A Study on the Nash Equilibrium of the Price of Insurance

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ABSTRACT

This note examines a situation where a risk-neutral insurer and a risk-averse individual (prospective insured) negotiate to reach an arbitration point of the price of insurance over the terms of an insurance contract in order to maximize their respective self-interests. The situation is modeled as a Nash bargaining problem. We analyze the dependence of the price of insurance, which is determined by the Nash solution, on the parameters such as the size of insured loss, the probability of a loss, the degree of risk-aversion of the insured, and the riskiness of loss distribution.

1. INTRODUCTION

Like other markets, an insurance market has its supplier (insurer) and demander (prospective insured), and both of these parties seek to achieve their respective interests through rational behavior. Their respective interests may be profit maximization, utility maximization, cost minimization, or whatever else either wants to achieve. Here we confine the term "insurance" to voluntary property insurance in the private sector of the economy. That is, it is assumed that any individual, to protect himself from a possible peril affecting his wealth, is willing to buy goods (insurance policies) from a private insurer. In order to maximize their respective self-interests, they may face a bargaining situation over the terms of an insurance contract. This bargaining situation is particularly applicable when the possible loss of the prospective insured is quite large and the loss is rather unique. Such an individual would rather negotiate to determine an insurance contract than take the price of insurance as given. In this note, we consider a situation where a risk-neutral insurer and a risk-averse individual (prospective insured) bargain about the price of insurance over the terms of an insurance contract in order to maximize their respective self-interests (expected underwriting profit to the former and expected utility of wealth to the latter). The situation can be modeled as a Nash bargaining problem. The goal of the model is to find an arbitration point of the price of insurance which maximizes a certain objective function. We analyze the dependence of the price of insurance, which is determined by the Nash solution, on various parameters which affect it. The parameters include the size of insured loss, the probability of a loss, the degree of risk-aversion of the prospective insured, and the riskiness of loss distribution.

2. Two-State Model: the effects of the size of loss and the probability of loss.

As a starting point, we examine the decision of whether to insure against a particular financial loss completely or not to insure at all. The insurance policy with which an individual may insure fully against the loss by paying a prescribed premium is called a full-cover policy. With this policy, the company promises the insured that it will pay any loss incurred over the term of a contract. Consider an individual with an initial wealth W facing the risk of a financial loss L (which is fixed) with probability q and a loss 0 with probability (1 - q), where L is assumed to be less than W. That is, the individual is assumed to face two states of nature in terms of his wealth position with the probabilities of these states being q and (1 - q), respectively. The insurance company's view of the individual's average amount of claims is then qL. The problem can be thought of as a two- outcome lottery. Without insurance, the individual's expected utility, written EU_0, is given by

$$EU_0 = qU(W - L) + (1 - q)U(W),$$

where $U(\cdot)$ denotes the individual's utility function of wealth with $U'(\cdot) > 0$, $U''(\cdot) < 0$. EU_0 is then interpreted as a guaranteed or threat payoff to the individual in the Nash sense.

However, suppose that by paying a premium p, the individual can insure fully against the possible loss L. Here the premium is defined as the pure premium (which is the mathematical expectation of claims payments) plus a positive loading. This definition is necessary for the company to maintain its solvency, pay related expenses, and make underwriting profit. As a risk averter, the individual may choose to pay an amount p for certain rather than suffer a loss L with probability q. Under the full-cover policy, the insured will be paid the amount of loss immediately to restore his wealth position to the level it was after he paid the premium. The individual's expected utility with a full-cover policy, written EU_1, is then given by

$$\begin{aligned} EU_1 &= qU(W - p) + (1 - q)U(W - p) \\ &= U(W - p). \end{aligned}$$

For an expected utility maximizer who chooses to have insurance, EU_1 should be no less than EU_0.

Now consider the company's side. By selling an insurance policy, the insurance company's expected (underwritng) profit will be

$$\begin{aligned} & q[p - L - c(p)] + (1 - q)[p - c(p)] \\ &= p - qL - c(p), \end{aligned}$$

where p is the premium charged, and c(p) is the company's expenses. The expenses, which are composed of various administrative and other operational costs, commissions to the agents, and premium taxes, are required in the production and servicing of the insurance. We assume that the amount of the expenses is assumed to be a fraction of the premium collected such that $c'(p) = \gamma (0 < \gamma < 1)$ and a boundary condition $c(0) = 0$. This assumption is reasonably realistic, since several of the expenses such as commissions to the agents and premium taxes actually increase proportionally with premiums collected by the company. The threat payoff to the insurer is assumed to be zero (i.e., no business).

Relating the individual's expected gain in utility and the company's expected gain in profit, relative to their respective threat payoffs, the Nash bargaining solution is then the premium p solving

$$V(p) = \max_{P} [U(W - P) - \{qU(W - L) + (1 - q)U(W)\}] \cdot [p - qL - c(p)]. \qquad (1)$$

That is, we would like to find the optimal p which maximizes the product of the individual's expected gain in utility and the company's expected gain in profit, relative to their respective threat payoffs. With assumption of the strict concavity of $U(\cdot)$, the next result, which is quite intuitive, follows.

PROPOSITION 1: Suppose an individual's utility function is strictly concave. Then (i) the premium is an increasing function of the size of loss, L; and (ii) the premium is an increasing function of the probability of loss, q.

PROOF: To get the first order condition, we differentiate V(p) with respect to p and get

$$- U'(W - p)(p - qL - c(p)) + [U(W - p) - qU(W - L) - (1 - q)U(W)](1 - c'(p)) = 0. \tag{1.1}$$

Thinking of p as a function of L and differentiating (1.1), we obtain

$$\frac{dp}{dL} = - \frac{qU'(W - p) + qU'(W - L)(1 - c'(p))}{U''(W - p)(p - qL - c(p)) - 2U'(W - p)(1 - c'(p))}. \tag{1.2}$$

However, by hypothesis, we know $U'(\cdot) > 0$, $U''(\cdot) < 0$, and $1 - c'(p) > 0$. Also, the premium collected should be, by definition, greater than the mathematical expectation of the company's claims payment and its expenses (i.e., $p > qL + c(p)$). Thus the numerator and the denominator on the right-hand-side of (1.2) are positive and negative, respectively, and $dp/dL > 0$. This completes the proof of part (i).

Similarily, to prove part (ii), we differentiate (1.1) with respect to q, and obtain

$$\frac{dp}{dq} = \frac{- U'(W - p)L + [U(W - L) - U(W)](1 - c'(p))}{U''(W - p)(p - qL - c(p)) - 2U'(W - p)(1 - c'(p))}. \tag{1.3}$$

By the same reasoning as in proving part (i) and the increasing nature of $U(\cdot)$ with the individual's wealth position, it easily follows that both the numerator and the denominator of (1.3) are negative, and $dp/dq > 0$. This completes the proof of part (ii). □

3. The effect of the degree of an individual's risk aversion

To be more realistic, the above model (1) may be generalized by relaxing the nature of the loss. Now the loss X is assumed to be random with probability distribution F(x). We also assume that the density $f(x) = F'(x)$ exists, and is continuous for all nonnegative x. Then we would like to find the premium p suggested by the Nash bargaining solution. To do so, we find p solving

$$V(p) = \max_p \left[U(W - p) - \int_0^\infty U(W - x)f(x)dx\right]\cdot\left[p - \int_0^\infty xf(x)dx - c(p)\right] \quad (2)$$

$$= \max_p \left[U(W - p) - EU(W - X)\right]\cdot\left[p - EX - c(p)\right],$$

where E denotes the expectation operator. As in the two-state model, this is actually a constrained optimization with the constraint $p > EX + c(p)$, since the insurance company would not accept a lower premium than its expected claims payment and expenses. Using the concept of Pratt's measure of absolute risk aversion, we can derive the following result from model (2).

PROPOSITION 2: The more risk-averse an individual is, the higher premium he is willing to pay. In order words, the company's expected profit is higher when dealing with a more risk-averse individual than when dealing with a less risk-averse individual.

Before proving the above Proposition, we need to describe one of the results shown by Pratt [2]. According to Pratt: if U_1 is a more risk-averse utility function than U_2, then

$$\frac{U_1(w) - U_1(v)}{U'_1(x)} > \frac{U_2(w) - U_2(v)}{U'_2(x)} \quad \text{for all } v, w, x \text{ with } v < w \leq x.$$

Using Pratt's result, we can now prove Proposition 2.

PROOF: To get the first order condition, we differentiate (2) with respect to p and get

$$\frac{\partial V}{\partial p} = - U'(W - p)(p - EX - c(p)) + [U(W - p) - EU(W - X)](1 - c'(p)) = 0. \qquad (2.1)$$

Let p^* be the premium which satisfies equation (2.1). By differetiating (2.1) with respect to p, using the concavity of U, and using the facts that $p > EX + c(p)$ and $1 - c'(p) > 0$, we see that for any p[1] $<$ (>) p^*, $\partial V/\partial p > (<) \ 0$. This implies that V(p) has a unique maximum at p^*. Now, rearranging the terms in (2.1), we obtain the condition that p^* should satisfy

$$\frac{p^* - EX - c'(p^*)}{1 - c'(p^*)} = \frac{U(W - p^*) - EU(W - X)}{U(W - p^*)} . \qquad (2.2)$$

Set $EU(W - X) = U(CE)$ where CE is the individual's certainty eqivalent of W - X. For an individual who chooses to purchase insurance, it is clear that $CE < W - p^*$.[2] Substituting U(CE) for EU(W - X) in (2.2), we get

$$\frac{p^* - EX - c(p^*)}{1 - c'(p^*)} = \frac{U(W - p^*) - U(CE)}{U'(W - p^*)} . \qquad (2.3)$$

Now, by Pratt's theorem, if $U(\cdot)$ is a more risk-averse utility function than $\mathcal{U}(\cdot)$, it follows that for $CE < W - p^*$,

$$\frac{U(W - p^*) - U(CE)}{U'(W - p^*)} > \frac{\mathcal{U}(W - p^*) - \mathcal{U}(CE)}{\mathcal{U}'(W - p^*)} . \qquad (2.4)$$

Substituting (2.3) into (2.4), we obtain

$$- \mathcal{U}'(W - p^*)(p^* - EX - c(p^*)) + [\mathcal{U}(W - p^*) - \mathcal{U}(CE)](1 - c'(p^*)) < 0. \qquad (2.5)$$

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1) As previously stated, "any" p means any p which is greater than $EX + c(p)$ by the definition of the premium.

2) If he chooses to pay p^* for insurance, rather than not have insurance, then $U(W - p^*) > EU(W - X) = U(CE)$. Since U is increasing, $W - p^* > CE$.

Now let $\mathcal{V}(p) = \underset{p}{\text{Max}}\ [\mathcal{U}(W - p) - E\mathcal{U}(W - X)]\cdot[p - EX - c(p)]$. To maximize $\mathcal{V}(p)$, we find $\hat{p}$ satisfying $\partial\mathcal{V}/\partial p = -\mathcal{U}'(W - p)(p - EX - c(p)) + [\mathcal{U}(W - p) - E\mathcal{U}(W - X)]\cdot(1 - c'(p)) = 0$, so that $\hat{p}$ is the optimal premium when the individual has utility function $\mathcal{U}(\cdot)$. Then $\mathcal{V}(p)$ also has a unique maximum at $\hat{p}$ since for any $p < (>)\ \hat{p}$, $\partial\mathcal{V}/\partial p > (<)\ 0$. However, if $\mathcal{U}(\mathcal{CE}) = E\mathcal{U}(W - X)$, then $CE < \mathcal{CE}$, and (2.5) is satisfied when CE is replaced by $\mathcal{CE}$. Therefore, (2.5) tells us that $\partial\mathcal{V}/\partial p$ is negative when it is evaluated at p^*. Thus, $p^* > \hat{p}$. This completes the proof. □

Assuming an insurer is risk-neutral, Proposition 2 implies that the insurer prefers to bargain with the more risk-averse of any two individuals since a more risk-averse individual is willing to pay more for the same insurance coverage than a less risk-averse individual.

We note that this result is similar to the result derived by Schlesinger [6]. Using a different Nash bargaining formulation, however, Schlesinger attempted to find the optimal expected profit of an insurer rather than the optimal premium. With the premium being the variable of bargaining interest, we also derived relationships between the optimal premium and the individual's size of loss, and the optimal premium and the probability of loss in a two-state model. In addition, using the generalized model (2), we will examine the dependence of the optimal premium on the riskiness of the loss distribution. The analysis follows.

4. The Effect of the Riskiness of the Loss Distribution

We now turn our attention to the riskiness of the loss X itself, and see how this factor affects the premium decision. According to Rothschild and Stiglitz [4,5], one random variable is called less risky than another random variable if the density function of the latter can be obtained from that of the former by applying a series of mean preserving spreads. A mean preserving spread (MPS) is defined as a step function s(x) which has two important properties over an interval (a,b):

$$\int_a^b s(x)dx = 0 \quad \text{and} \quad \int_a^b xs(x)dx = 0.$$

Thus if f(x) is a density function defined over (a,b), and if $g(x) = f(x) + s(x) \geq 0$, then g(x) is also a density function defined over the same interval, it has the same expectation, and it is more risky than f(x). Rothschild and Stiglitz [4] show that if an outcome Y with probability density $g(\cdot)$ is riskier than an outcome X with probability density $f(\cdot)$, then $EU(X) \geq EU(Y)$ for all concave $U(\cdot)$; that is, every risk averter prefers X to Y.

Hence, if the distribution F(x) is less risky than G(x), assuming that their respective density functions f(x) and g(x) exist, then it follows that:

$$\int_0^{\infty} U(W - x)f(x)dx \geq \int_0^{\infty} U(W - x)g(x)dx.$$

Now, from the previously generalized model (2), the next result easily follows.

PROPOSITION 3: The less risky the loss (distribution) is, the smaller premium will be charged.

PROOF: As in the proof of Proposition 2, let p^* satisfy the first order condition

$$\frac{\partial V(p^*)}{\partial p} = - U'(W - p^*)(p^* - EX - c(p^*)) + [U(W - p^*) - EU(W - X)](1 - c'(p^*)) = 0. \tag{3.1}$$

If we let $y = EU(W - X)$, then as the loss distribution becomes less risky, y increases, but EX does not change. To see the effect on p^*, we differentiate (3.1) with respect to y. The result is

$$\frac{dp^*}{dy} = \frac{1 - c'(p^*)}{U''(W - p^*)(p^* - EX - c(p^*)) - 2U'(W - p^*)(1 - c'(p^*))} < 0,$$

where the inequality follows from the strict concavity of $U(\cdot)$ and the facts that $1 - c'(p^*) > 0$ and $p^* > EX + c(p^*)$. Therefore, as the loss distribution becomes less risky, y increases, and p^* decreases. This completes the proof. □

5. Summary

In this note, we examined an insurance market where two parties (a risk-neutral insurer and a risk-averse individual) negotiate to reach an arbitration point of the price of insurance in order to maximize their respective self-interests. The interest to the former is its expected gain in underwriting profit by selling an insurance policy and the interest to the latter is his expected gain in utility of wealth by purchasing the policy. The Nash bargaining scheme was employed to formulate the model.

Specifically, we examined the dependence of the optimal price of insurance which maximizes the product of an insurer's expected gain in underwriting profit and an individual's expected gain in utility of his wealth, relative to their respective threat payoffs, on various parameters which affect it.

First, assuming that only one value of loss is possible, we analyzed the effect of the size of the insured loss and the effect of the probability of a loss on the Nash arbitration point. The following results were derived from the two -state model: the premium increases with the size of the insured loss; and the premium increases with the probability of a loss.

Second, assuming that the size of the loss is random. The previous two-state model can be generalized. From this generalized model, we showed how the degree of risk aversion of the insured affects the Nash solution. In particular, we showed that a more risk-averse individual is willing to pay a higher premium, and thus, the insurer's expected underwriting profit will be higher when negotiating with a more risk-averse individual. That is, the risk-neutral insurer would prefer to bargain with a more risk-averse individual than a less risk-averse individual since a more risk-averse person is willing to pay more for the same insurance coverage.

Third, the effect of the riskiness of the loss distribution on the optimal premium was examined. It was shown that a person whose loss (distribution) is less risky is charged a smaller premium than a person whose loss (distribution) is more risky.

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