Stochastic Models for Random Request Availability

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Key words - Random Request Availability

Reader Aids -

Purpose : Present a method of analysis
Special math needed for explanation : Markov processes, probability theory
Special math needed to use results : Same
Results useful to : Reliability Theoreticians and analysts

Abstract - A system is considered which is required to perform several tasks which arrive randomly during the fixed mission duration. A new availability measure called random request availability is proposed. An analytic model is derived and illustrated by a numerical example.

요약 - 일정기간 동안 임의로 도착하는 여러개의 임무를 수행하는 시스템이 고려된다. "random request availability"라 불리우는 새로운 가용도 측도가 제시된다. 이의 분석 모형이 유도되고 실례를 들어 보았다.

1. Introduction

For repairable systems a fundamental quantity of interest is the availability. Three different widely used measures are : point availability, interval availability, and steady - state availability. Their definitions can be seen in various literatures [1,2,3]. Those measures well represent a fraction of time that a system is in an operational state.

In this study a system is considered which is required to perform several tasks which arrive randomly during the fixed mission duration. For a such system some availability measure needs to be newly proposed which incorporates random task arrival rate into the model. As an extension of previous study [4], a new availability measure called random request availability is proposed. An analytic model for random request availability is derived and illustrated by a numerical example. System description, assumptions, and notations follow those of previous study.

2. Random Request Availability

Random request availability is defined to be the expected fractional amount of a number of job arrivals at which the system is in operational state.

2.1 (Additional) Notation

$O(N(T))$ number of job arrivals at which the system is in operational state out of $N(T)$

$A(\lambda(t),T)$ random request availability with job arrival rate $\lambda(t)$ and mission duration T

2.2 Derivation

In defining random request availability, the probability of "no task arrival" needs special consideration. This case can either be regared as a system being available, the logic being that if there are no tasks to perform, the system did its job, or a mission can be defined to occur only when there are task arrivals. Expressions for random request availability are given for both instances. However, for purpose of derivation, only the first of these is considered in detail.

$A(\lambda(t),T)$ can be expressed as

$$A(\lambda(t),T) = E\left[\frac{O(N(T))}{N(T)}\right] \tag{1}$$

$$= E\left[E\left\{\frac{O(N(T))}{N(T)} \mid N(T)=K\right\}\right]$$

$$= Pr[\,N(T)=0\,] + \sum_{K=1}^{\infty}\frac{e^{-m(T)}\cdot m(T)^K}{K!}\cdot E\left[\frac{O(N(T))}{N(T)} \mid N(T)=K\right]$$

$$= e^{-m(T)} + \sum_{K=1}^{\infty}\frac{e^{-m(T)}\cdot m(T)^k}{K!}\cdot\frac{1}{K}\cdot E[O(K)]$$

$$E(O(K))=K\left[\int\int_{0<t_1<\cdots<t_k<T}\cdots\int P_r[Z(t_1)=1,Z(t_2)=1,\cdots,Z(t_k)=1]\cdot f(t_1,t_2,\cdots t_k)dt_1dt_2\cdots dt_k\right]$$

$$+(K-1)\left[\int\int_{0<t_1<\cdots<t_k<T}\cdots\int\{P_r[Z(t_1)=0,Z(t_2)=1,Z(t_3)=1,\cdots,Z(t_k)=1] \tag{2}\right.$$

$$+P_r[Z(t_1)=1,Z(t_2)=0,Z(t_3)=1,\cdots,Z(t_k)=1]$$

$$\left.+\cdots P_r[Z(t_1)=1,Z(t_2)=1,Z(t_3)=1,\cdots,Z(t_k)=0]\}f(t_1,t_2,\cdots t_k)dt_1dt_2dt_k]\right]$$

$$+\cdots$$

$$+1\cdot\left[\int\int_{0<t_1<\cdots<t_k<T}\cdots\int\{P_r[Z(t_1)=1,Z(t_2)=0,Z(t_3)=0,\cdots,Z(t_k)=0]\right.$$

$$+P_r[Z(t_1)=0,Z(t_2)=1,Z(t_3)=0,\cdots,Z(t_k)=0]$$

$$+\cdots$$

$$\left.+P_r[Z(t_1)=0,Z(t_2)=0,Z(t_3)=0,\cdots,Z(t_k)=1]\}f(t_1,t_2,\cdots,t_k)dt_1dt_2\cdots dt_k\right]$$

For instance, E(O(2)) can be computed as

$$E(O(2))=2\cdot\left[\iint_{0<t_1<t_2<T} P_r[Z(t_1)=1,Z(t_2)=1]\cdot f(t_1,t_2)dt_1dt_2\right] \quad (3)$$

$$+1\cdot\left[\iint_{0<t_1<t_2<T}\{P_r[Z(t_1)=1,Z(t_2)=0]+P_r[Z(t_1)=0,Z(t_2)=1]\}f(t_1,t_2)dt_1dt_2\right]$$

where

$$f(t_1,t_2) = 2!\ \lambda(t_1)\lambda(t_2)\ /\ m(T)^2$$

$$P_r[Z(t_1)=1,Z(t_2)=1]=\left(\frac{\beta}{\alpha+\beta}+\frac{\alpha}{\alpha+\beta}\cdot e^{-(\alpha+\beta)t_1}\right)\left(\frac{\beta}{\alpha+\beta}+\frac{\alpha}{\alpha+\beta}\cdot e^{-(\alpha+\beta)(t_2-t_1)}\right)$$

$$P_r[Z(t_1)=1,Z(t_2)=0]=\left(\frac{\beta}{\alpha+\beta}+\frac{\alpha}{\alpha+\beta}\cdot e^{-(\alpha+\beta)t_1}\right)\left(\frac{\alpha}{\alpha+\beta}-\frac{\alpha}{\alpha+\beta}\cdot e^{-(\alpha+\beta)(t_2-t_1)}\right)$$

$$P_r[Z(t_1)=0,Z(t_2)=1]=\left(\frac{\alpha}{\alpha+\beta}-\frac{\alpha}{\alpha+\beta}\cdot e^{-(\alpha+\beta)t_1}\right)\left(\frac{\beta}{\alpha+\beta}-\frac{\beta}{\alpha+\beta}\cdot e^{-(\alpha+\beta)(t_2-t_1)}\right)$$

If mission effectiveness is considered only when at least one task arrives, then $A(\lambda(t),T)$ can be obtained from (1) by subtracting $\exp(-m(T))$ and dividing by $1-\exp(-m(T))$, these quantities being the probability that $N(T)$ is equal to zero and the probabilyty that $N(T)$ is greater than zero, respectively.

$$A(\lambda(t),T) = \sum_{k=1}^{\infty}\frac{e^{-m(T)}\cdot m(T)^k}{K!}\cdot\frac{1}{K}E[O(k)]\ \Big/\ (1-e^{-m(T)}) \quad (4)$$

3. Numerical Example

Assumptions

1. Task arrival times is a homegeneous Poisson Process with intensity 0.04
2. Mission duration time is $T = 10$, which produces an average of 0.4 task arrival during the mission.
3. sojourn times in on and off states of the system follow the negative exponential distributions with $\alpha = 1$ and $\beta = 5$ respectively
4. Two situations are considered : A) No-task arrrival is regarded as a system being available, B) A mission occurs iff there is at least one task arrival.

Assume two tasks arrive during the mission

$$E(O(2)) = 2 \cdot \left[\int_0^{10}\int_0^{t_2} \left(\frac{5}{6} + \frac{1}{6} e^{-6t_1}\right)\left(\frac{5}{6} + \frac{1}{6} e^{-6(t_2-t_1)}\right) \cdot 2! \cdot 0.1^2 dt_1\, dt_2\right]$$

$$+ 1 \cdot \left[\int_0^{10}\int_0^{t_2} \left\{\left(\frac{5}{6} + \frac{1}{6} e^{-6t_1}\right)\left(\frac{1}{6} - \frac{1}{6} e^{-6(t_2-t_1)}\right)\right.\right.$$

$$\left.\left. + \left(\frac{1}{6} - \frac{1}{6} e^{-6t_1}\right)\left(\frac{5}{6} - \frac{5}{6} e^{-6(t_2-t_1)}\right)\right\} 2! \cdot 0.1^2 dt_1 dt_2\right]$$

$$= 1.682$$

Table 1 summarizes the results for $K \geq 0$. Since $P_r[N(T) \geq 3]$ is almost negligible, $A(0.04, 10)$ for situation A can be approximated as 0.938 with the error bound 0.01

$$A(0.04, 10) = 0.670 + 0.268 \cdot 0.836 + 0.053 \frac{1.682}{2} + \cdots \approx 0.938$$

For situation B :

$$A(0.04, 10) \approx \frac{0.938 - 0.670}{1 - 0.670} = 0.812$$

Table 1

Example Results

Number of task arrivals during the mission(k)	Pr[N(T)=K]	Expected number of job arrivals at which the system is in on - state out of K (E(O(k))
0	0.670	
1	0.268	0.836
2	0.053	1.682
3 a mone	negligible	

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