sufficient conditions for a nonlinear system with the output to be transformed to a controllable and decouplable linear system via state feedback and transformation. We characterize each of the nonlinear systems satisfying these conditions by a set of parameters which are invariant under the group action of state feedback and transformation. We also show that this set of parameters determines one and the only one of the canonical forms of decouplable and controllable linear systems, to which a nonlinear system can be transformed via state feedback and transformation. Finally, we present the necessary and sufficient conditions for our problem to be solvable with internal stability, that is, for stable decoupling. Section IV contains some concluding remarks.

II. Preliminaries

We consider nonlinear systems of the form:

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad y_i = h_i(x), \quad i = 1, \cdots, m \qquad (2.1)$$

where f, $g_1, \cdots, g_m$ are $\mathcal{C}^\infty$-vector fields defined on $\Re^n$; $h_1, \cdots, h_m$ are $\mathcal{C}^\infty$-functions defined on $\Re^n$; and $u_i(t) \in \Re$, $y_i(t) \in \Re$ are the ith components of the input $u(t) \in \Re^m$, the output $y(t) \in \Re^m$, respectively. Let $g \triangleq [g_1 \cdots g_m]$ and $h \triangleq [h_1 \cdots h_m]^T$. We denote by $\{f, g, h\}$ the system in (2.1) with $n \geq m$. In this paper, we study only the local dynamic behavior of the system $\{f, g, h\}$ around the origin and assume without loss of generality that $f(0) = 0$, $h(0) = 0$.

For conciseness of our presentation, we further specialize the class of nonlinear systems considered in this paper as follows. For a system $\{f, g, h\}$, define the relative degrees, $d_1, \cdots, d_m$ by the nonnegative integers satisfying the following m-row vector conditions on an open neighborhood of the origin:

$$[\mathrm{L}_{g_1}\mathrm{L}_f^k h_i(x) \cdots \mathrm{L}_{g_m}\mathrm{L}_f^k h_i(x)] = 0, \ k = 0, 1, \cdots, d_i - 1 \ \text{ if } \ d_i \geq 1,$$

but $D_i^*(0) \neq 0$ where $D_i^*(x) \triangleq [\mathrm{L}_{g_1}\mathrm{L}_f^{d_i} h_i(x) \cdots \mathrm{L}_{g_m}\mathrm{L}_f^{d_i} h_i(x)]$. We denote as $\mathcal{S}$ the set that consists of all systems $\{f, g, h\}$ satisfying the following assumption:

(C.1) the relative degrees, d_i, $i = 1, \cdots, m$ are well-defined on an open neighborhood of the origin.

Most of notations and symbols used in our development follow those in [14] and other literature. However, we still need some new notations and definitions to make our presentation clear and concise. Let χ be an open subset of $\Re^n$. Let T be a $\mathcal{C}^\infty$-mapping from χ into $\Re^n$. A $\mathcal{C}^\infty$-function $\hat{\phi} : \Re^n \to \Re$ is T-related to a $\mathcal{C}^\infty$-function $\phi : \Re^n \to \Re$ if $\phi(p) = \hat{\phi} \circ T(p)$, $p \in \chi$ where $\circ$ denotes the function composition. A $\mathcal{C}^\infty$-vector field $\hat{f}$ on $\Re^n$ is T-related to a $\mathcal{C}^\infty$-vector field f if $T_{*p}f(p) = \hat{f}(T(p))$, that is, $\hat{f} \circ T(p) = \frac{\partial T}{\partial x} f(p)$, $p \in \chi$. The class of control laws we consider in this paper has the form : $u = \alpha(x) + \beta(x)\hat{u}$ where χ is an open neighborhood of the origin; $\alpha : \chi \to \Re^m$ and $\beta : \chi \to \Re^{m \times m}$ are $\mathcal{C}^\infty$; and $\beta(0)$ is nonsingular. We denote by $\mathcal{F}$ the set of all such feedback pairs $\{\alpha, \beta\}$. We denote by $\{f, g, h\}^{\alpha,\beta}$ the feedback system of $\{f, g, h\}$ corresponding to a control law $u = \alpha(x) + \beta(x)\hat{u}$ with $\{\alpha, \beta\} \in \mathcal{F}$. In other words, $\{f, g, h\}^{\alpha,\beta}$ stands for the feedback system $\{\hat{f}, \hat{g}, \hat{h}\}$ defined on an open neighborhood of 0 where $\hat{f}(\hat{x}) \triangleq f(\hat{x}) + \sum_{i=1}^{m} \alpha_i(\hat{x})g_i(\hat{x})$, $\hat{g}_j(\hat{x}) \triangleq \sum_{i=1}^{m} \beta_{ij}(\hat{x})g_i(\hat{x})$, and $\hat{h}_j(\hat{x}) = h_j(\hat{x})$, $j = 1, \cdots, m$. Here, α_i is the ith component of α and β_{ij} is the (i,j)th component of β. Define a mapping $J : \chi \times \Re^m \to \chi \times \Re^m$ by

$$J(x, u) \triangleq \begin{bmatrix} T(x) \\ -[\beta(x)]^{-1}\alpha(x) + [\beta(x)]^{-1}u \end{bmatrix}, (x, u) \in \chi \times \Re^m$$

where χ is an open neighborhood of 0, T is a $\mathcal{C}^\infty$-diffeomorphism, and $\{\alpha, \beta\} \in \mathcal{F}$. Let $\mathcal{G}$ be the set of such mappings J. Then, $\mathcal{G}$ is a group with a right composition $\circ$ defined by

$$J_2 \circ J_1(x, u) \triangleq \begin{bmatrix} T_2(T_1(x)) \\ -[\beta_2(T_1(x))]^{-1}\{\alpha_2(T_1(x)) + [\beta_1(x)]^{-1}\alpha_1(x)\} \\ +[\beta_2(T_1(x))]^{-1}[\beta_1(x)]^{-1}u \end{bmatrix}$$

for $J_1, J_2 \in \mathcal{F}$, where two mappings J_1, J_2 are identified if they agree on some open neighborhood of 0. We often write $J = \{\alpha, \beta, T\}$.

Using these definitions, we give relations on the set $\mathcal{S}$, which concern state feedback and transformation between the systems in $\mathcal{S}$.

Definition 2.1 : $\{\hat{f}, \hat{g}, \hat{h}\}$ is *feedback-equivalent at the origin to* $\{f, g, h\}$ if there exists a mapping $J \triangleq \{\alpha, \beta, T\} \in \mathcal{G}$ such that the following properties hold for $\{\tilde{f}, \tilde{g}, \tilde{h}\} \triangleq \{\hat{f}, \hat{g}, \hat{h}\}^{\alpha,\beta}$:

(i) the vector fields $\tilde{f}, \tilde{g}_1, \cdots, \tilde{g}_m$ are T-related to the vector fields $f, g_1, \cdots, g_m$, respectively,

(ii) the functions $\tilde{h}_1, \cdots, \tilde{h}_m$ are T-related to the functions $h_1, \cdots, h_m$, respectively.

In particular when the above properties hold with $\alpha = 0$ and $\beta = I_m$, $\{\hat{f}, \hat{g}, \hat{h}\}$ is *state-equivalent at the origin to* $\{f, g, h\}$. □

Each of the above relations is actually an equivalent relation on the set $\mathcal{S}$ and will partition the set $\mathcal{S}$ into equivalent classes. Note that two systems $\{f, g, h\}$, $\{\tilde{f}, \tilde{g}, \tilde{h}\}$ which belong to one of the equivalent classes generated by the state-equivalent relation will have the identical input-output dynamic characteristics if $\tilde{x}(0) = T(x(0))$, $t \geq 0$.

Let $\{A, B, C\}$ denote a controllable linear system of the form:

$$\dot{x} = Ax + \sum_{i=1}^{m} B_i u_i, \quad y_i = C_i x, \quad i = 1, \cdots, m$$

where $A \in \Re^{n \times n}$ and $B_i \in \Re^{n \times 1}$, $C_i \in \Re^{1 \times n}$ are the ith components of $B \in \Re^{n \times m}$, $C \in \Re^{m \times n}$, respectively. Let $\mathcal{L}$ be the set of such controllable linear systems. Then the problem of "input-output linearization with state equivalence" is described as: to find the conditions on a system $\{f, g, h\}$, which guarantee the existence of the mappings α, β, and T such that the feedback system $\{f, g, h\}^{\alpha,\beta}$ is state-equivalent on $\Re^n$ to a linear system $\{A, B, C\} \in \mathcal{L}$. The necessary and sufficient conditions for the local existence of such mappings α, β, and T already have been known [4]. As the conditions are closely related to our work, we state them here in terms of our definitions.

To do so, we need to introduce other notations, which are used in [4]. For $\{f, g, h\}$ in $\mathcal{S}$, define Toeplitz matrices $M_k : \Re^n \to \Re^{(k+1)m \times (k+1)m}$, $k = 0, 1, \cdots, 2n - 1$ by

$$M_k(x) = \begin{bmatrix} H_0(x) & H_1(x) & \cdots & H_k(x) \\ 0 & H_0(x) & \cdots & H_{k-1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H_0(x) \end{bmatrix}$$

where each $H_k(x)$ is an $m \times m$ matrix function with $\mathrm{L}_{g_j}\mathrm{L}_f^k h_i(x)$ as its (i,j)th component. We say that $x_0 \in \Re^n$ is a regular point for M_k if the rank of $M_k(x)$ is constant around x_0 and we denote its rank by $r_\rho(M_k)$. On the other hand, the dimension of the vector space generated over the field $\Re$ by the rows of M_k is denoted by $r_R(M_k)$. We also define the distributions L_j, $j = 1, \cdots, n$ by

$$L_j \triangleq \mathrm{span}\{\mathrm{ad}_f^k g_i : i = 1, \cdots, m, k = 0, 1, \cdots, j - 1\}.$$

Now, we are ready to state Theorem 8 in [4].

Theorem 2.1 : Let $\{f,g,h\} \in \mathcal{S}$. Then, it is feedback-equivalent at the origin to a linear system $\{A,B,C\}$ in $\mathcal{L}$ if and only if the following properties hold on an open neighborhood of χ of the origin:

(i) $\dim L_j = \nu_j$, $j=1,\cdots,n$ and $\nu_n = n$ where the ν_j are nonnegative integers,

(ii) for all $k=0,1,\cdots,2n-1$, the origin is a regular point of M_k and $r_R(M_k) = r_\rho(M_k)$,

(iii) there exists a solution pair $\{\alpha,\beta\}$ of

$$\begin{array}{rcl}[\mathrm{L}_{g_1}\Gamma\cdots\mathrm{L}_{g_m}\Gamma]\alpha &=& -\mathrm{L}_f\Gamma \\ [\mathrm{L}_{g_1}\Gamma\cdots\mathrm{L}_{g_m}\Gamma]\beta &=& [I_{r_{2n-1}} \quad 0]\end{array} \tag{2.2}$$

such that $\{\tilde{f},\tilde{g},\tilde{h}\} \triangleq \{f,g,h\}^{\alpha,\beta}$ with $\{\alpha,\beta\} \in \mathcal{F}$ satisfies

$$[\mathrm{ad}^q_{\tilde{f}}\tilde{g}_i, \mathrm{ad}^r_{\tilde{f}}\tilde{g}_j] \doteq 0,\ i,j=1,\cdots,m, q+r=0,1,\cdots,2n-1 \tag{2.3}$$

where the vector function $\Gamma : \chi \rightarrow \Re^{r_{2n-1}}$ and the nonnegative integer r_{2n-1} are determined via the structure algorithm in [12].

Note that the equation (2.2), in general, has infinitely many solution pairs $\{\alpha,\beta\}$ [12]. Thus, it is not an easy task to check (2.3) for all possible solution pairs of (2.2). As is shown below, however, when the system $\{f,g,h\}$ can be decoupled, the solution pair of (2.2) is unique and can be obtained easily. As the result, it becomes easy to check the properties (ii), (iii) in Theorem 2.1.

It is well known from [8], [20], [21] that $\{f,g,h\} \in \mathcal{S}$ can be decoupled at the origin by a control law $u = \alpha(x) + \beta(x)\hat{u}$ with $\{\alpha,\beta\} \in \mathcal{F}$ if and only if it satisfies that

(C.2) $D^*(0)$ is nonsingular.

Furthermore, the control law $u = \alpha^*(x) + \beta^*(x)\hat{u}$ given by

$$\alpha^*(x) \triangleq -[D^*(x)]^{-1}A^*(x), \quad \beta^*(x) \triangleq [D^*(x)]^{-1} \tag{2.4}$$

decouples the system $\{f,g,h\}$ at 0, where $D^* : \Re^n \rightarrow \Re^{m\times m}$ and $A^* : \Re^n \rightarrow \Re^m$ are defined as follows:

$$D^*(x) \triangleq \begin{bmatrix} D^*_1(x) \\ \vdots \\ D^*_m(x)\end{bmatrix}, \quad A^*(x) \triangleq \begin{bmatrix} \mathrm{L}_f^{d_1+1}h_1(x) \\ \vdots \\ \mathrm{L}_f^{d_m+1}h_m(x)\end{bmatrix}. \tag{2.5}$$

Note that (C.2) assures that the property (ii) in Theorem 2.1 holds with $r_R(M_k) = r_\rho(M_k) = \mu_k, k=0,1,\cdots,2n-1$ where μ_k is the cardinal number of the set $\{i : d_i \le k, i = 1,\cdots,m\}$. Moreover, the computation of Γ through use of the structure algorithm in [12] shows that (1) $\mathrm{L}_f\Gamma$ in (2.2) is just the row permutation of A^* and (2) $r_{2n-1} = m$. Without loss of generality, we can assume that the output of the system $\{f,g,h\}$ is ordered so that:

$$\mathrm{L}_f\Gamma(x) = A^*(x), \quad [\mathrm{L}_{g_1}\Gamma(x)\cdots\mathrm{L}_{g_m}\Gamma(x)] = D^*(x).$$

Therefore, the solution pair of (2.2) is unique and is given by (2.4). Furthermore, $\{\alpha^*,\beta^*\} \in \mathcal{F}$. Hence we see that the properties (ii), (iii) in Theorem 2.1 can be checked easily in the case when the system $\{f,g,h\}$ can be decoupled. In other words, the conditions for the system $\{f,g,h\}$ to be feedback-equivalent to a linear system $\{A,B,C\}$ can be easily verified under the additional assumption (C.2).

The problem we consider in this paper is more restrictive than the problem of input-output linearization with state equivalence, which was solved by Cheng, *et al.* [4]. We request that a linear system $\{A,B,C\} \in \mathcal{L}$, to which a system $\{f,g,h\}$ is feedback-equivalent, be decouplable. We denote the set of all decouplable systems in $\mathcal{L}$ by $\mathcal{L}^d$. We also denote by $\mathcal{S}^d$ the set of all systems in $\mathcal{S}$ which are feedback-equivalent at the origin to linear systems in $\mathcal{L}^d$. It has been commonly recognized that decoupling control is one of the most efficient control methods for MIMO systems. In the next section, we present the necessary and sufficient conditions for a system $\{f,g,h\}$ to be in $\mathcal{S}^d$. The conditions can be easily deduced from Theorem 2.1 and can be easily verified, as was suggested in the preceding paragraph.

Even when a system $\{f,g,h\}$ is found to be feedback-equivalent to a decouplable linear system $\{A,B,C\}$, however, we still face the very important question whether the linear system $\{A,B,C\}$ can be stably decoupled (that is, decoupled with internal stability). Of course, we can answer this question by finding a mapping T that transforms $\{f,g,h\}^{\alpha^*,\beta^*}$ into a decoupled linear system $\{A^*,B^*,C^*\}$ and by applying Theorem 5 in [7] to the decoupled linear system $\{A^*,B^*,C^*\}$. However, such a mapping T can be obtained, in general, by solving a set of partial differential equations. Therefore, it would be more economic if we could know as much about the system as possible without solving the set of partial differential equations. This is just the main issue we attempt to resolve in this paper.

Recently, we have presented the canonical forms of decouplable and controllable linear systems, which are determined uniquely by a complete set of invariances [10]. We introduce here the set of canonical forms of decouplable and controllable linear systems defined in [10] since we refer to it frequently in the next section.

Definition 2.2 : The set denoted by $\mathcal{C}^d$ is a collection of controllable and decouplable linear systems $\{A^+,B^+,C^+\}$ determined uniquely by the set of parameters, which consists of integers $\{\bar{d}_i, i=1,\cdots,m\}$, $\{\bar{p}_i, i=1,\cdots,m+1\}$, $\bar{q}$, $\{\bar{m}_1,\cdots,\bar{m}_q\}$, $\{\bar{n}_i, i=\bar{m}_1,\cdots,\bar{m}_q\}$, and real numbers $\{\bar{\beta}_{ij}, j=1,\cdots,\bar{p}_i-\bar{d}_i-1, i=1,\cdots,m\}$, $\{\bar{\alpha}_{ijk}, k=0,1,\cdots,\bar{h}_{ji}, j=\bar{m}_1,\cdots,\bar{m}_q, i=\bar{m}_1,\bar{m}_1+1,\cdots,m\}$, where $\bar{h}_{ji} \triangleq \min(\bar{n}_i,\bar{n}_j-1)$ if $j<i$ and $\bar{h}_{ji} \triangleq \min(\bar{n}_i,\bar{n}_j)-1$ if $j \ge i$, in the following way.

(i) $1 \le \bar{m}_1 < \bar{m}_2 < \cdots < \bar{m}_q \le m$, $\bar{n}_i \ge 1$, $i=\bar{m}_1,\cdots,\bar{m}_q$,

$$\bar{p}_i \ge \bar{d}_i + 1 \ge 1, \quad i=1,\cdots,m,$$

$$\bar{n}_{\bar{m}_1} + \bar{n}_{\bar{m}_2} + \cdots + \bar{n}_{\bar{m}_q} = \bar{p}_{m+1} \ge 0, \quad \sum_{i=1}^{m+1}\bar{p}_i = n.$$

(ii) The matrices A^+, B^+, and C^+ have the partitioned forms ;

$$A^+ \triangleq \begin{bmatrix} \bar{A}_1 & 0 & \cdots & 0 & 0 \\ 0 & \bar{A}_2 & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \bar{A}_m & 0 \\ \bar{A}^c_1 & \bar{A}^c_2 & \cdots & \bar{A}^c_m & \bar{A}_{m+1}\end{bmatrix},$$

$$B^+ \triangleq [B^+_1\cdots B^+_m] = \begin{bmatrix} \bar{b}_1 & 0 & \cdots & 0 \\ 0 & \bar{b}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{b}_m \\ 0 & 0 & \cdots & 0\end{bmatrix},$$

$$C^+ \triangleq \begin{bmatrix} C^+_1 \\ \vdots \\ C^+_m\end{bmatrix} = \begin{bmatrix} \bar{c}_1 & 0 & \cdots & 0 & 0 \\ 0 & \bar{c}_2 & & 0 & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \bar{c}_m & 0\end{bmatrix}$$

where $\bar{A}_i, \bar{A}^c_i, \bar{b}_i$, and $\bar{c}_i$ are $(\bar{p}_i \times \bar{p}_i)$, $(\bar{p}_{m+1} \times \bar{p}_i)$, $(\bar{p}_i \times 1)$, and $(1 \times \bar{p}_i)$ matrices, respectively, for $i=1,\cdots,m$ and $\bar{A}_{m+1}$ is a $(\bar{p}_{m+1} \times \bar{p}_{m+1})$ matrix.

(iii) For $i=1,\cdots,m$,

$$\bar{A}_i \triangleq \begin{bmatrix} 0 & I_{(\bar{p}_i-1)} \\ 0 & 0 \end{bmatrix}, \; \bar{b}_i \triangleq \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}, \; \bar{c}_i \triangleq [\bar{\beta}_{i\bar{r}_i} \cdots \bar{\beta}_{i1} \; 1 \; 0 \cdots 0]$$

where $\bar{r}_i \triangleq \bar{p}_i - \bar{d}_i - 1$.

(iv) If $\bar{p}_{m+1} \geq 1$,

(a) $\bar{A}_i^c \triangleq [\; \bar{b}_i^c \;\; 0 \;\; \cdots \;\; 0 \;]$, $i = 1, \cdots, m$, and

$$\bar{A}_{m+1} \triangleq \begin{bmatrix} \bar{A}_{\bar{m}_1 \bar{m}_1} & \bar{A}_{\bar{m}_1 \bar{m}_2} & \cdots & \bar{A}_{\bar{m}_1 \bar{m}_q} \\ \bar{A}_{\bar{m}_2 \bar{m}_1} & \bar{A}_{\bar{m}_2 \bar{m}_2} & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ \bar{A}_{\bar{m}_q \bar{m}_1} & \cdots & \cdots & \bar{A}_{\bar{m}_q \bar{m}_q} \end{bmatrix}$$

where

$$\bar{b}_i^c \triangleq [\underbrace{0 \cdots \bar{\sigma}_{1i}}_{\bar{n}_{\bar{m}_1}} \vdots \underbrace{0 \cdots \bar{\sigma}_{2i}}_{\bar{n}_{\bar{m}_2}} \vdots \cdots \vdots \underbrace{0 \cdots \bar{\sigma}_{qi}}_{\bar{n}_{\bar{m}_q}}]^T,$$

$$\bar{\sigma}_{ji} \triangleq \begin{cases} \bar{\alpha}_{i\bar{m}_j 0} & \text{if} \;\; \bar{m}_j < i \;\; \text{and} \;\; i \notin \{\bar{m}_1, \cdots, \bar{m}_q\} \\ 1 & \text{if} \;\; \bar{m}_j = i \\ 0 & \text{otherwise}, \end{cases}$$

$$\bar{A}_{ii} \triangleq \begin{bmatrix} \bar{\alpha}_{ii(\bar{n}_i-1)} & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & & 1 \\ \bar{\alpha}_{ii0} & 0 & \cdots & 0 \end{bmatrix}, \quad \bar{A}_{ji} \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \bar{\alpha}_{ij\bar{n}_j} & \cdot & & \cdot \\ \vdots & \vdots & & \vdots \\ \bar{\alpha}_{ij0} & 0 & \cdots & 0 \end{bmatrix},$$

(b) rank $[\bar{B}_{m+1}^i \;\; \bar{A}_{m+1} \bar{B}_{m+1}^i \; \cdots \; \bar{A}_{m+1}^{\bar{p}_{m+1}-1} \bar{B}_{m+1}^i] = \bar{p}_{m+1}$, $i = 1, \cdots, m$ where $\bar{B}_{m+1}^i \triangleq [\bar{b}_1^c \; \bar{b}_2^c \cdots \bar{b}_{i-1}^c \; \bar{b}_{i+1}^c \cdots \bar{b}_m^c]$. □

It is clear that if a system $\{f, g, h\}$ is in $\mathcal{S}^d$, then it is also feedback-equivalent to a canonical form $\{A^+, B^+, C^+\}$ in $\mathcal{C}^d$. Therefore, all properties of the system $\{f, g, h\}$, which are invariant under the group action of state feedback and transformation are possessed by the corresponding canonical form $\{A^+, B^+, C^+\}$. Stable decouplability is also an invariant property. In the next section, we show that a system $\{f, g, h\}$ in $\mathcal{S}^d$ can be characterized by a set of parameters. This set of parameters can be obtained algebraically without solving a set of partial differential equations and directly determines the canonical form $\{A^+, B^+, C^+\}$ in $\mathcal{C}^d$, which the system $\{f, g, h\}$ is feedback-equivalent to. Therefore, we can study all invariant properties of the system $\{f, g, h\}$ through use of its simplest representer, $\{A^+, B^+, C^+\}$ without solving a set of partial differential equations.

III. Main Results

Throughout this section, $\hat{f}$ and $\hat{g}_i$, $i = 1, \cdots, m$ denote the vector fields corresponding to the feedback system $\{f, g, h\}^{\alpha^*\beta^*}$. At each $x \in \Re^n$, we denote $\kappa_i(x)$, $i = 1, \cdots, m$ by the smallest nonnegative integers such that $\mathrm{ad}_{\hat{f}}^{\kappa_i(x)} \hat{g}_i(x)$ linearly depends on its antecedents in the ordered set $\{\hat{g}_1(x), \cdots, \hat{g}_m(x), \mathrm{ad}_{\hat{f}} \hat{g}_1(x), \cdots, \mathrm{ad}_{\hat{f}} \hat{g}_m(x), \cdots, \mathrm{ad}_{\hat{f}}^{n-1} \hat{g}_1(x), \cdots, \mathrm{ad}_{\hat{f}}^{n-1} \hat{g}_m(x)\}$. Recall that this definition parallels to the definition of Kronecker indices for linear systems (see e.g. [10],[19]). Also, note that if $\mathrm{ad}_{\hat{f}}^l \hat{g}_i$ linearly depends on its antecedents in the ordered set on an open subset of $\Re^n$, then $\mathrm{ad}_{\hat{f}}^k \hat{g}_i$, $k = l+1, l+2, \cdots$ do. Now, we are ready to state the following theorem.

Theorem 3.1 : Let $\{f, g, h\} \in \mathcal{S}$. Then, it is in $\mathcal{S}^d$ if and only if the following properties with (C.2) hold on an open neighborhood χ of the origin:

(i) $\kappa_i(x), i = 1, \cdots, m$ are constant and $\sum_{i=1}^m \kappa_i = n$,

(ii) for all nonnegative integers q, r such that for $q + r = 0, 1, \cdots, \kappa_i + \kappa_j - 1$,

$$[\mathrm{ad}_{\hat{f}}^q \hat{g}_i, \mathrm{ad}_{\hat{f}}^r \hat{g}_j] = 0, \quad i, j = 1, \cdots, m.$$

The necessary and sufficient conditions for a nonlinear system to be feedback-equivalent to a controllable and decouplable linear system can be easily derived from Theorem 2.1. However, they are more explicit and can be checked more easily, though more stringent, than the necessary and sufficient conditions for a nonlinear system to be feedback-equivalent to a controllable linear system. In particular, when $m = 1$, our conditions are reduced directly to the necessary and sufficient conditions in [17] for a SISO nonlinear system to be feedback-equivalent to a SISO controllable linear system.

By Theorem 3.1, we can easily check if a system $\{f, g, h\}$ have a feedback control law $u = \alpha(x) + \beta(x)\hat{u}$ and a state transformation $\hat{x} = T(x)$ which take the system to a decouplable and controllable linear system $\{A, B, C\}$. Such a feedback control law is given by (2.4). In the special case when $n = \sum_{i=1}^m (d_i + 1)$, the desired state transformation T can be also easily obtained and is given by

$$T(x) = \begin{bmatrix} T_1(x) \\ \vdots \\ T_m(x) \end{bmatrix}, \quad T_i(x) = \begin{bmatrix} h_i(x) \\ \mathrm{L}_f h_i(x) \\ \vdots \\ \mathrm{L}_f^{d_i} h_i(x) \end{bmatrix}.$$

However, in the general case when $n > \sum_{i=1}^m (d_i + 1)$, it is obtained usually by solving a set of partial differential equations. Therefore, the detailed structure of the linear system $\{A, B, C\} \in \mathcal{L}^d$ which is state-equivalent to $\{f, g, h\}^{\alpha^*\beta^*}$ can be known only after solving such a set of partial differential equations. As is mentioned in Section II, we had better know it, in advance, before struggling with the set of partial differential equations.

The next two lemmas characterize the nonlinear systems in $\mathcal{S}^d$ by a set of parameters. The set of parameters can be determined without solving a set of partial differential equations but identifies completely the canonical form $\{A^+, B^+, C^+\}$ in $\mathcal{C}^d$, to which a nonlinear system $\{f, g, h\}$ in $\mathcal{S}^d$ is feedback-equivalent. To state these lemmas, we define the sets Λ_i and the distributions Δ_i, $i = 1, \cdots, m$ by $\Lambda_i(\{f, g, h\}) \triangleq \{\mathrm{ad}_{\hat{f}}^k \hat{g}_j : k = 0, 1, \cdots, n-1, \; j = 1, \cdots, m, \; j \neq i\}$ and $\Delta_i(\{f, g, h\}) \triangleq LC\{\xi : \xi \in \Lambda_i(\{f, g, h\})\}$, respectively where LC denotes linear combinations of the elements in Λ_i over $\Re$.

Lemma 3.1 : Let $\{f, g, h\} \in \mathcal{S}^d$. For $i = 1, \cdots, m$, then, the following properties hold on an open neighborhood χ of the origin;

(i) $\Delta_i^\perp(\{f, g, h\})$ has constant dimension p_i such that

$$d_i + 1 \leq p_i \leq \kappa_i. \tag{3.1}$$

(ii) If $r_i \triangleq p_i - d_i - 1 \geq 1$, there exist unique real numbers, $\beta_{i1}, \cdots, \beta_{ir_i}$ such that

$$\mathrm{ad}_{\hat{f}}^{p_i} \hat{g}_i + \sum_{j=1}^{r_i} (-1)^j \beta_{ij} \mathrm{ad}_{\hat{f}}^{p_i - j} \hat{g}_i \in \Delta_i(\{f, g, h\}). \tag{3.2}$$

Proof: (i) Omitted.

(ii) Since $\dim(\Delta_i) = n - p_i$, we can take $(n - p_i)$ linearly independent vector fields, $\eta_{i1}, \cdots, \eta_{i(n-p_i)}$ from the set Λ_i. Then, by (i) of Theorem 3.1 and (3.1),

$$\text{the set } \bar{\Lambda}_i \triangleq \{\mathrm{ad}_{\hat{f}}^{p_i - 1} \hat{g}_i, \cdots, \hat{g}_i, \eta_{i1}, \cdots, \eta_{i(n-p_i)}\} \text{ is linearly independent on } \chi \text{ and spans } \chi. \tag{3.3}$$

Let $\Delta_i^* \triangleq \mathrm{span}\{\mathrm{ad}_{\hat{f}}^{p_i - 2} \hat{g}_i, \cdots, \hat{g}_i, \eta_{i1}, \cdots, \eta_{i(n-p_i)}\}$. By the property (ii) in Theorem 3.1, Lemma A.2 in the Appendix holds trivially with $\Delta = \Delta_i^*$ and $d = 1$. Therefore, there exists the solution T_{i1} satisfying the following set of partial differential equations on an open neighborhood χ of 0:

$$\mathrm{L}_{\mathrm{ad}_f^{p_i-1}\hat{g}_i} T_{i1} = (-1)^{p_i-1} \tag{3.4a}$$
$$\mathrm{L}_\zeta T_{i1} = 0, \quad \zeta \in \Delta_i^* \tag{3.4b}$$

Since (3.4) implies that $d\mathrm{L}_{\hat{g}_i}\mathrm{L}_f^j T_{i1} = 0$ on χ, $j = 0, 1, \cdots, p_i - 2$, it follows from Lemma A.1 in the Appendix that

$$\mathrm{L}_{\mathrm{ad}_f^j \hat{g}_i}\mathrm{L}_f^l T_{i1} = (-1)^{j-k}\mathrm{L}_{\mathrm{ad}_f^k \hat{g}_i}\mathrm{L}_f^{j+l-k}T_{i1}, \quad k = 0, 1, \cdots, j+l, j+l = 0, 1, \cdots, p_i - 1. \tag{3.5}$$

Define p_i-covector fields T_{ij}, $j = 1, \cdots, p_i$ by $T_{ij} \triangleq \mathrm{L}_f^{j-1}T_{i1}$. Then, by (3.4b), $dT_{ij} \in \Delta_i^\perp, j = 1, \cdots, p_i$. Moreover, by using (3.4) and (3.5). it is straightforward to show that

$$\{dT_{ij}, j = 1, \cdots, p_i\} \text{ is a basis of } \Delta_i^\perp \text{ at each } x \in \chi. \tag{3.6}$$

Since $d\mathrm{L}_f^{p_i}T_{i1} \in \Delta_i^\perp$, this implies that there exist $\mathcal{C}^\infty$-functions β_{ij}, $j = 1, \cdots, p_i$, $i = 1, \cdots, m$ such that

$$d\mathrm{L}_f^{p_i}T_{i1}(x) + \sum_{j=1}^{p_i}\beta_{ij}(x)d\mathrm{L}_f^{p_i-j}T_{i1}(x) = 0, \quad x \in \chi. \tag{3.7}$$

Post-multiplying $\mathrm{ad}_f^k\hat{g}_i(x)$, $k = 0, 1, \cdots, p_i - 1$ successively to (3.7) and using (3.3), (3.4), and (3.5) lead to the fact that β_{ik}, $k = 1, \cdots, p_i$ are constant and unique on χ. On the other hand, it is well known that (C.2) and (2.4) imply that for $i, j = 1, \cdots, m$, $k = 0, 1, \cdots$,

$$\mathrm{L}_{\mathrm{ad}_f^k \hat{g}_j} h_i(x) = \begin{cases} (-1)^k & \text{if } \; k = d_i \; \text{ and } \; i = j \\ 0 & \text{otherwise} \end{cases} \tag{3.8}$$

and that $dh_i, \cdots, d\mathrm{L}_f^{d_i}h_i$ are linearly independent on χ. And, by (3.8) and Lemma A.1 in the Appendix,

$$d\mathrm{L}_f^k h_i \in \Delta_i^\perp, \quad k = 0, 1, \cdots, d_i, i = 1, \cdots, m. \tag{3.9}$$

Thus, (3.8) and (3.9) lead to $\beta_{ir_i+1} = \beta_{ir_i+2} = \cdots = \beta_{ip_i} = 0$. Hence, the property (ii) has been verified.

In the proof of Lemma 3.1, we have shown that, if a system $\{f, g, h\}$ is in $\mathcal{S}^d$, the β_{ij} are constant and unique around 0 and hence we can determine these parameters only by considering (3.2) at $x = 0$. Also, the arguments in the proof of the above lemma suggest that if the $\mathcal{C}^\infty$-functions T_{i1}, $i = 1, \cdots, m$, which are the unique solutions of (3.4) are known, the condition in (3.2) can be replaced by

$$\mathrm{L}_f^{p_i}T_{i1}(x) + \sum_{j=1}^{r_i}\beta_{ij}\mathrm{L}_f^{p_i-j}T_{i1}(x) = 0, \quad x \in \chi. \tag{3.10}$$

From (3.8) and (3.4), we can see that $T_{i1} = h_i$, $i = 1, \cdots, m$ in the special case when $p_i = d_{i+1}$, $i = 1, \cdots, m$. We define T_i and $\hat{F}_i$, $i = 1, \cdots, m$ by

$$T_i(x) \triangleq [T_{i1}(x) \cdots T_{ip_i}(x)]^T \in \Re^{p_i}, \quad \hat{F}_i \triangleq [\, 0 \cdots 0 \; \beta_{ir_i} \cdots \beta_{i1}] \in \Re^{1\times p_i}. \tag{3.11}$$

Let q be the number of elements in the set $\mathcal{M} \triangleq \{i : i \in \{1, \cdots, m\}, \kappa_i > p_i\}$. We denote the elements in $\mathcal{M}$ by $m_i, i = 1, \cdots, q$ in the order of $m_1 < \cdots < m_q$. We define the vector fields $\tilde{f}$, $\tilde{g}_i$, $i = 1, \cdots, m$ by

$$\tilde{f}(x) \triangleq \hat{f}(x) + \sum_{i=1}^m \hat{g}_i(x)\hat{F}_iT_i(x), \quad \tilde{g}_i(x) \triangleq \hat{g}_i(x), \tag{3.12}$$

respectively. Here, (3.12) represents the vector fields corresponding to the new closed-loop system $\{\tilde{f}, \tilde{g}, \tilde{h}\}$ consisting of the system $\{f, g, h\}^{\alpha^*\beta^*}$ and the new control law $\hat{u}_i = \hat{F}_i\hat{x}_i + \tilde{u}_i$, $i = 1, \cdots, m$ where $\hat{x}_i \triangleq T_i(x)$, $i = 1, \cdots, m$. Direct calculations show that on an open neighborhood of the origin,

$$\mathrm{ad}_{\tilde{f}}^k\tilde{g}_i = \mathrm{ad}_f^k\hat{g}_i + \sum_{j=1}^{\min(k, r_i)}(-1)^j\beta_{ij}\mathrm{ad}_f^{k-j}\hat{g}_i, \; k = 0, 1, \cdots, \kappa_i - 1. \tag{3.13}$$

By (i) and (ii) of Theorem 3.1, and (3.13), we see that on an open neighborhood of 0,

$$\text{the set } \tilde{\Lambda} \triangleq \{\hat{g}_1, \cdots, \mathrm{ad}_f^{\kappa_1-1}\hat{g}_1, \cdots, \hat{g}_m, \cdots, \mathrm{ad}_f^{\kappa_m-1}\hat{g}_m\} \text{ is linearly independent,} \tag{3.14}$$

$$[\mathrm{ad}_f^k\hat{g}_i, \mathrm{ad}_f^l\hat{g}_j] = 0, \; k + l = 0, 1, \cdots, \kappa_i + \kappa_j - 1, \; i, j = 1, \cdots, m. \tag{3.15}$$

Also, using (3.4), (3.5), (3.10), and (3.13), we can easily show that the functions T_{ij}, $j = 1, \cdots, p_i$, $i = 1, \cdots, m$ satisfy that

$$\mathrm{L}_{\mathrm{ad}_f^k \hat{g}_l} T_{ij} = \begin{cases} (-1)^k & \text{if } \; i = l \; \text{ and } \; j = p_i - k \\ 0 & \text{otherwise} \end{cases} \tag{3.16}$$

for $k = 0, 1, \cdots$, $l = 1, \cdots, m$.

Now, we are ready to state the other lemma.

Lemma 3.2 : Let $\{f, g, h\} \in \mathcal{S}^d$. If $p_{m+1} \triangleq n - \sum_{i=1}^m p_i \geq 1$, then there exist unique constant parameters α_{ijk}'s such that for $i = 1, \cdots, m$,

$$(-1)^{p_i+n_i}\mathrm{ad}_f^{p_i+n_i}\hat{g}_i = \sum_{j\in\mathcal{M}}\sum_{k=0}^{h_{ji}}(-1)^{k+p_j}\alpha_{ijk}\mathrm{ad}_f^{k+p_j}\hat{g}_j, \tag{3.17}$$

on an open neighborhood χ of the origin, where

$$n_i \triangleq \kappa_i - p_i, \quad h_{ji} \triangleq \begin{cases} \min(n_i, n_j - 1) & \text{if } \; j < i \\ \min(n_i, n_j) - 1 & \text{otherwise.} \end{cases}$$

Now we show that the parameters introduced in Lemma 3.1 and Lemma 3.2 determine uniquely the canonical form $\{A^+, B^+, C^+\} \in \mathcal{C}^d$ which $\{f, g, h\} \in \mathcal{S}^d$ is feedback-equivalent to.

Theorem 3.2 : Let $\{f, g, h\} \in \mathcal{S}^d$. Then, it is feedback-equivalent at the origin to the linear system $\{A^+, B^+, C^+\}$ in Definition 2.2 with $\bar{p}_i = p_i$, $\bar{n}_i = n_i$, $\bar{d}_i = d_i$, $i = 1, \cdots, m$, $\bar{q} = q$, $\bar{m}_i = m_i$, $i = 1, \cdots, q$, $\bar{\beta}_{ij} = \beta_{ij}$, $j = 1, \cdots, \bar{p}_i - \bar{d}_i - 1$, $i = 1, \cdots, m$, and $\bar{\alpha}_{ijk} = \alpha_{ijk}$, $k = 0, \cdots, \bar{h}_{ji}$, $j = \bar{m}_1, \cdots, \bar{m}_{\bar{q}}$, $i = \bar{m}_1, \bar{m}_1 + 1, \cdots, m$.

Proof: By the arguments in the proof for the property (ii) of Lemma 3.1, there exist unique $\mathcal{C}^\infty$-functions, T_{i1}, $i = 1, \cdots, m$ satisfying (3.4). From (3.10) and (3.11), it then follows that on an open neighborhood χ of 0,

$$\mathrm{L}_{\tilde{f}}T_i(x) = (A_i - b_i\hat{F}_i)T_i(x), \quad x \in \chi, i = 1, \cdots, m \tag{3.18}$$

where

$$A_i \triangleq \begin{bmatrix} 0 & I_{p_i-1} \\ 0 & 0 \end{bmatrix}, \quad b_i \triangleq \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Letting $k = 0$ in (3.16) shows that

$$\mathrm{L}_{\hat{g}_i}T_j(x) = \begin{cases} b_i & \text{if } \; i = j \\ 0 & \text{otherwise.} \end{cases} \tag{3.19}$$

By (3.13), (3.18), and (3.19), we can write

$$L_{\tilde{f}}T_i(x) = A_iT_i(x), \quad i = 1,\cdots,m. \tag{3.20}$$

On the other hand, (3.9) and (3.6) imply that there exist C^∞-functions ρ_{ij}, $j = 1,\cdots,p_i$, $i = 1,\cdots,m$ such that:

$$dh_i(x) = \sum_{j=1}^{p_i}\rho_{ij}(x)dT_{i(p_i-j+1)}(x),\ x \in \chi, i = 1,\cdots,m. \tag{3.21}$$

Then, for $k = 0,1,\cdots,p_i-1$,

$$L_{\mathrm{ad}_{\tilde{f}}^k\hat{g}_i}h_i(x) = \sum_{j=1}^{p_i}\rho_{ij}(x)L_{\mathrm{ad}_{\tilde{f}}^k\hat{g}_i}T_{i(p_i-j+1)}(x). \tag{3.22}$$

Applying the properties in (3.8), (3.4), and (3.5) to (3.22) for $k = 0,1,\cdots,d_i$ yields

$$\rho_{ij} = 0,\ j = 1,\cdots,d_i,\ \rho_{i(d_i+1)} = 1 \ \text{ on } \chi, i = 1,\cdots,m. \tag{3.23}$$

And applying (3.8), (3.4), and (3.10) to (3.22) for $k = d_i + 1,\cdots,p_i-1$ then yields that

$$\rho_{ij}(x) = \beta_{i(j-d_i-1)}, \quad j = d_i+2,\cdots,p_i, i = 1,\cdots,m. \tag{3.24}$$

It follows from (3.21), (3.23), and (3.24) that for $i = 1,\cdots,m$,

$$dh_i(x) = \beta_{ir_i}dT_{i1}(x) + \cdots + \beta_{i1}dT_{ir_i}(x) + dT_{i(r_i+1)}(x).$$

Since $T_{ij}(0) = 0$ and $h_i(0) = 0$, this implies that

$$h_i(x) = c_iT_i(x), \quad i = 1,\cdots,m \tag{3.25}$$

where $c_i \triangleq [\beta_{ir_i}\cdots\beta_{i1}\ 1\ 0\cdots 0]$.

Next, observe that (3.15) implies that the distribution $\tilde{\Delta} \triangleq \mathrm{span}\{\xi : \xi \in \tilde{\Lambda}\}$ is involutive and invariant under $\mathrm{ad}_{\tilde{f}}^k\hat{g}_i$ for each $k = p_i,\cdots,\kappa_i-1$ and $i = m_1,\cdots,m_q$. Furthermore, $[\mathrm{ad}_{\tilde{f}}^k\hat{g}_i, \mathrm{ad}_{\tilde{f}}^l\hat{g}_j] = 0$, $k = p_i,\cdots,\kappa_i-1, l = p_j,\cdots,\kappa_j-1$, $i,j = m_1,\cdots,m_q$. Thus, Lemma A.2 holds with $\Delta = \tilde{\Delta}$ and $d = p_{m+1}$, where ξ_i, $i = 1,\cdots,d$ correspond to $\mathrm{ad}_{\tilde{f}}^{\kappa_{m_1}-1}\hat{g}_{m_1},\cdots,\mathrm{ad}_{\tilde{f}}^{p_{m_1}}\hat{g}_{m_1},\cdots,\mathrm{ad}_{\tilde{f}}^{\kappa_{m_q}-1}\hat{g}_{m_q},\cdots,\mathrm{ad}_{\tilde{f}}^{p_{m_q}}\hat{g}_{m_q}$, respectively. Therefore, there exist (p_{m+1}) C^∞-functions $\lambda_{i1},\cdots,\lambda_{in_{m_i}}$, $i = 1,\cdots,q$ defined on an open neighborhood $\hat{\chi}$ of 0 such that $\lambda_{ij}(0) = 0$ and

$$L_{\mathrm{ad}_{\tilde{f}}^k\hat{g}_l}\lambda_{ij}(x) = \begin{cases} (-1)^k & \text{if } l = m_i \text{ and } k = p_{m_i} + n_{m_i} - j \\ 0 & \text{otherwise} \end{cases} \tag{3.26}$$

for $k = 0,1,\cdots,\kappa_l-1$, $l = 1,\cdots,m$, $j = 1,\cdots,n_{m_i}$ and $x \in \hat{\chi}$. Let $\tilde{\chi} \triangleq \chi \cap \hat{\chi}$. Define $T : \tilde{\chi} \to \Re^n$ by

$$T(x) \triangleq \begin{bmatrix} T_1(x) \\ \vdots \\ T_m(x) \\ T_{m+1}(x) \end{bmatrix} \text{ where } T_{m+1}(x) \triangleq \begin{bmatrix} \lambda_{11}(x) \\ \vdots \\ \lambda_{1n_{m_1}}(x) \\ \vdots \\ \lambda_{q1}(x) \\ \vdots \\ \lambda_{qn_{m_q}}(x) \end{bmatrix}. \tag{3.27}$$

Then, it follows from (3.16) and (3.26) that

$$\begin{aligned}\mathrm{rank}[&T_*\mathrm{ad}_{\tilde{f}}^{p_1-1}\hat{g}_1(x)\cdots T_*\hat{g}_1(x)\cdots T_*\mathrm{ad}_{\tilde{f}}^{p_m-1}\hat{g}_m(x)\cdots T_*\hat{g}_m(x)\\ &T_*\mathrm{ad}_{\tilde{f}}^{\kappa_{m_1}-1}\hat{g}_{m_1}(x)\cdots T_*\mathrm{ad}_{\tilde{f}}^{p_{m_1}}\hat{g}_{m_1}(x)\cdots T_*\mathrm{ad}_{\tilde{f}}^{\kappa_{m_q}-1}\hat{g}_{m_q}(x)\\ &\cdots T_*\mathrm{ad}_{\tilde{f}}^{p_{m_q}}\hat{g}_{m_q}(x)] = n, \quad x \in \tilde{\chi}.\end{aligned} \tag{3.28}$$

By (3.14) and (3.28), we see that T is a diffeomorphism on an open neighborhood of 0.

Also, it is clear from (3.28) that there exist C^∞-functions a_{ijkl} and b_{ijm_kl} such that

$$dL_{\tilde{f}}\lambda_{ij}(x) = \sum_{k=1}^{m}\sum_{l=1}^{p_k}a_{ijkl}(x)dT_{kl}(x) + \sum_{k=1}^{q}\sum_{l=1}^{n_{m_k}}b_{ijm_kl}(x)d\lambda_{kl}(x) \tag{3.29}$$

for $j = 1,\cdots,n_{m_i}$, $i = 1,\cdots,q$. Successively apply $\mathrm{ad}_{\tilde{f}}^k\hat{g}_i$, $k = 0,1,\cdots,\kappa_i-1$, $i = 1,\cdots,m$ to (3.29) and use the relations (3.16), (3.17), (3.26), and Lemma A.1, then the functions a_{ijkl} and b_{ijm_kl} can be found as constants after quite tedious but straightforward calculations. Now, define A_i^c, $i = 1,\cdots,m$ and A_{m+1} by $\bar{A}_i^c$, $i = 1,\cdots,m$ and $\bar{A}_{m+1}$ in (iv) of Definition 2.2, respectively, with $\bar{p}_{m+1} = p_{m+1}$, $\bar{m}_i = m_i$, $\bar{q} = q$, $\bar{\alpha}_{ijk} = \alpha_{ijk}$, and $\bar{h}_{ji} = h_{ji}$. Then, it follows that

$$dL_{\tilde{f}}T_{m+1}(x) = \sum_{i=1}^{m}A_i^cdT_i(x) + A_{m+1}dT_{m+1}(x).$$

Since $T(0) = 0$ and $\tilde{f}(0) = 0$, this implies that

$$L_{\tilde{f}}T_{m+1}(x) = \sum_{i=1}^{m}A_i^cT_i(x) + A_{m+1}T_{m+1}(x). \tag{3.30}$$

Let

$$A^* \triangleq \begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & A_m & 0 \\ A_1^c & A_2^c & \cdots & A_m^c & A_{m+1} \end{bmatrix},$$

$$B^* \triangleq [B_1^*\cdots B_m^*] = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & b_m \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$C^* \triangleq \begin{bmatrix} C_1^* \\ \vdots \\ C_m^* \end{bmatrix} = \begin{bmatrix} c_1 & 0 & \cdots & 0 & 0 \\ 0 & c_2 & & \vdots & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & c_m & 0 \end{bmatrix}.$$

Then, from (3.20) and (3.30), we easily see that

$$L_{\tilde{f}}T(x) = A^*T(x). \tag{3.31}$$

And from (3.19) and (3.26), we also see that

$$L_{\hat{g}_i}T(x) = B_i^*, \quad i = 1,\cdots,m. \tag{3.32}$$

Let

$$\tilde{\alpha}^*(x) \triangleq \alpha^*(x) - \beta^*(x)\begin{bmatrix} \hat{F}_1T_1(x) \\ \vdots \\ \hat{F}_mT_m(x) \end{bmatrix}.$$

It is then clear from (3.12), (3.19), (3.25), (3.31), and (3.32) that $\{f,g,h\}^{\tilde{\alpha}^*,\beta^*}$ is state-equivalent at the origin to the above linear system $\{A^*,B^*,C^*\}$.

Now, what remains to complete the proof is to show that $\{A^*,B^*,C^*\}$ has the property (iv)-(b) in Definition 2.2. By (3.19) and (3.31), we have $T_*\tilde{f} = A^*T$, $T_*\hat{g}_i = B_i^*$, $i = 1,\cdots,m$ and thus

$$T_*\mathrm{ad}_{\tilde{f}}^k\hat{g}_i = A^{*k}B_i^*, \quad k = 0,1,\cdots, i = 1,\cdots,m. \tag{3.33}$$

Through direct calculation, we also see that

$$A^{*p_i+k}B_i^* = \begin{bmatrix} 0 \\ \vdots \\ A_{m+1}^k b_i^c \end{bmatrix}, \quad k = 0, 1, \cdots. \tag{3.34}$$

Let $B_{m+1}^i \triangleq [b_1^c \; \cdots \; b_{i-1}^c \quad b_{i+1}^c \; \cdots \; b_m^c] \in \Re^{p_{m+1}\times(m-1)}$, $i = 1, \cdots, m$. Then, the property (i) of Lemma 3.1, (3.33), and (3.34) lead to the property (iv)-(b) in Definition 2.2.

We denote by $\Im(\{f,g,h\})$ the set of the parameters p_i, n_i, d_i, q, m_i, β_{ij}, α_{ijk} generated by a system $\{f,g,h\} \in \mathcal{S}^d$. As will be shown soon, these parameters are invariant under state feedback and transformation. Moreover, two systems $\{f,g,h\}$, $\{\hat{f},\hat{g},\hat{h}\}$ in $\mathcal{S}^d$ are feedback-equivalent at the origin if and only if the parameters generated by two systems are identical. To state these facts in a standard form, we introduce some definitions which can be viewed as the nonlinear extension of those in [19].

Definition 3.1 : A function Θ defined on the set $\mathcal{S}$ into a set is called a *$\mathcal{G}$-invariance on $\mathcal{S}\times\mathcal{G}$* if $\Theta(\{f,g,h\}) = \Theta(\{\hat{f},\hat{g},\hat{h}\})$ for any two systems $\{f,g,h\}$, $\{\hat{f},\hat{g},\hat{h}\}$ in $\mathcal{S}$ which are feedback-equivalent at the origin. □

Definition 3.2 : A subset $\mathcal{C}^d$ of $\mathcal{S}^d$ is called *a set of canonical forms of $\mathcal{S}^d$ under $\mathcal{G}$* if all elements of each equivalent class are feedback equivalent at the origin to one and only one $\{A^+,B^+,C^+\} \in \mathcal{S}^d$ and a collection $\Im$ of $\mathcal{G}$-invariances $\theta_1, \cdots, \theta_k$ is called *a complete set of $\mathcal{G}$-invariances* if $\theta_i(\{f,g,h\}) = \theta_i(\{\hat{f},\hat{g},\hat{h}\})$, $i = 1, \cdots, k$ imply that $\{f,g,h\}$ and $\{\hat{f},\hat{g},\hat{h}\}$ are feedback-equivalent at the origin. □

Now, we are ready to state the following Theorem.

Theorem 3.3 :
(i) $\Im$ is a complete set of invariances under $\mathcal{G}$.
(ii) $\mathcal{C}^d$ is a set of canonical forms of $\mathcal{S}^d$ under $\mathcal{G}$.

Proof : (i) By Lemma 3.1 and Lemma 3.2, the parameters in $\Im$ are unique for each of the systems in $\mathcal{S}^d$. Therefore, each of these parameters can be viewed as a function well-defined on $\mathcal{S}^d$. First, suppose that $\{\bar{f},\bar{g},\bar{h}\}$ in $\mathcal{S}^d$ is feedback-equivalent at 0 to $\{f,g,h\}$ in $\mathcal{S}^d$. Then there exists a mapping $J = \{\alpha,\beta,T\}$ in $\mathcal{G}$ such that on an open neighborhood of 0,

$$T_*(f + \sum_{i=1}^m \alpha_i g_i) = \bar{f}\circ T, \quad T_*(\sum_{i=1}^m \beta_{ij} g_i) = \bar{g}_j \circ T, \quad \text{and} \quad h_j = \bar{h}_j \circ T, j = 1, \cdots, m. \tag{3.35}$$

Let $\{\bar{\alpha}^*, \bar{\beta}^*\}$ be determined by (2.4) and (2.5) by replacing $\{f,g,h\}$ by $\{\bar{f},\bar{g},\bar{h}\}$. By Lemma 2.1 in [9], we then have the following relationship,

$$\begin{aligned} \bar{\alpha}^*(T(x)) &= -[D^*(x)\beta(x)]^{-1}[A^*(x) + D^*(x)\alpha(x)], \\ \bar{\beta}^*(T(x)) &= [D^*(x)\beta(x)]^{-1}. \end{aligned} \tag{3.36}$$

Let $\{\hat{f},\hat{g},\hat{h}\}$ and $\{\check{f},\check{g},\check{h}\}$ represent the feedback systems, $\{f,g,h\}^{\alpha^*\beta^*}$ and $\{\bar{f},\bar{g},\bar{h}\}^{\bar{\alpha}^*\bar{\beta}^*}$, respectively. We then have

$$\hat{f} = f + \sum_{i=1}^m \alpha_i^* g_i, \; \hat{g}_j = \sum_{i=1}^m \beta_{ij}^* g_i, \; \hat{h}_j = h_j, \; j = 1, \cdots, m. \tag{3.37}$$

From (2.4) and (3.35) – (3.37), we can see that the systems $\{\hat{f},\hat{g},\hat{h}\}$ and $\{\check{f},\check{g},\check{h}\}$ are T-related. As the result, we have

$$T_*\left(\mathrm{ad}_{\hat{f}}^k \hat{g}_i\right) = \left(\mathrm{ad}_{\check{f}}^k \check{g}_i\right)\circ T, \; \mathrm{L}_{\mathrm{ad}_{\hat{f}}^k \hat{g}_i}\hat{h}_j = \left(\mathrm{L}_{\mathrm{ad}_{\check{f}}^k \check{g}_i}\check{h}_j\right)\circ T, \quad k = 0, 1, \cdots, i, j = 1, \cdots, m. \tag{3.38}$$

Since T is an C^∞-diffeomorphism on an open neighborhood χ of 0, this also implies that at each $x \in \chi$, $\Delta_i(\{f,g,h\})_x$, $i = 1, \cdots, m$ are isomorphic to $\Delta_i(\{\bar{f},\bar{g},\bar{h}\})_{T(x)}$, $i = 1, \cdots, m$, respectively. Using this fact and (3.38), it is not difficult to see that the parameters generated by two systems $\{f,g,h\}$, $\{\bar{f},\bar{g},\bar{h}\}$ are identical.

Next, let $\{f,g,h\}$, $\{\bar{f},\bar{g},\bar{h}\}$ be in $\mathcal{S}^d$. By Theorem 3.2, there then exist two canonical forms $\{A^+,B^+,C^+\}$, $\{A^*,B^*,C^*\}$ in $\mathcal{C}^d$ such that $\{f,g,h\}$, $\{\bar{f},\bar{g},\bar{h}\}$ are feedback-equivalent at 0 to $\{A^+,B^+,C^+\}$, $\{A^*,B^*,C^*\}$, respectively. If $\Im\{f,g,h\}) = \Im(\{\bar{f},\bar{g},\bar{h}\})$, then $A^* = A^+$, $B^* = B^+$, and $C^* = C^+$. Since the feedback-equivalence defined in Definition 2.1 is an equivalence relation, this implies that $\{f,g,h\}$ is feedback-equivalent at 0 to $\{\bar{f},\bar{g},\bar{h}\}$.

(ii) Let $\{A^+,B^+,C^+\}$ be the linear system indicated in Definition 2.2. Clearly, $\{A^+,B^+,C^+\} \in \mathcal{S}^d$. Furthermore, the arguments similar to those used for the proof of Theorem 3.2 in [10] show that $\mathcal{C}^d$ is a set of canonical forms of $\mathcal{S}^d$ under $\mathcal{G}$.

Theorem 3.3 shows that each of the equivalence classes of the set $\mathcal{S}^d$ generated by the feedback-equivalence relation can be completely characterized by its unique representer in the set $\mathcal{C}^d$ of canonical forms. Therefore, we can study the invariant properties of a system in $\mathcal{S}^d$ through use of its representer with simple structure. First of all, it can be readily seen from the structure of $\{A^+,B^+,C^+\}$ in Definition 2.2 that the internal stability of $\{A^+,B^+,C^+\}$ as a decoupled system completely relies on the $\bar{A}_{m+1}$ component, which is invariant under feedback and state transformations.

Theorem 3.4 : Let $\{f,g,h\} \in \mathcal{S}^d$. Define the matrix A_{m+1} by $\bar{A}_{m+1}$ in (iv) of Definition 2.2 with $\bar{p}_{m+1} = p_{m+1}$, $\bar{\alpha}_{ijk} = \alpha_{ijk}$, $k = 0, 1, \cdots, (\bar{n}_i - 1)$, $\bar{h}_{ji} = h_{ji}$, $i, j = m_1, \cdots, m_q$. Then it can be decoupled at the origin with internal stability if and only if A_{m+1} is Hurwitz.

The result similar to Theorem 3.4 could be also obtained from the work by Isidori and Grizzle [13] where a more general class of nonlinear systems than $\mathcal{S}^d$ is considered. They have found the existence of the *fixed dynamics* that are invariant under any regular feedback control law used to achieve decoupling and have shown that stable decoupling of a nonlinear system depends on the stability of this fixed dynamics. Unfortunately, their method of identifying the fixed dynamics of a system requests that a set of partial differential equations be solved to find the desired state transformation (see e.g. (4.5) in [13]), although it can be applied to a more general class of nonlinear systems than $\mathcal{S}^d$. The fixed dynamics of a system $\{f,g,h\}$ in $\mathcal{S}^d$ are given by

$$\dot{z}_{m+1} = A_{m+1} z_{m+1}, \quad z_{m+1}(t) \in \Re^{p_{m+1}},$$

and can be determined here without solving a set of partial differential equations.

IV. Conclusion

In this paper, we have investigate all aspects of the nonlinear systems which are feedback-equivalent locally to decouplable and controllable linear systems. The conditions for a nonlinear system with the output to be feedback-equivalent to a decouplable and controllable linear system are apparently more restrictive than those for a nonlinear system with no output to be feedback-equivalent to a controllable linear system. Nonetheless, many practical nonlinear systems happen to fall in the set $\mathcal{S}^d$. The various results we have obtained so far can be utilzed in real applications as follows.

Suppose we have a system $\{f,g,h\}$ in $\mathcal{S}$. Using Theorem 3.1, we can check if it is in $\mathcal{S}^d$, in other words, if it is feedback-equivalent at the origin to a linear system $\{A,B,C\}$ in $\mathcal{L}^d$.

When it turns out to be in S^d, Theorem 3.4 can tell us whether or not it can be stably decoupled. Also, using Theorem 3.2, we can find the canonical form $\{A^+, B^+, C^+\}$ in C^d which is feedback-equivalent at the origin to the system $\{f, g, h\}$. Based on the structure of the canonical form, we can choose the $\mathcal{C}^\infty$-functions η_i, λ_i, $i = 1, \cdots, m$ so that the closed-loop system has the desired input-output dynamic characteristics.

Note that all these procedures can be performed algebraically without finding the state transformation T that can be obtained usually by solving the set of partial differential equations given by (3.4) and (3.26). However, we have to solve at least the set of partial differential equations given by (3.4) in order to determine the complete form of a decoupling control law.

Appendix

Lemma A.1 : Let q be any nonnegative integer, ξ, ζ be $\mathcal{C}^\infty$-vector fields on $\Re^n$, and θ be a $\mathcal{C}^\infty$-function on $\Re^n$. Then, for $i = 0, 1, \cdots, j$ and $j = 0, 1, \cdots, q+1$,

$$\mathrm{L}_{\mathrm{ad}_\xi^j \zeta}\theta(x) = (-1)^{j-i}\mathrm{L}_{\mathrm{ad}_\xi^i \zeta}\mathrm{L}_\xi^{j-i}\theta(x) = (-1)^j \mathrm{L}_\zeta \mathrm{L}_\xi^j \theta(x), \quad x \in \Re^n,$$

if $d\mathrm{L}_\zeta \mathrm{L}_\xi^j \theta(x) = 0$, $j = 0, 1, \cdots, q$, $x \in \Re^n$.

Lemma A.2 : Let $\xi_i, i = 1, \cdots, n$ be linearly independent $\mathcal{C}^\infty$-vector fields on $\Re^n$. Then there exist an open neighborhood χ of x_0 and a $\mathcal{C}^\infty$-diffeomorphism $\theta : \chi \to \Re^n$ such that for any integer $1 \le d \le n$ and for all $x \in \chi$,

$$\mathrm{L}_{\xi_i}\theta_j(x) = \begin{cases} \delta_{ij} & \text{if } i, j = 1, \cdots, d, \\ 0 & \text{if } i = 1, \cdots, d \text{ and } j = d+1, \cdots, n, \\ 0 & \text{if } i = d+1, \cdots, n \text{ and } j = 1, \cdots, d \end{cases}$$

where δ_{ij} is the Kronecker delta and $\theta(x) \triangleq [\theta_1(x), \cdots, \theta_n(x)]^T$ if and only if, on χ,

(i) the distribution $\Delta \triangleq \mathrm{span}\{\xi_{d+1}, \cdots, \xi_n\}$ is involutive,
(ii) Δ is ξ_i-invariant, $i = 1, \cdots, d$,
(iii) $\mathrm{ad}_{\xi_i}\xi_j = 0, \quad i, j = 1, \cdots, d$.

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