DIFFUSION OF A STEADY LINE SOURCE IN TURBULENT SHEAR FLOWS

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INTRODUCTION

Vertical mixing is the initial stage of mixing in rivers. So, the rapid vertical mixing contributes to the whole mixing process by shortening the time before the succeeding stages, i.e., lateral diffusion (in case of point source) and longitudinal dispersion. In addition, vertical mixing itself is important for the atmospheric diffusion process, where the boundary layer thickness is of the order of hundred meters.

For a laterally uniform steady flow and a uniformly distributed continuous line source, the vertical diffusion problem can be described by the two-dimensional steady-state advective diffusion equation. Mathematical solutions for the problem with constant velocity and diffusivity have long been available (Carslaw and Jaeger 1959). Yeh and Tsai(1976) obtained an analytic solution for a power-law velocity and power-law diffusivity, which is unrealistic since the measured velocity distribution is approximately logarithmic, and this implies a parabolic diffusivity distribution. McNulty and Wood(1984) used Aris' method of moments for the case of logarithmic velocity and parabolic diffusivity distributions but the comparison they made with the solution for constant velocity and diffusivity is incorrect. Nokes, et al.(1984) solved the problem analytically by reducing it to an eigenvalue problem following the approach of Smith(1982), who dealt with the problem on where to put the discharge in meandering rivers. On the other hand, Coudert(1970) solved the problem numerically by using a finite difference method but his stated initial condition is dimensionally incorrect.

In this paper, a two-dimensional steady-state advective diffusion equation is solved numerically to simulate the diffusion process for turbulent shear flow in a channel. The

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results are compared with those for the case of constant velocity and diffusivity. The numerical model is used to investigate the sensitivity of the vertical diffusion process to the friction factor. The effect of varying the source position on the downstream concentration distribution is also studied to find the best position for the most rapid vertical mixing.

FORMULATION

The advective diffusion equation for a steady horizontal line source in a turbulent shear flow can be written as

$$u \frac{\partial c}{\partial x} + \frac{\partial q_z}{\partial z} = 0 \qquad (1)$$

where $c = c(x, z)$ = mass concentration; q_z = vertical mass flux by turbulent diffusion; $u = u(z)$ = longitudinal flow velocity; z = vertical coordinate; and x = longitudinal coordinate. Diffusive transport in the x direction is not included in Eq. (1) because it is negligibly small compared with the advective transport (Fischer et al. 1979). If a logarithmic velocity distribution is assumed except for the region very near the bottom, u can be written as

$$u = U + \frac{u_*}{\kappa} \left\{ 1 + \ln\frac{z}{d} \right\}, \quad \text{for } z_0 \le z \le d \qquad (2)$$

where κ = von Karman constant; d = water depth; U = mean flow velocity; and the shear velocity (u_*) is defined such that

$$u_*^2 = \frac{\tau_0}{\rho} \qquad (3)$$

in which τ_0 = bottom shear stress; and ρ = water density. For a hydraulically smooth channel, z_0 is the thickness of the laminar sublayer in which the velocity distribution is linear and given by

$$u = \frac{z u_*^2}{\nu}, \quad \text{for } 0 \le z \le z_0 \qquad (4)$$

where ν is the kinematic viscosity of water. For a rough channel, z_0 can be taken as the distance to the position where the velocity given by Eq. (2) is zero, and the velocity is taken as zero for the region $0 \le z \le z_0$. The vertical diffusive flux is modelled as

$$q_z = -\varepsilon \frac{\partial c}{\partial z} \quad (5)$$

where $\varepsilon = \varepsilon(z)$ = turbulent diffusion coefficient. Eq. (1), on substitution for q_z from Eq. (5), can be written as

$$u \frac{\partial c}{\partial x} - \frac{\partial}{\partial z} \left(\varepsilon \frac{\partial c}{\partial z} \right) = 0 \quad (6)$$

The vertical turbulent diffusion coefficient can be obtained from the Reynolds analogy that the turbulent diffusivities for momentum and mass are the same (Fischer et al. 1979). The momentum diffusivity can be derived from the following expression for the turbulent shear stress:

$$\tau = -\rho\varepsilon \frac{du}{dz} \quad (7)$$

where $\tau = \tau(z)$ = turbulent shear stress. As the shear stress has a linear distribution, $\tau(z)$ can be expressed in terms of bottom shear stress (τ_0) as

$$\tau(z) = \frac{d-z}{d} \tau_0 \quad (8a)$$

$$= \frac{d-z}{d} \rho u_*^2 \quad (8b)$$

From Eqs. (2), (7) and (8b), it can be derived that

$$\varepsilon = \kappa u_* z \left(1 - \frac{z}{d}\right) \quad (9)$$

In a laminar sublayer for a smooth channel, ε is the same as the kinematic viscosity of water as it can be derived from Eqs. (4), (7) and (8b).

Boundary conditions at the bottom and on the water surface are given by no-flux conditions as

$$\frac{\partial c}{\partial z}(x, 0) = 0 \quad (10)$$

$$\frac{\partial c}{\partial z}(x, d) = 0 \tag{11}$$

The initial condition for the source introduced at x = 0 through the area (Δz) between $z = d_s + \Delta z/2$ and $z = d_s - \Delta z/2$, can be expressed as

$$c(0, z) = c_0 = \frac{c_s q_s}{u(d_s)\Delta z} \quad \left(d_s - \frac{\Delta z}{2} < z < d_s + \frac{\Delta z}{2} \right) \tag{12a}$$

$$= 0 \quad (\text{ otherwise }) \tag{12b}$$

where $c_s q_s$ is the source strength in mass per unit volume per unit width.

RESULTS AND ANALYSIS

The diffusion equation and the initial and boundary conditions are normalized in terms of following dimensionless variables and a Crank-Nicholson scheme is adopted for the solution.

$$z' = \frac{z}{d} \tag{13}$$

$$x' = \frac{x}{d} \tag{14}$$

$$u' = \frac{u}{u_*} \tag{15}$$

$$\varepsilon' = \frac{\varepsilon}{u_* d} \tag{16}$$

$$c' = \frac{c}{c_\infty} \tag{17}$$

where c_∞ is the equilibrium concentration, i.e. the concentration far downstream where the source mass is completely mixed over the depth:

$$c_\infty = \frac{c_s q_s}{Ud} \tag{18}$$

The results are presented in terms of the dimensionless variables defined above.

Comparison with the Solution for Constant Velocity and Diffusivity

Illustrated in Fig. 1 are concentration distributions at two downstream positions for the source released at $d_s' = 0.01$. Case 1 in Fig. 1 and Fig. 2 represents the concentration distribution computed by the numerical model and Case 2 is the analytic solution for the case of constant velocity and diffusivity. Fig. 2 represents the normalized crossing distance (X_c') and mixing distance (X_m') for various friction factors. The crossing distance is defined as the longitudinal distance required for the source mass to spread across the depth and the mixing distance is the distance for the mass to be completely mixed. These definitions are essentially the same as those defined by Holley, et al. (1972) for the case of transverse mixing. X_c' and X_m' in Fig. 2 are taken as the distances where the concentration at the water surface first becomes larger than 2 % and 98 %, respectively, of the concentration at the bottom.

It is seen that the result for constant velocity and diffusivity overestimates the rate of vertical mixing. The mixing distance for Case 2 is about half as short as that for Case 1. This contradicts the conclusion of McNulty and Wood(1984), who used $u_* d$ as the average diffusivity instead of the correct $u_* d/6$. Also observed in Fig. 2 is that the crossing and mixing distances become shorter for larger friction factor. This means a more rapid vertical mixing for a larger friction factor.

Sensitivity to Friction Factor

A series of numerical simulations was carried out to see the effect of friction factor on the vertical diffusion in turbulent shear flows. Friction factors of 0.01, 0.02, 0.04, and 0.08 were tried. The source position was taken as $d_s' = 0.01$. Fig. 3 shows concentration distributions at downstream positions, $x' = 16$ and $x' = 32$ for various friction factors. The initial concentration is higher for larger friction factor because given the same mean velocity, the velocity near the bottom is lower for a larger friction factor, consequently giving the higher initial concentration as it is given by Eq. (12a). However, as the source mass travels downstream, it is mixed more rapidly in case of larger friction factor. One can observe that the concentration distribution at $x' = 16$ for $f = 0.08$ is very close to that at $x' = 32$ for $f = 0.02$, which means that the degree of vertical mixing for $f = 0.08$ is twice as large as it is for $f = 0.02$. Similar relationship is observed between f

= 0.04 and f = 0.01. Considering that a vertical turbulent velocity component for a given mean flow velocity is proportional to the square root of friction factor, increasing a friction factor by 4 times would give twice as large a turbulent velocity, resulting in approximately twice as rapid a diffusion process as observed in the above comparison. By comparing concentration distributions for various friction factors at different downstream positions, it is concluded that the rate of vertical diffusion varies approximately as the square root of the friction factor.

Sensitivity to Source Position

The effect of source position on the vertical diffusion was investigated by computing concentration distributions at downstream positions for various initial conditions. Fig. 4 shows the simulation results for the following three different source positions: (1) d_s' = 0.01 (near the bottom); (2) d_s' = 0.99 (near the water surface); (3) d_s' = 0.50 (at the mid-depth). For all cases f = 0.04 was used. The high initial concentration for d_s' = 0.01 in Fig. 4-(a) is due to the low flow velocity near the bottom. As shown in Fig. 4-(b) and Fig. 4-(c), d_s' = 0.50 is the best among the three for the rapid mixing, and d_s' = 0.99 results in the slowest mixing. The vertical diffusivity, which has a parabolic distribution, is equally low both near the surface and near the bottom of the channel, but the longitudinal advection is higher at the surface due to the higher flow velocity. Hence, to achieve the same degree of mixing, source mass introduced near the water surface need more time than that introduced near the bottom. Moreover, the location of the maximum concentration for d_s' = 0.05, due to the slow longitudinal advection as well as the low vertical diffusivity near the bottom, moves toward the channel bottom. The fact that the concentration is higher in the region closer to the bottom implies that the best source position, which gives the most rapid uniform mixing, exists somewhere between the mid-depth and the water surface.

Optimum Source Position

The best source position for rapid mixing was found for different friction factors and the result is shown in Fig. 5. Various source positions were tried for each friction factor and the one which gives the shortest mixing distance was taken as the optimum source position.

One can see that the optimum source position is above the mid-depth as is expected. Furthermore, the best source position moves toward the water surface as the friction factor increases since the velocity profile becomes steeper for a larger friction factor, which means a larger difference between the longitudinal advection near the surface and that near the bottom.

CONCLUSIONS

As is seen from a normalization of the advective diffusion equation, the vertical diffusion process depends solely on the friction factor. The analytic solution for constant velocity and diffusivity overestimates the degree of vertical mixing. The rate of vertical mixing varies approximately as the square root of the friction factor, which is reasonable considering that the turbulent velocity component varies with the square root of the friction factor. The best source position, which gives the most rapid mixing moves toward the water surface as the fricion factor increases since the velocity profile becomes steeper for a larger friction factor.

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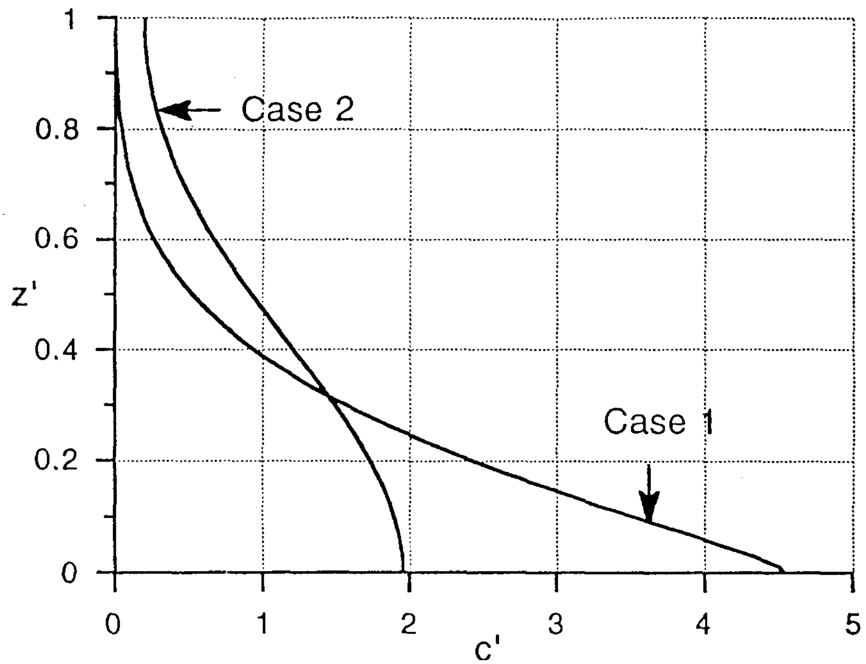
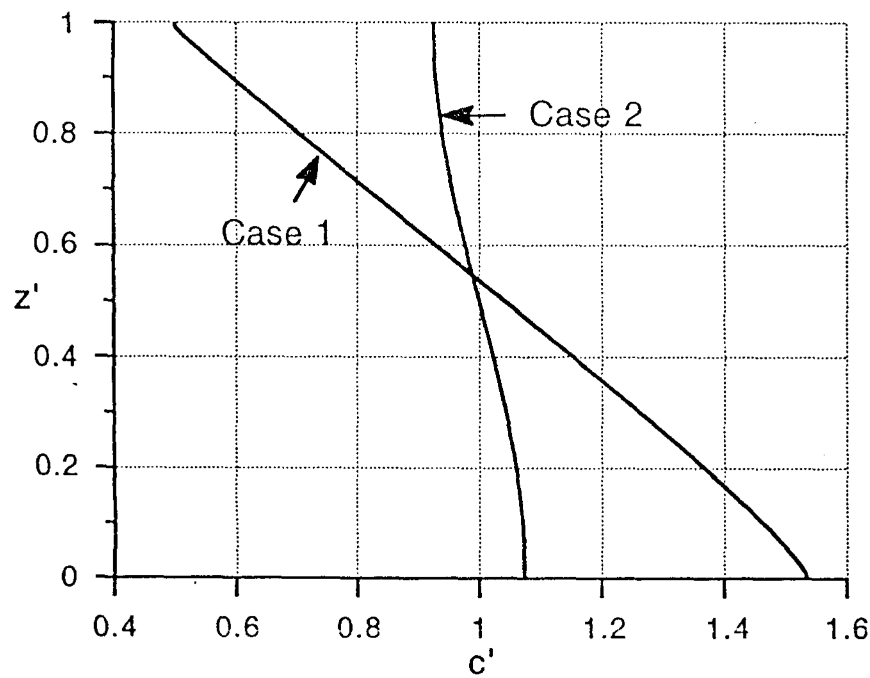
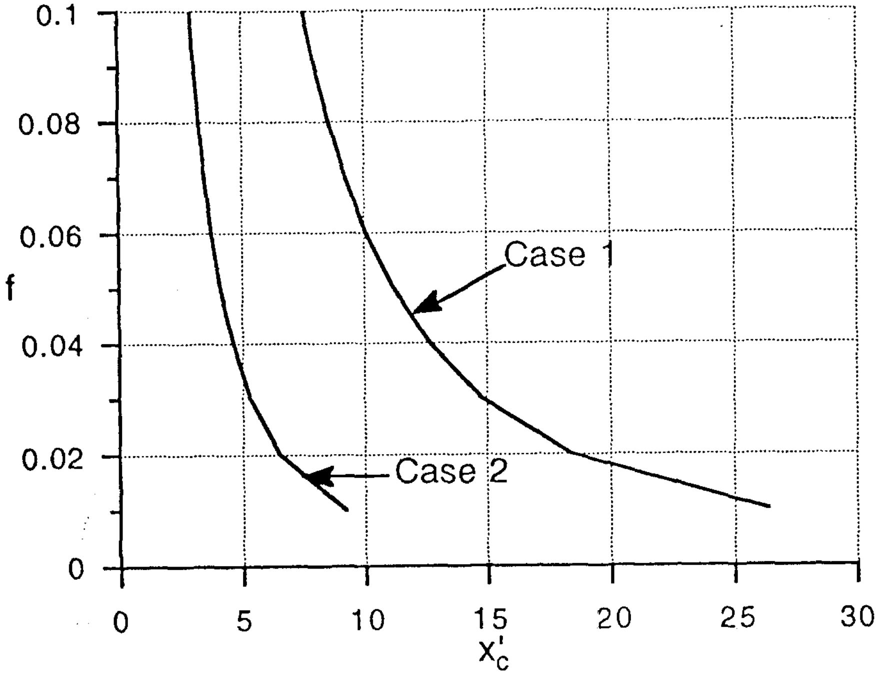


(a) x' = 10

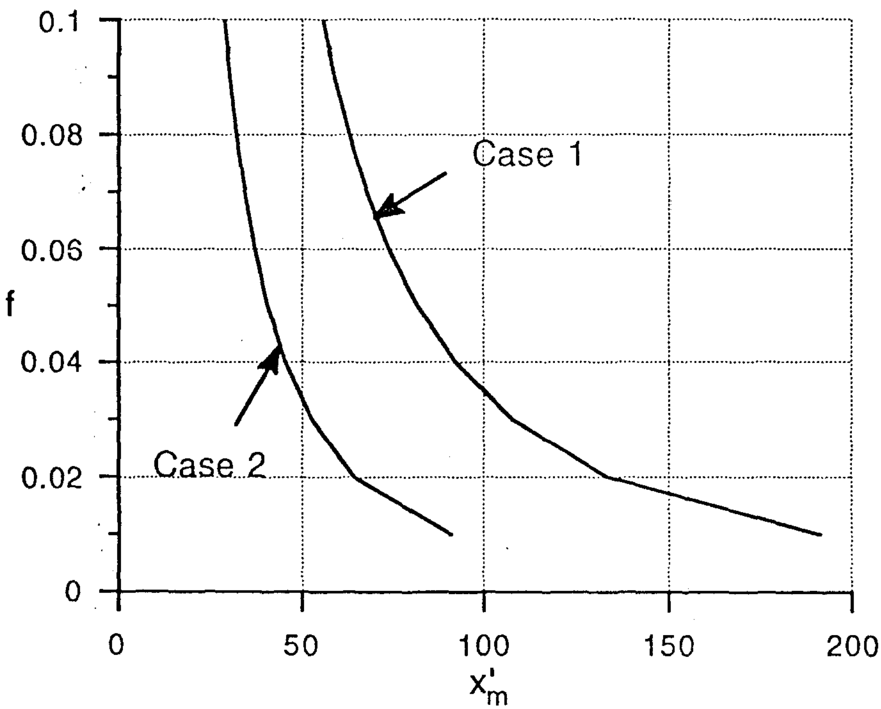


(b) x' = 40

Fig. 1 Concentration Distribution at Downstream Positions

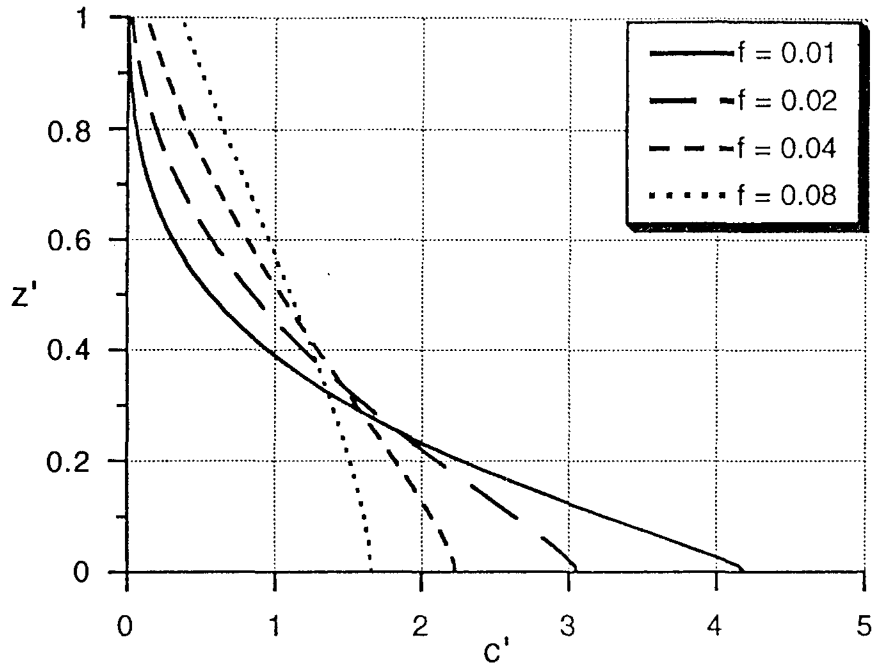


(a) Crossing Distance

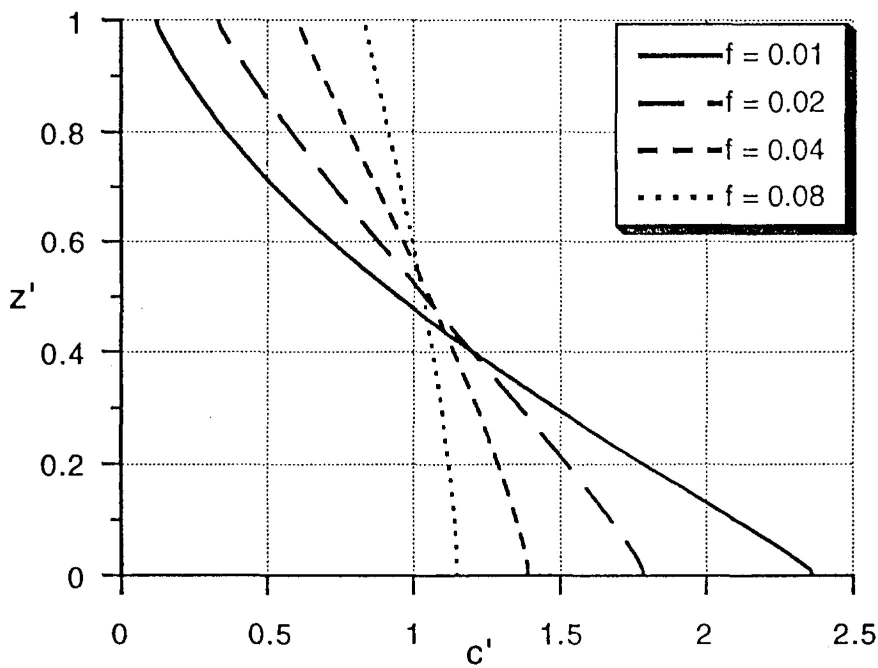


(b) Mixing Distance

Fig. 2 Crossing and Mixing Distances

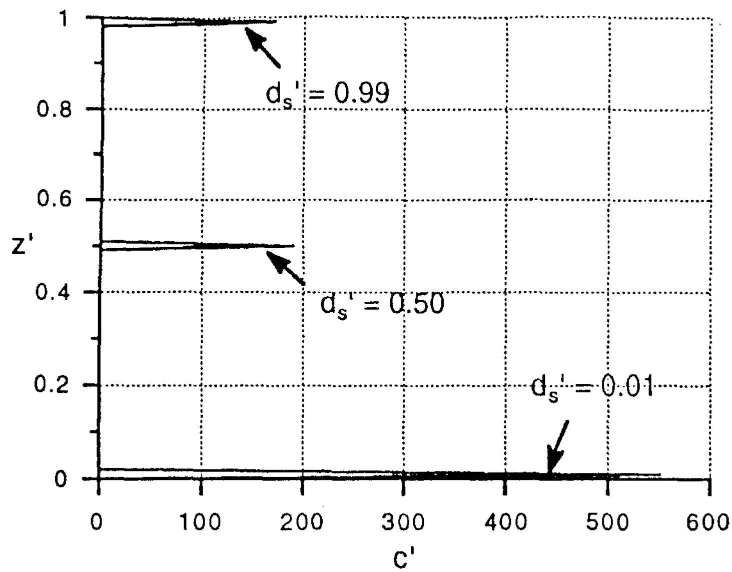


(a) x' = 16



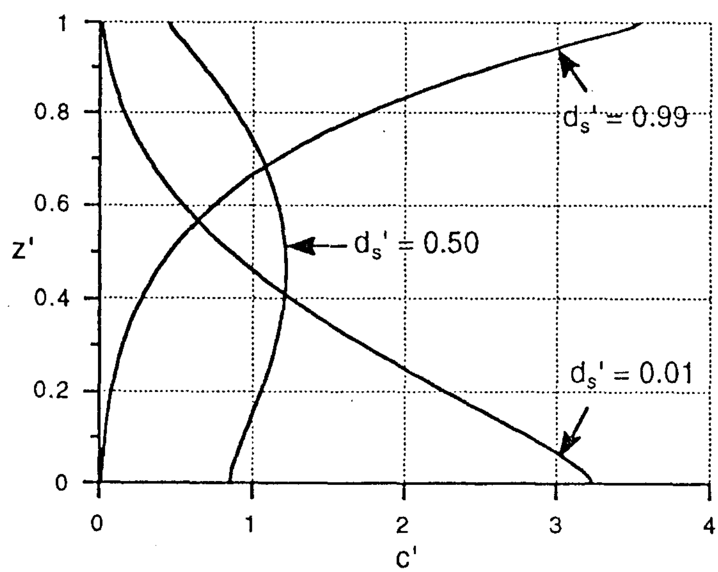
(b) x' = 32

Fig. 3 Concentration Distribution for Various Friction Factors



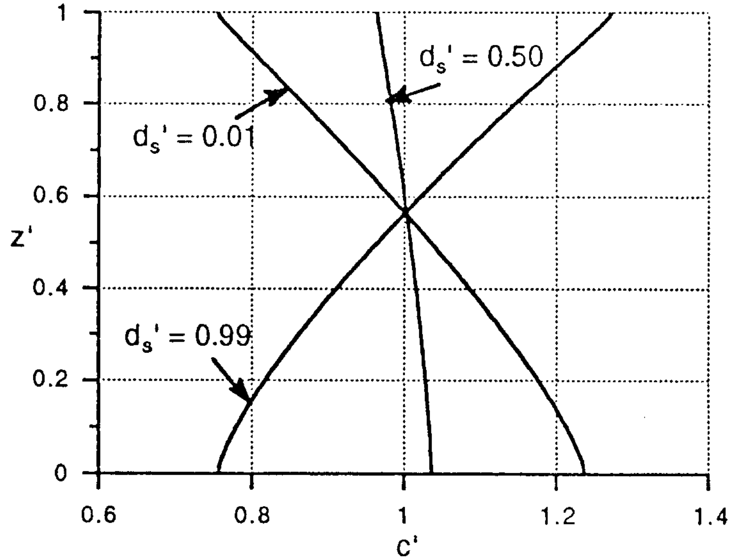
1
0.8
0.6
0.4
0.2
0
z'
0
100
200
300
400
500
600
c'
d_s' = 0.99
d_s' = 0.50
d_s' = 0.01

(a) x' = 0



1
0.8
0.6
0.4
0.2
0
z'
0
1
2
3
4
c'
d_s' = 0.99
d_s' = 0.50
d_s' = 0.01

(b) x' = 10



(c) x' = 40

Fig. 4 Concentration Distribution for Various Source Positions

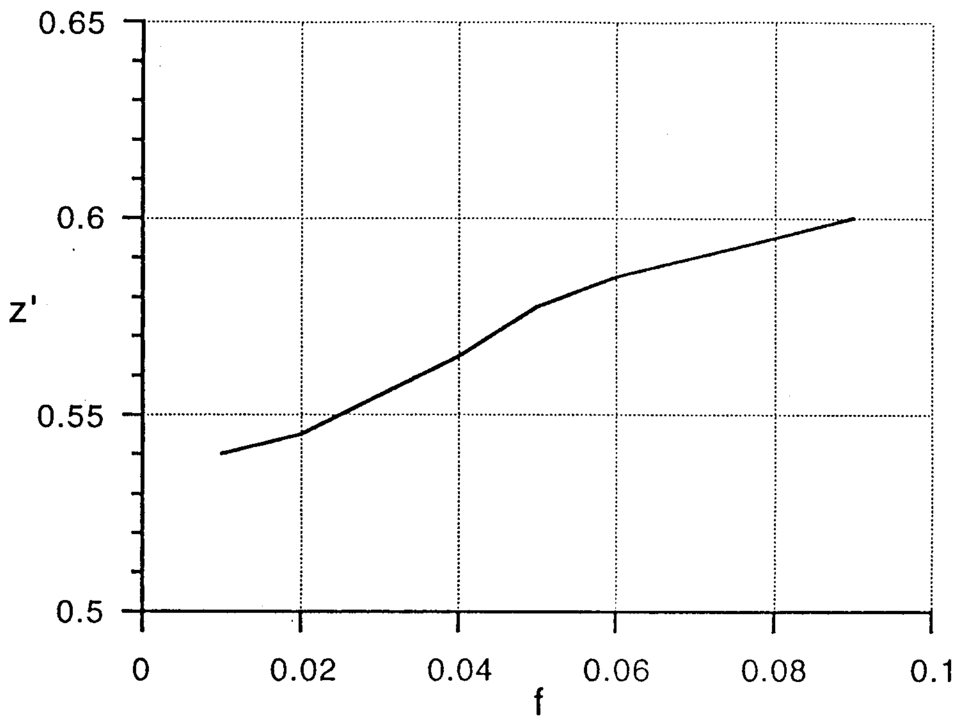


Fig. 5 Optimum Source Position for Various Friction Factors