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Exponential Convergence of A Learning Scheme for Unknown Linear Systems

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Abstract

In this paper the issue of convergence rate is introduced for a learning control scheme we have developed and applied for tracking of unknown linear systems. A sufficient condition under which the output trajectory converges exponentially fast is obtained using the controllability grammian of controllable linear systems. Under the same condition it is also shown that the learning control input converges exponentially with the same rate as the rate of output convergence. A numerical example with computer simulation results is presented to show the feasibility of the scheme.

1 Introduction

In parallel to the classical learning control methods which correct coefficients or parameters of learning system model, a number of new learning schemes which are directly related to object dynamics have been proposed for uncertain systems. One common feature of these learning schemes is that the control input is updated so as to improve the performance of the controlled system based on the observation of system response. Among the developed learning schemes for uncertain linear systems, Arimoto *et.al.*[2] used the time-derivative of system output in updating the control input and established convergence under the condition $||I - CB\Gamma||_\infty < 1$, where C, B, Γ represent the output matrix, the input matrix and the learning gain matrix, respectively. Oh *et.al.*[5] combined a least square parameter estimator with an input correction scheme and constructed the estimated system model which was used in generating the update signal for learning input. Their scheme converges if the estimator does and the defined linear operators L, P and $\tilde{P}$ satisfy $||L||||P - \tilde{P}|| < 1$. On the other hand, Togai[8] investigated several learning schemes and provided some of the convergence conditions for the investigated discrete system models. One condition looks like $||I - BG|| < 1$, where B and G denote the input matrix and the learning gain matrix, respectively. In both of Oh's and Togai's algorithm, the time-derivative of state error has been used to generate the learning control input. Whereas these learning schemes are simple and straightforward, the convergence of each of them hinges upon the prudent choice of the learning gain which satisfies one of the norm conditions such as given above. Strictly speaking, this implies that they are not applicable to truly unknown systems whose matrices are not known a priori.

Recently, we have proposed a new learning method which can be applied to a class of unknown linear systems[7]. Although the class of target systems is somwhat restricted, the proposed technique does not use any norm conditions for convergence such as given above but relies on the most basic conditions such as controllability and observability conditions. In generating the learning signal, this scheme uses the system outputs directly without any derivative terms, which makes the learning controller more practical. On the other hand, the issue of convergence rate was not introduced in [7]. In this paper, with the aid of controllability condition of unknown linear systems, we derive a sufficient condition for exponential convergence of the learning systems. The analysis uses the nonsingular controllability grammian and is based on the descreasing index functional given in [7]. Finally, a simulation results is given to demonstrate feasibility of the learning control scheme.

2 An Output Learning Control System

The class of unknown linear systems considered in [7] is written as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Bd(t) \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where the state vector $x(t) \in R^n$, the input vector $u(t) \in R^m$, the output vector $y(t) \in R^m$, the bounded unknown disturbance vector $d(t) \in R^m$ and rank$(B) = m$. The completely controllable/observable system matrices $\{A, B, C\}$ are time-invariant or time-varying with continuous elements and satisfy the constraints

$$\dot{P} + PA + A^T P = -aC^T C \tag{2}$$

$$PB = C^T, \tag{3}$$

where a denotes a positive constant and $P = P^T \in R^{n\times n}$ is a bounded positive definite matrix.

Assuming that a feasible input/output pair $\{u_d, y_d\}$ for $t \in [0, t_f]$ exists for the unknown system (1), a continuous-time output tracking problem was solved by using the following output learning rule.

$$u^{j+1}(t) = u^j(t) + \Gamma e_o^j(t). \tag{4}$$

As for an initial condition, we set $u^1(t) = 0$ for all $t \in [0, t_f]$ and $e^j(0) = 0$ for all $j = 1, 2, ...$ The gain matrix Γ is symmetric positive definite and $\Gamma < aI$.

The closed-loop error system at the jth iteration is then

$$\begin{aligned} \dot{e}^j(t) &= Ae^j(t) + B\tilde{u}^j(t) \\ e_o^j(t) &= Ce^j(t), \end{aligned} \tag{5}$$

(6)

where $\tilde{u}^j(t) \equiv u_d(t) - u^j(t)$.

The following theorem briefly summerizes the convergence results presented in [7].

Theorem 1,2[7]: Assume that the desired input u_d is bounded. Then, the output learning rule (4) for the class of unknown linear systems (1) converges uniformly as follows:

$$\begin{aligned} &i) \quad V^{j+1}(t) \le V^j(t) \\ &ii) \quad \lim_{j\to\infty} e_o^j(t) = 0 \\ &iii) \quad \lim_{j\to\infty} \tilde{u}^j(t) = 0, \end{aligned}$$

where

$$V^j(t) = \int_0^t \tilde{u}^{j^T}(\tau)\Gamma^{-1}\tilde{u}^j(\tau)d\tau$$

for all $j = 1, 2, ...$ and for all $t \in [0, t_f]$.

The theorems were proven by showing that the inequality holds for the learning system

$$V^{j+1} - V^j = -e^{j^T}(t)Pe^j(t) - \int_0^t e_o^{j^T}(\tau)(aI - \Gamma)e_o^j(\tau)d\tau \le 0. \tag{7}$$

This inequality is essential in analyzing convergence property of the learning system as shown in the sequel.

3 Exponential Convergence of the Learning Systems

Based on the inequality (7) obtained, we now investigate the rate of convergence of the learnine systems. If we define the controllability grammian of the system (1) as

$$W(0,t) \equiv \int_0^t \Phi(t,\tau)BB^T\Phi^T(t,\tau)d\tau,$$

where Φ is the transition matrix of A, det$W(0,t) \neq 0$ for all $t \in [0, t_f]$, since the pair $\{A, B\}$ is completely controllable. Similary, we definde the matrix

$$W_\Gamma(0,t) \equiv \int_0^t \Phi(t,\tau)B\Gamma^{-1}\Phi^T(t,\tau)B^T d\tau,$$

where Γ is the symmetric positive definite learning gain matrix.

Then, as verified in the following lemma, the matrix $W_\Gamma(0,t)$ is nonsingular if and only if $W(0,t)$ is nonsingular.

Lemma 1: $W_\Gamma(0,t)$ is nonsingular if and only if $W(0,t)$ is nonsingular.

Proof: Let $\Gamma^{-1} = KK^T$, where $K \in R^{m\times m}$ represents a nonsingular matrix. Then, $W_\Gamma(0,t)$ is the controllability gramian of the system $\{A, \bar{B}\}$, where $\bar{B} = BK$ and rank$(\bar{B}) = m$. To prove the lemma, we are going to show that the rows of $\Phi(t)B$ are linearly independent if and only if the rows of $\Phi(t)\bar{B}$ are linearly independent. Then, the lemma follows, since the controllability grammian $W_\Gamma(0,t)$ is nonsingular if and only if the n rows of $\Phi(t)\bar{B}$ are linearly independent[4].

(Necessity) By way of contradiction. Assume that the n rows of $\Phi(t)\bar{B}(t)$ are linearly independent, but the n rows of $\Phi(t)B(t)$ are not. Then, there exists a nonzero row vector v such that $0 = v\Phi(t)B = v\Phi(t)\bar{B}$ for any $t \in [0, t_f]$, which contradicts the assumption.

(Sufficiency) Similarly, assume that the n rows of $\Phi(t)B$ are linearly independent, but the n rows of $\Phi(t)\bar{B}$ are linearly dependent. Then, there exists a nonzero row vector v such that $0 = v\Phi(t)\bar{B} = v\Phi(t)BK$ for any $t \in [0, t_f]$. Let $\bar{v} \equiv (v\Phi(t)B)^T$. Then, it becomes $K\bar{v} = 0$, which results in a trivial solution $\bar{v} = 0$ with a nonsingular matrix K. This is a contradiction. Q.E.D.

Now, let $\bar{W}(0,t) = W^{-T}(0,t)W_\Gamma(0,t)W^{-1}(0,t)$. Then, *Lemma 1* implies that $\bar{W}(0,t)$ is nonsingular, since the uncertain system is completely controllable for all $t \in [0, t_f]$. With this, the following lemma is obtained which is useful in deriving the conditions for exponential convergence of the learning systems.

Lemma 2: The performance indices V^j in *Theorem 1,2* satisfies the following equation

$$V^j(t) = e^{j^T}(t)\bar{W}(0,t)e^j(t). \qquad (8)$$

Proof: Because the solutions $e^j(t)$ of the error equation (5) is written as $e^j(t) = \int_0^t \Phi(t,\tau)B\tilde{u}^j(\tau)d\tau$, $e^j(0) = 0$, the input error $\tilde{u}^j(\tau)$ can be solved from $e^j(t)$ as $\tilde{u}^j(\tau) = B^T\Phi^T(t,\tau)W^{-1}(0,t)e^j(t)$ for $t, \tau \in [0, t_f]$, where $\tau \le t$.

Then, the cost functional $V^j(t)$ given in *Theorem 1,2* becomes

$$\begin{aligned} V^j(t) &= \int_0^t \tilde{u}^j(\tau)\Gamma^{-1}\tilde{u}^j(\tau)d\tau \\ &= \int_0^t e^{j^T}(t)W^{-T}(0,t)\Phi(t,\tau)B\Gamma^{-1}B^T\Phi^T(t,\tau) \\ &\quad W^{-1}(0,t)e^j(t)d\tau \\ &= e^{j^T}(t)W^{-T}(0,t)\int_0^t \Phi(t,\tau)B\Gamma^{-1}B^T\Phi^T(t,\tau)d\tau \\ &\quad W^{-1}(0,t)e^j(t) \\ &= e^{j^T}(t)W^{-T}(0,t)W_\Gamma(0,t)W^{-1}(0,t)e^j(t) \\ &= e^{j^T}(t)\bar{W}(0,t)e^j(t). \end{aligned}$$

This completes the proof. Q.E.D.

The output learning rule is now shown to be exponentially convergent in the following theorem.

Theorem 1(Exponential Convergence of Learning Systems): Assume that the learning gain Γ in (4) is chosen such that $\lambda_{min}(P) < \lambda_{max}(\bar{W}(0,t))$. Then, the output learning control systems converge exponentially as follows:

i) $\lim_{j\to\infty} V^j(t) = 0$, such that $V^j(t) \le V^1\rho^j$

ii) $\lim_{j\to\infty} e^j(t) = 0$, such that $|e^j(t)| \le \sqrt{\frac{V^1}{\lambda_{min}(\bar{W})}}\rho^{j/2}$

iii) $\lim_{j\to\infty} E_o^j(t) = 0$, such that $E_o^j(t) \le V^1\rho^j$,

where $\rho(t) = (1 - \frac{\lambda_{min}(P)}{\lambda_{max}(\bar{W})})$ and $E_o^j \equiv \int_0^t e_o^{j^T}(\tau)(aI - \Gamma)e_o^j(\tau)d\tau$.

Proof: From the inequality (7) and *Lemma 2*, we obtain

$$\begin{aligned} V^{j+1} &\le V^j - e^{j^T}Pe^j \\ &\le (1 - \frac{\lambda_{min}(P)}{\lambda_{max}(\bar{W})})V^j \\ &\le \rho^j V^1, \end{aligned}$$

which confirms *i)*. With this, the inequality (7) implies *ii)*, since from *Lemma 1* $\bar{W}(0,t)$ is nonsingular. Similarly, *iii)* follows from the inequality (7). Q.E.D.

Corolary 1: Under the same condition as in *Theorem 1*, the learning input sequence $\{u^j\}$ in the output learning systems converge exponentially as j increases.

Proof: From the error equation, it becomes $|\tilde{u}^j(\tau)| \le ||B^T\Phi(t,\tau)W^{-1}(0,t)|||e^j(t)|$. With this, *Theorem 1* implies that $\tilde{u}^j$ converges exponentially to zero as j increases. Q.E.D.

Remark: Since ρ depends on the learning gain Γ, a sufficiently small value of Γ can be chosen to make ρ less than unity. A simple choice is to let $\Gamma = \gamma I$ with which the convergence rate function becomes $\rho = 1 - \gamma\frac{\lambda_{min}(P)}{\lambda_{min}(W)}$.

4 A Numerical Example

As an example, consider the second order system $\ddot{x} + 2\dot{x} + x = u$, which comes from a simple model of DC motor[3], where x and u denote angular velocity and input votage, respectively.

Describing the equation in state space form, we obtain

$$\begin{aligned} \dot{z} &= Az + bu \\ y &= cz, \end{aligned}$$

where

$$z \equiv \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \; A \equiv \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \; b \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; c^T \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and the system $\{A, b, c\}$ forms a controllable and observable system. Because A is strictly stable, the positivity of the solution matrix P of the Lyapunov equation (2) is guaranteed with the observable pair $\{A, c\}$. Note also from constraint (3) that the output matrix c should satisfy $cb \neq 0$, which implies that the relative degree of the present SISO system is one. The existence of a positive constant a can be shown by solving the equations (2) and (3), from which $a = 4$ is obtained.

When performing the simulation, the desired output $y_d(t)$ is set as[3],

$$y_d(t) = 12t^2(1-t) \quad \text{for} \quad t \in [0, 1].$$

Figure 1, 2 and 3 show respectively $x^j, \dot{x}^j$ and u^j trajectories of the learning system using the prediction learning rule with $\Gamma = 2.5$. Figure 4 shows that the trajectory x^j is sufficiently close to the desired trajectory at around the $40th$ iteration. Note in Figure 4 that the rms error of the system trajectory reduces to less than one percent of the initial error.

The learning rule with different learning gains up to a have been tested with and without the input disturbance d. In every case, the system converges relatively fast with these learning control schemes. The results show that the reasonably fast convergence rate can be achieved even without using the time-derivative of output/state error for the learning rule [2, 5, 8]. Note also that the output represents the angular acceleration of a DC motor.

5 Conclusion

In this paper, we have derived a sufficient condition under which the learning control method presented in [7] converges exponentially fast. Under the condition, the tracking error of learning system as well as the learning input converges with exponential rate which depends on the learning gain of learning rule. A simulation example shows feasibility of the learning method without time-derivative terms in the learning signal.

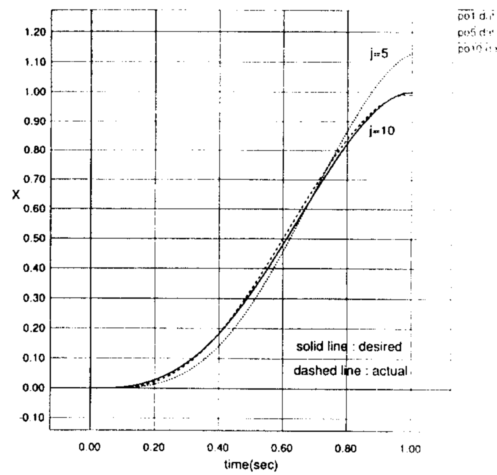

Figure 1: Trjectories of x^j's after the $5th$ and the $10th$ iteration

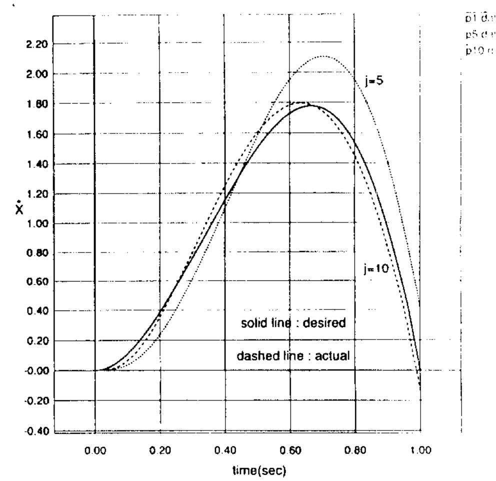

Figure 2: Trjectories of $\dot{x}^j$'s after the $5th$ and the $10th$ iteration

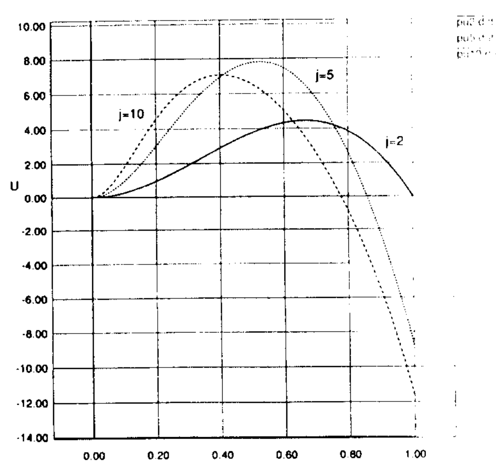

Figure 3: Trjectories of u^j's after the $2nd$, the $5th$ and the $10th$ iteration

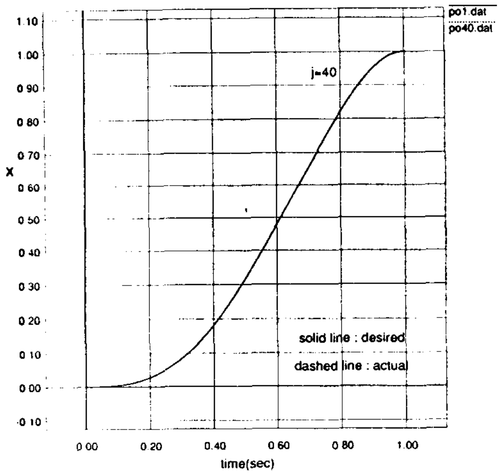

Figure 4: Trjectory of x^j after the $40th$ iteration

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