Latticed Set Existence Conditions in the Plane

Valery V. STAROVOITOV

Institute of Engineering Cybernetics,
Belarusian Academy of Sciences,
6 Surganov St., 220012, Minsk, Republic of Belarus

Abstract

Point sets in the Euclidean and digital planes are discussed. The local necessary and sufficient conditions are suggested for pointed lattice extraction from these sets are presented. A number of theorems and corollaries are given. The regular and "almost" regular point sets are studied. The results can be used in automatic control of textured textile images by both general and multiprocessing systems.

1. Introduction

We consider a set of points in R^2 and Z^2. The results were obtaned in the process of solving a particular task of texture recognition [8]. The point sets are formed by replacing the structural texture patterns by their centers using eigen-filters, for example [12]. It is required to define the structure of this set by local analysis of its points. The problem is complicated and special cases have been stadied before [1,3,4,10]. Thus, Grenander [6] analyzed point images on which the deformations of a "jittered crystal" type influence.

How to find conditions which define the latticed sets? There are theorems, connecting the packing properties with the latticed set and its density [6,10]. The so-called "right" n-dimensional point This point system is infinite (from c-condition) and in any limited area there is a finite number of points of the system (from r-condition).

r is selected as the largest one from the available numbers and c as the smallest one. Note, the ratio r/c can be more than 1, but never exceeds 2. An (r,c)-system is called <u>right system</u>, if latticed sets are studied in crystallography [2-6].

Relations between adjacent points of the arbitrary set in the plane providing its latticed properties are considered.

2. Background

Is $S \in R^2$ the latticed set or not? It is required to formulate the local conditions permitting to define that. The local analysis assumes a simple test and can be easily realized by multiprocessing systems.

Let r and c be arbitrary positive numbers ($r,c \in R$). The point system $S = \{p_i,\ i=1,2,\dots\}$ in R^n is called an <u>(r,c)-system</u> or <u>point system</u> if the following conditions are satisfied [2]:

(i) the distance between any two points p_i and p_j is not less than r (r-condition);

(ii) the arbitrary located circle of radius c includes or touches at least one point p_i (c-condition).

(iii) for any two points of the set exists the movement matching the first one with the second and the whole system itself with its own.

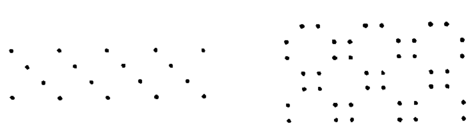
Hilbert founded [5] the right systems in R^n which are not point lattice (Fig.1).

Let $\bar{\alpha}$ and $\bar{\beta}$ be two noncollinear vectors of R^2 starting from one point. The <u>point lattice</u> L in R^2 is the set of the extremities $\bar{x}$ that satisfy:

$\bar{x} = m\bar{\alpha} + n\bar{\beta}$, where m, n are integers.

L_0 and L_H are called the <u>orthogonal</u> and <u>hexagonal point lattices</u> respectively. $L_0, L_H \subset R^2$. L_0 is defined by $\bar{\alpha}_0 = (0,a)$, $\bar{\beta}_0 = (a,0)$ and L_H is defined by $\bar{\alpha}_h = (a,0)$, $\bar{\beta}_h = (0.5a, 0.5\sqrt{3}a)$. The number a is called the <u>spacing</u> of L. Let us consider the case $r = ka$ (k is an integer and a is the length of the unit vector).

We shall use the Euclidean distance in R^2, and the city or square one in Z^2. The set is called a <u>square point lattice</u> if vectors $\bar{\alpha}$ and $\bar{\beta}$ are equal and orthogonal each other. Any plane lattice is the right system. We can see that the square lattice with spacing r is the right system and $c = 0.5\sqrt{2}r$. Remove the c-condition. Let us call this set as an infinite point <u>r-system</u>. The set S is the square lattice if $\forall\, p_i \in S\ \exists$ four points $p_j \in S$ (j=1,2,3,4) nearest to p_i and they form a square. Note, if



a) b)

Fig.1. Right point systems: the latticed set (a), the nonlatticed set (b)

Fig.2. Nonlatticed set having 4 points at the distance r near every point

every point of the set has only four neighbours at the distance r it is not sufficient condition of the square lattice existence (Fig.2).

3. The right point system criterion existence

Consider the arbitrary (r,c)-system of points in R^n. Join by segments all points of the system with each other. Such segment sets are called the <u>global spiders</u>. If the segment length is limited by a number r, then the segment sets are called <u>local</u> <u>r-spiders</u>. (r,c)-system is right only in the case when the global spiders of all its points are congruent between each other [5].

Let in (r,c)-system r-spiders of all its points be congruent in pairs with some r. The system may be not right if r $\leqslant \mathrm{r} < 2c$, but there are so-called <u>stable spiders</u> for the $\mathrm{r} \geqslant r+2c$. It is known [3]:

THEOREM 1. **Let S be the (r,c)-system in R^n. It is the right one if and only if the stable spiders of all its points are congruent.**

It gives us a local conditions of the right (r,c)-system. But, it does not give the radius r of the stable spiders. The spider of the lattice squares is constructed in a neighbourhood with radius $\mathrm{r} = r(1+\sqrt{2})$ and 21 points of (r,c)-system in it. It is required for all points to check the congruentability of spiders, i.e. we must find the movement matching them completely for all stable spiders. We can form a sample spider. All other spiders of the given sizes are compared with this one. In both cases we must estimate the segments lengths and angles.

The radius r can be decreased up to r or r+e, then the spider is called <u>prestable</u>. The spiders may consist of four perpendicular segments in the case of the square lattice. It is requires the c-condition and angles calculation. We use not the angle estimation.

4. Local criteria of lattice existence in R^2

Let a point set P be infinite and satisfy only r-condition. We shell call such set as r-system. Then following Theorem is hold.

THEOREM 2. **Let P be a point r-system in R^2. P is the square point lattice with the spacing r if and only if for $\forall p_i \in P$ $\exists$ no less than eight points of P inside the circle with the radius $r\sqrt{2}$ and located in p_i.**

The necessity is proved by contradictions [8]. The sufficience is obvious. From Theorem 2 follows

COROLLARY 1. **There are stable spiders with $r \geqslant r\sqrt{2}$ for the square lattice with the spacing r in R^2.**

The requirement for availability of no less than eight points follows from the existence of sets which are not square lattices and are shown in Figure 3.

Theorem 2 gives the conditions of square lattice availability: P is the square lattice, if for every $p_i \in P$ four nearest points are situated at the distance r from p_i and four - at the distance $r\sqrt{2}$. Theorem 3 attenuates these restrictions.

THEOREM 3. **Let P be a point r-system in R^2. It is the square point lattice if and only if $\forall p_i \in P$ $\exists$ four points of P at the distance r from p_i and three points of P at the distance $r\sqrt{2}$ from p_i.**

The sufficience is obvious, the necessity is proved in [9]. It is not possible to attenuate the condition of

a) **b)**

Fig.3. Right point, but not latticed systems. Every point has 6 (a) or 7 (b) set points at the distance r or $r\sqrt{2}$

Theorem 3. There are examples of point systems which are not latticed in Fig.3.

Thus in order the lattice be square it is needed that there will be four adjacent points at the distance r for each lattice point. This condition is not sufficient in Euclidean distance.

The hexagonal lattice is a right system with c = 2r/3. According to Theorem 1, $r = 7r/3$. There are 19 points of the r-system in the neighbourhood of any p_i. Next obvious Theorem is hold.

THEOREM 4. **Let P be a point r-system in R^2. P is the hexagonal lattice with spacing r if and only if $\forall p_i \in P$ $\exists$ no less than six nearest points at the distance r from p_i.**

Following to Theorem 4 we have

COROLLARY 2. **There are stable spiders with $r \geqslant r$ for the hexagonal lattice with the spacing r in R^2.**

5. Local criterion in Z^2

Theorems 2 and 4 are hold in Z^2, but Theorem 3 is not hold in Z^2 (Figure 4). It is transformed in

THEOREM 5. **Let P be a point r-system in Z^2. P is the square lattice with the spacing r if and only if for $\forall p_i$ e P $\exists$ equally four or eight points of P inside the circle with the radius r centered in p_i for the city distance and chess one, respectively.**

This Theorem give us

COROLLARY 3. **There are stable spiders with $r \geqslant r$ for the lattice in Euclidean sense with the spasing r in Z^2.**

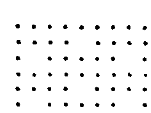


Fig.4. Every point has 7 neighbors at the distance r or $r\sqrt{2}$, but the set is not right and not latticed

6. The case of "almost" square lattice

In practice, because of aliasiang, calculation errors and for other reasons the points of lattice deflect from their ideal positions at some **e**. Consider (r-**e**)-system of points in $\mathbf{R}^2$, for every point of which no less than eight neighbours are in the circle with the radius $r\sqrt{2}+\mathbf{e}$. Let us call this restriction as (r,**e**)-condition. Evaluate the consequence of the ring extension at the value **e** in the both directions.

Can the square lattice points deviate from their ideal position on **e**? Such point system with the precision up to **e** is "almost" square lattice and satisfies the conditions of Theorem 2 for (r-**e**)-system in the circle with radius $\sqrt{2}(r + \mathbf{e})$. It is difficult to answer the question are there latticed sets locally satisfying the extended conditions of Theorem 2? It can be the square lattice with the spacing (r+**e**), but in global consideration they is not any square lattice with constant error.

We investigated the examples of such sets in $\mathbf{R}^2$, which differs in maximal from the square lattice, but locally satisfies the condition (r,**e**). We build the set next manner. Let numbers m, n form the regular lattice with the vectors $\bar{a}$, $\bar{b}$, and spacing **r**. Permit to numberize all the lattice points and introduce the new coordinate system. Let the origin point has coordinates (0,0). We shall call <u>i-layer</u> around the point (0,0) the points having one of the coordinates equal to ir. Let us rotate around the point (0,0) every i-layer of points relatively to the previous one at some angle α_i. As a result we receive a new point set which locally (for each point) is "almost" square lattice (with the precision up to some **e**), but in general significantly differs from it (Figure 5). It is a determinate and irregular point system, but it is not the right one. By using the rectangle and polar coordinates we evaluate the maximal possible every layer rotation angles for the fixed **e**. The corner point of right i-layer has next polar and rectangle coordinates:

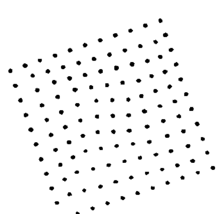


Fig.5. A spiral lattice locally is almost square lattice

$$\begin{cases} x = ir, \\ y = ir, \end{cases} \quad \begin{cases} \rho = \sqrt{ir^2 + ir^2} = ir\sqrt{2}, \\ \varphi = \operatorname{arctg} \frac{ir}{ir} = 45^\circ. \end{cases}$$

Introduce the restriction for the value **e** in the i-layer corner point.

$$r - e_i + ri\sqrt{2} = \sqrt{ir^2 + ((i+1)r)^2},$$

or

$$e_i = r(1 + i\sqrt{2} - \sqrt{i^2 + (i+1)^2}).$$

They mean that point ((i+1)r, ir) must touch (r-**e**)-neigbourhood of point (ir,ir) for optional i and **r**. The other estimation gives us

$$(r\sqrt{2} + e') - (r - e') < r \;\Rightarrow\; e' < (1 - 0{,}5\sqrt{2})r,$$

$$e_{max} = \min\{e_i, e'\}.$$

From the one side, we can rotate the next (i+1)-layer at a large angle until the points, surrounding the point (ir,ir), do not go out the boundaries of the corresponding circle and don't violate (r,**e**)-conditions. From another side, it is needed to take into account that points of (i+2)th and the next layers can be in (r,**e**)- circle of point (ir,ir) even without rotation of the layer. It is complicated to evaluate the restrictions followed from this fact. We can estimate following case.

Turn every i-layer at the angle α_i in such manner, that points (ir,ir) would not be apart from its ideal position no more than on **e**. Then

$$e^2 = 2(ir)^2 + 2(ir)^2 - 4 - (ir)^2 \cdot \cos\alpha_i$$

or $$\alpha_i = \arccos\left(1 - \frac{e^2}{4(ir)^2}\right).$$

It is known, that $\sum_{i=1}^{\infty} \alpha_i$ forms the divergent series, i.e. the cornerpoints of rotated layers form a spiral. We call this point set a <u>spiral lattice</u> in R^2.

7. Discussion

Consider the possibility of using the obtained results. Let the system in the process of image input receive a set of point. Proved Theorems give the possibility to check if the given point set is the square lattice with the spacing r.

It is obvious that the conditions of Theorems are carried out for inside points of the lattice. Let point $\mathbf{p}_i$ belong to the studied point set **P** and the conditions are not carried out for it. If will be a point $\mathbf{p}'_i$ in the neighbourhood of point $\mathbf{p}_i$ with radius $\mathbf{r}\sqrt{2}$, for which the conditions are carried out, point $\mathbf{p}_i$ will be called a <u>border point</u> of the square lattice. If **P** is the finite square lattice we can give its compressed description. For that all the border points are followed and the obtained lines are approximated.

The conditions of Theorems in R^2 are invariant to the lattice turn and to the errors of the angles between any adjacent points. The conditions are simplified, if forming lattice vectors are parallel to the coordinate axis, and we use the square distance. It is required to check the existence of 7 points exactly at the distance **r** from each point of the analyzed set instead of points existence in a circle.

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