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A Pole Assignment Control Design for Single-Input Double-Output Nonlinear Mechanical Systems

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Abstract

This paper discusses a design of a nonlinear control for a class of single-input double-output nonlinear mechanical systems. When conventional linearization methods are applied to the mechanical systems, some problems of oscillation and unstable phenomena arise. The proposed nonlinear control system resolves these problems. In this design the eigenvalues of the closed-loop nonlinear system are assigned to desired locations and local asymptotic stability of the closed-loop system is guaranteed. The design method is applied to an inverted pendulum system with a moving weight mechanism. Experimental results show that the proposed nonlinear controller is more effective for stability than the usual linear controller.

1 Introduction

Nonlinear control design using nonlinear feedback has been discussed in several papers [1]-[5]. The main idea of the approach is to transform a nonlinear system into a decoupled linear system by the cancellation of nonlinear terms so that standard linear control methods can be applied. If the magnitude of the control input is within a certain limit, it is possible to assure the stability of the nonlinear closed-loop system. These systems, however, have drawbacks such as oscillation and unstable phenomena due to mis-linearization. Moreover, in the input-output linearization approach described above, the number of input signals must be equal to, or more than, the number of output signals.

In mechanical systems there are many nonlinear systems with more output signals than input signals. Examples of these are inverted pendulums [7][8], container cranes [9], and rotary cranes [10]. In these examples, Lyapunov's linearization method is applied at the equilibrium point and the linear state feedback controller is used to stabilize the systems for small range motions.

The purpose of this paper is to outline the design of a nonlinear controller based on the pole assignment method for a class of single-input and double-output nonlinear mechanical systems. The proposed nonlinear control system consists of three parts: nonlinear feedback input for canceling nonlinear terms, nonlinear state feedback input for assigning the eigenvalues of the closed loop to the desired locations, and stepwise reference input. The controller ensures the stability of the closed-loop nonlinear system. The nonlinear control design is applied to an inverted pendulum system with a moving weight. The control objective of the inverted pendulum system is to follow the cart position to the stepwise reference input and to stabilize the inverted pendulum to the upright position when the weight of the pendulum is moving up and down. Experimental results show the effectiveness of the proposed controller compared with a conventional linear controller.

The paper is organized as follows. Section 2 presents the formulation of a single-input and double-output nonlinear mechanical system. In Section 3, a nonlinear control design is developed based on the pole assignment scheme. The proof of local asymptotic stability of the closed-loop nonlinear system is given in Section 4. The experimental results are shown in Section 5.

2 Formulation of a Single-Input Double-Output Nonlinear System

This paper considers a single-input double-output mechanical system described by the pair of second-order nonlinear time-variable differential equations

$$\begin{bmatrix} 1 & J_1(\boldsymbol{x},t) \\ J_2(\boldsymbol{w},t) & 1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} = \begin{bmatrix} F_1(\boldsymbol{x},t) + G(\boldsymbol{x},t)u(t) \\ F_2(\boldsymbol{w},t) \end{bmatrix} \quad (1)$$

$$y_1(t) = x_1(t), \quad y_2(t) = x_2(t) \quad (2)$$

where $u(t)$ is the control input and $y_1(t)$, $y_2(t)$ are the outputs. $\boldsymbol{x}(t)$ is the 4-dimensional state vector and $\boldsymbol{w}(t)$ is the part of the vector $\boldsymbol{x}(t)$ as follows.

$$\boldsymbol{x}(t) = [x_1(t),\ \dot{x}_1(t),\ x_2(t),\ \dot{x}_2(t)]^T \quad (3)$$

$$\boldsymbol{w}(t) = [x_2(t),\ \dot{x}_2(t)]^T \quad (4)$$

In equation (1), it is assumed that the nonlinear function $J_i(\cdot)$, $F_i(\cdot)$ (i=1,2), and $G(\cdot)$ are smooth and bounded for all $\boldsymbol{x}, t$, and satisfy the following equations.

$$1 - J_1(\boldsymbol{x},t)J_2(\boldsymbol{w},t) \neq 0 \quad (5)$$

$$G(\boldsymbol{x},t) \neq 0 \quad (6)$$

$$J_2(\boldsymbol{w},t) \neq 0 \quad (7)$$

The nonlinear function $F_2(\boldsymbol{w},t)$ is assumed to be composed of a linear combination of the vector $\boldsymbol{w}(t)$ such that

$$F_2(\boldsymbol{x},t) = f_{23}(\boldsymbol{w},t)x_2(t) + f_{24}(\boldsymbol{w},t)\dot{x}_2(t) \quad (8)$$

where

$$f_{23}(\boldsymbol{w},t) \neq 0 \quad (9)$$

and

$$f_{2i}(\boldsymbol{w},t) = \sum_{k=1}^{q_i} a_{F2ik} \cdot b_{F2ik}(t) \cdot c_{F2ik}(\boldsymbol{w}) \quad (i=3,4) \quad (10)$$

in which the time function $b_{F2ik}(t)$, and the nonlinear function $c_{F2ik}(\boldsymbol{w})$ satisfy

$$\lim_{t\to\infty} |b_{F2ik}(t) - b_{F2ike}| = 0 \quad (i=3,4;\ k=1\sim q_i) \quad (11)$$

$$\lim_{\|\boldsymbol{w}\|\to 0} |c_{F2ik}(\boldsymbol{w}) - 1| = 0 \quad (i=3,4;\ k=1\sim q_i) \quad (12)$$

In addition, it is assumed that the nonlinear function $J_2(\boldsymbol{w},t)$ is given by

$$J_2(\boldsymbol{w},t) = \sum_{k=1}^{r} a_{J2k} \cdot b_{J2k}(t) \cdot c_{J2k}(\boldsymbol{w}) \quad (13)$$

where the time function $b_{J2k}(t)$ and the nonlinear function $c_{J2k}(\boldsymbol{w})$ satisfy

$$\lim_{t\to\infty} |b_{J2k}(t) - b_{J2ke}| = 0 \quad (k=1\sim r) \quad (14)$$

$$\lim_{\|\boldsymbol{w}\|\to 0} |c_{J2k}(\boldsymbol{w}) - 1| = 0 \quad (k=1\sim r) \quad (15)$$

In the multi-input multi-output nonlinear mechanical system, there are many systems with more output signals than input signals. System formulation (1) satisfies a class of single-input double-output mechanical systems such as the inverted pendulum and container crane. In Section 5, it will be shown that inverted pendulum nonlinear dynamics can be described in form (1). The objective of the control design for the mechanical system is to achieve one system output $y_1(t)$ that follows a desired stepwise reference input while another system output $y_2(t)$ converges to zero.

3 Control Design for a Single-Input Double-Output Nonlinear System

This section discusses a nonlinear control design for equation (1) with two outputs and only one input. It is assumed that the state vector $\boldsymbol{x}(t)$ is available for feedback control.

First, the input-output linearization is considered between the output y_1 and input u. The nonlinearity of the first equation in (1) is canceled using the following nonlinear feedback input

$$\begin{aligned} u(t) &= u_L(t) - \frac{1}{G(\boldsymbol{x},t)}(F_1(\boldsymbol{x},t) - J_1(\boldsymbol{x},t)F_2(\boldsymbol{w},t)) \\ &\stackrel{def}{=} u_L(t) - u_N(\boldsymbol{x},t) \end{aligned} \quad (16)$$

where $u_L(t)$ is a new input to be determined. Substituting the above input into equation (1) produces the nonlinear state-space form

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & f_{23}(\boldsymbol{w},t) & f_{24}(\boldsymbol{w},t) \end{bmatrix} \boldsymbol{x}(t) \\ &+ \begin{bmatrix} 0 \\ 1 \\ 0 \\ -J_2(\boldsymbol{w},t) \end{bmatrix} \frac{G(\boldsymbol{x},t)}{1 - J_1(\boldsymbol{x},t)J_2(\boldsymbol{w},t)} u_L(t) \\ &\stackrel{def}{=} A_o(\boldsymbol{w},t)\boldsymbol{x}(t) + B_o(\boldsymbol{x},t)u_L(t) \end{aligned} \quad (17)$$

There is a linear time-invariant state-space equation to which a standard linear controller can be applied if the matrices $A_o(\boldsymbol{w},t)$ and $B_o(\boldsymbol{w},t)$ are linearized around the equilibrium point. However, based on the linearization control method, there are some problems, especially in single-input and double-output systems, such as oscillation, unstable phenomena, and a small range of operation.

Therefore, stabilization of the state-space equation (17) is considered with no linearization. The new input u_L as a nonlinear state feedback input $u_F(\boldsymbol{x},t)$, and a reference input $u_R(\boldsymbol{x},t)$ is introduced

$$\begin{aligned} u_L(t) &= \frac{1 - J_1(\boldsymbol{x},t)J_2(\boldsymbol{w},t)}{G(\boldsymbol{x},t)} \\ &\quad \times (K(\boldsymbol{w},t)\boldsymbol{x}(t) + h(\boldsymbol{w},t)v(t)) \\ &\stackrel{def}{=} u_F(\boldsymbol{x},t) + u_R(\boldsymbol{x},t) \end{aligned} \quad (18)$$

where $K(\boldsymbol{w},t)$ is a nonlinear state feedback gain matrix, $h(\boldsymbol{w},t)$ is a reference gain, and $v(t)$ is a stepwise input. Defining the feedback gain $K(\boldsymbol{w},t)$ by

$$K(\boldsymbol{w},t) = [k_1(\boldsymbol{w},t),\ k_2(\boldsymbol{w},t),\ k_3(\boldsymbol{w},t),\ k_4(\boldsymbol{w},t)] \quad (19)$$

the nonlinear closed-loop system is given by (the notation $(\boldsymbol{w},t)$ is removed for simplicity)

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ k_1 & k_2 & k_3 & k_4 \\ 0 & 0 & 0 & 1 \\ -J_2k_1 & -J_2k_2 & f_{23} - J_2k_3 & f_{24} - J_2k_4 \end{bmatrix} \boldsymbol{x}(t) \\ &+ \begin{bmatrix} 0 \\ h \\ 0 \\ -J_2h \end{bmatrix} v(t) \\ &\stackrel{def}{=} A_c(\boldsymbol{w},t)\boldsymbol{x}(t) + B_c(\boldsymbol{w},t)v(t) \end{aligned} \quad (20)$$

The time-variable characteristic equation of $A_c(\boldsymbol{w},t)$ is given by

$$P_c(\sigma(t)) = \det(\sigma(t)I - A_c(\boldsymbol{w},t)) \quad (21)$$

The desired characteristic equation is given by

$$\begin{aligned} P_M(\sigma) &= (\sigma + \lambda_{M1})(\sigma + \lambda_{M1})(\sigma^2 + 2\zeta\omega\sigma + \omega^2) \\ &\stackrel{def}{=} \sigma^4 + P_{M3}\sigma^3 + P_{M2}\sigma^2 + P_{M1}\sigma + P_{M0} \end{aligned} \quad (22)$$

so as to achieve a desired response. The feedback matrix $K(\boldsymbol{w},t)$ is determined by matching the coefficients in the time-variable characteristic equation $P_c(\sigma(t))$ and in the desired characteristic equation $P_M(\sigma)$.

$$K(\boldsymbol{w},t) = [P_{M3} + f_{24}(\boldsymbol{w},t),\ P_{M2} + f_{23}(\boldsymbol{w},t),\ P_{M1},\ P_{M0}] \times \begin{bmatrix} 0 & -1 & f_{24}(\boldsymbol{w},t) & f_{23}(\boldsymbol{w},t) \\ -1 & f_{24}(\boldsymbol{w},t) & f_{23}(\boldsymbol{w},t) & 0 \\ 0 & J_2(\boldsymbol{w},t) & 0 & 0 \\ J_2(\boldsymbol{w},t) & 0 & 0 & 0 \end{bmatrix}^{-1} \quad (23)$$

Thus, the time-variable poles of closed-loop system (20) are assigned to the desired constant locations. The reference gain is given by

$$h(\boldsymbol{w},t) = -\frac{P_{M0}}{f_{23}(\boldsymbol{w},t)} \quad (24)$$

Note that the square matrix in equation (23) corresponds to the controllability matrix of $\{A_o(\boldsymbol{w},t),\ B_o(\boldsymbol{x},t)\}$ when equation (17) is considered to be the linear time-invariant system. From the assumption in (7) and (9), the determinant of the square matrix, $J_2(\boldsymbol{w},t)^2 f_{23}(\boldsymbol{w},t)^2$, is not zero for all $\boldsymbol{w}, t$.

The closed-loop system is represented in the block diagram in **Figure 1**. The inner loop achieves linearization of the first equation in (1) and the outer loop achieves stabilization of the closed-loop nonlinear system (20).

4 Proof of Asymptotic Stability for the Closed-loop System

This section considers local stability of the proposed nonlinear control system discussed in Section 3. A lemma is used which guarantees local asymptotic stability of a nonlinear system.

Lemma[6]: The null solution of the nonlinear time-variable system

$$\dot{\boldsymbol{x}}(t) = (A + B(t) + C(\boldsymbol{x},t))\boldsymbol{x}(t) \quad (25)$$

where the matrices $B(t), C(\boldsymbol{x},t)$ are continuous for all t, is exponential and asymptotically stable if

① All eigenvalues of the matrix A have negative real parts.

② $\lim_{t\to\infty} \|B(t)\| = 0$

③ $\lim_{\|\boldsymbol{x}\|\to 0} \|C(\boldsymbol{x},t)\| = 0$

(holds uniformly with respect to t)

Theorem 1: The null solution of the free dynamic system $\dot{\boldsymbol{x}}(t) = A_c(\boldsymbol{w},t)\boldsymbol{x}(t)$ of the closed-loop system (20) is exponentially asymptotically stable.

Proof: It will be shown that the free system of closed equation (20) can be transformed into equation (25), so that the transformed matrices A, $B(t)$, and $C(\boldsymbol{x},t)$ satisfy the lemma condition ①~③.

Substituting nonlinear functions (10) and (13) into feedback matrix gain (23), $k_i(\boldsymbol{w},t)$ $(i = 1 \sim 4)$ are given by

$$k_i(\boldsymbol{w},t) \stackrel{def}{=} \frac{\sum_{j=1}^{r_i} a_{Rij} b_{Rij}(t) c_{Rij}(\boldsymbol{w})}{\sum_{d=1}^{h} a_{Hij} b_{Hij}(t) c_{Hij}(\boldsymbol{w})}$$

$$(i = 1 \sim 4) \quad (26)$$

Also, the elements of the 4-th row in $A_c(\boldsymbol{w},t)$ are expressed as

$$f_i(\boldsymbol{w},t) - J_2(\boldsymbol{w},t)k_i(\boldsymbol{w},t) \stackrel{def}{=} \frac{\sum_{f=1}^{p_i} a_{Pif} b_{Pif}(t) c_{Pif}(\boldsymbol{w})}{\sum_{d=1}^{h} a_{Hij} b_{Hij}(t) c_{Hij}(\boldsymbol{w})}$$

$$(i = 1 \sim 4; f_1(\cdot) = 0, f_2(\cdot) = 0) \quad (27)$$

Each function in (26) and (27) can be represented under assumptions (11) and (14) as

$$b_{Rij}(t) = b_{Rije} + \bar{b}_{Rij}(t) \qquad (\bar{b}_{Rij}(\infty) = 0) \quad (28)$$

$$b_{Hid}(t) = b_{Hide} + \bar{b}_{Hid}(t) \qquad (\bar{b}_{Hid}(\infty) = 0) \quad (29)$$

$$b_{Pif}(t) = b_{Pife} + \bar{b}_{Pif}(t) \qquad (\bar{b}_{Pif}(\infty) = 0) \quad (30)$$

$$i = 1 \sim 4,\ j = 1 \sim r_i,\ d = 1 \sim h,\ f = 1 \sim p_i$$

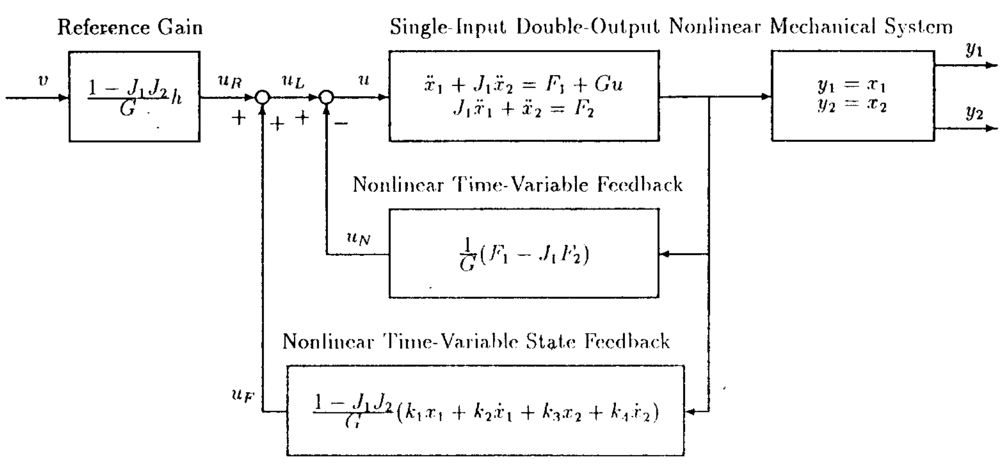

Fig. 1: Nonlinear control structure for the single-input double-output system

Also, from assumptions (12) and (15)

$$\lim_{\|\boldsymbol{w}\|\to 0} \|c_{Rij}(\boldsymbol{w}) - 1\| = 0 \quad (31)$$

$$\lim_{\|\boldsymbol{w}\|\to 0} \|c_{Hid}(\boldsymbol{w}) - 1\| = 0 \quad (32)$$

$$\lim_{\|\boldsymbol{w}\|\to 0} \|c_{Pif}(\boldsymbol{w}) - 1\| = 0 \quad (33)$$

$$i = 1\sim 4,\ j = 1\sim r_i,\ d = 1\sim h,\ f = 1\sim p_i$$

Substituting (26) and (27) into (20) gives $A_c(\boldsymbol{w},t)$ as follows,

$$\begin{aligned} A_c(\boldsymbol{w},t) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ k_1 & k_2 & k_3 & k_4 \\ 0 & 0 & 0 & 1 \\ -J_2k_1 & -J_2k_2 & f_{23}-J_2k_3 & f_{24}-J_2k_4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & 0 & 1 \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ B_{21}(t) & B_{22}(t) & B_{23}(t) & B_{24}(t) \\ 0 & 0 & 0 & 0 \\ B_{41}(t) & B_{42}(t) & B_{43}(t) & B_{44}(t) \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ C_{21}(\boldsymbol{w},t) & C_{22}(\boldsymbol{w},t) & C_{23}(\boldsymbol{w},t) & C_{24}(\boldsymbol{w},t) \\ 0 & 0 & 0 & 0 \\ C_{41}(\boldsymbol{w},t) & C_{42}(\boldsymbol{w},t) & C_{43}(\boldsymbol{w},t) & C_{44}(\boldsymbol{w},t) \end{bmatrix} \\ &\stackrel{def}{=} A^* + B^*(t) + C^*(\boldsymbol{w},t) \end{aligned} \quad (34)$$

Functions A_{2i}, B_{2i}, C_{2i} in A^*, $B^*(t)$, $C^*(\boldsymbol{w},t)$ are given by

$$A_{2i} = \frac{\sum_{j=1}^{r_i} a_{Rij} b_{Rije}}{\sum_{d=1}^{h} a_{Hid} b_{Hide}} \quad (35)$$

$$B_{2i}(t) = \frac{\sum_{j=1}^{r_i} a_{Rij}(\bar{b}_{Rij}(t) + b_{Rije})}{\sum_{d=1}^{h} a_{Hid}(\bar{b}_{Hid}(t) + b_{Hide})} - \frac{\sum_{j=1}^{r_i} a_{Rij} b_{Rije}}{\sum_{d=1}^{h} a_{Hid} b_{Hide}} \quad (36)$$

$$C_{2i}(\boldsymbol{w},t) = \frac{\sum_{j=1}^{r_i} a_{Rij} b_{rij}(t)}{\sum_{d=1}^{h} a_{Hid} b_{Hid}(t)} \left(\frac{c_{Rij}(\boldsymbol{w})}{c_{Hid}(\boldsymbol{w})} - 1 \right) \quad (37)$$

$$i = 1 \sim 4$$

Functions A_{4i}, $B_{4i}(t)$, $C_{4i}(\boldsymbol{w},t)$ $(i = 1\sim 4)$ can be expressed similarly as given in (35), (36), and (37).

It can be shown that $\det(\sigma I - A^*) = P_M(\sigma)$, and A^* satisfies lemma condition ①.

To show that $B^*(t)$ satisfies lemma condition ②, introduce the following matrix norm

$$\|B^*(t)\| = \max_i \sum_{j=1}^{4} |B_{ij}(t)| \qquad (i = 1\sim 4) \quad (38)$$

Substituting equations (28), (29), and (30) into (36), $B_{ij}(t) = 0$ as $t \to \infty$. Thus, $\|B^*(t)\| = 0$ as $t \to \infty$ and lemma condition ② is shown.

To show that $C^*(\boldsymbol{w},t)$ satisfies lemma condition ③, introduce the following matrix norm

$$\|C^*(\boldsymbol{w},t)\| = \max_i \sum_{j=1}^{4} |C_{ij}(\boldsymbol{w},t)| \qquad (i = 1\sim 4) \quad (39)$$

Substituting equations (31), (32), and (33) into (37), $C_{ij}(\boldsymbol{w},t) = 0$ as $\|\boldsymbol{w}\| \to 0$. Thus $\|C^*(\boldsymbol{w},t)\| = 0$ as $\|\boldsymbol{w}\| \to 0$ and lemma condition ③ is shown.

The next theorem shows that the closed-loop system (20) with the external reference input $v(t)$ achieves the control objective mentioned in Section 2.

Theorem 2: In the closed-loop system (20) where reference input $v(t) = d$ the state $[d\ 0\ 0\ 0]^T$ is a local exponential asymptotic equilibrium point.

Proof: The closed-loop system with the stepwise reference input $v(t) = d$ is described by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ k_1 & k_2 & k_3 & k_4 \\ 0 & 0 & 0 & 1 \\ -J_2k_1 & -J_2k_2 & f_{23}-J_2k_3 & f_{24}-J_2k_4 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ h \\ 0 \\ -J_2h \end{bmatrix} d \quad (40)$$

Since $h(\boldsymbol{w},t) = -k_1(\boldsymbol{w},t)$, there is the free system

$$\dot{\hat{\boldsymbol{x}}}(t) = A_C(\boldsymbol{w},t)\hat{\boldsymbol{x}}(t) \quad (41)$$

where the new state vector is $\hat{\boldsymbol{x}}(t) = [\boldsymbol{x}_1(t) - d,\ (\boldsymbol{x}_1(t) - d),\ \boldsymbol{x}_2(t),\ \dot{\boldsymbol{x}}_2(t)]^T$. According to theorem 1, the null solution of equation (41) is locally asymptotically stable.

5 Experimental Results of an Inverted Pendulum with a Moving Weight

In this section, the proposed nonlinear controller for single-input double-output is applied to a cart-pendulum system whose pendulum has a moving weight mechanism. The control objective of the system is to stabilize the inverted pendulum in its vertical position and to follow the cart to the stepwise reference input. The stability of the cart-pendulum system with the moving weight mechanism is an important problem in the study of walking robots.

5.1 Modeling of the Inverted Pendulum with a Moving Weight Mechanism

Figure 2 shows the cart-pendulum system with a moving weight mechanism. The system consists of a cart, a pendulum with a moving weight, a belt-pulley transmission system, and a direct drive motor.

$y(t)$: Position of the cart (m)
$\theta(t)$: Angle of the pendulum (rad)
$u(t)$: Input voltage (V)
M : Mass of the cart (kg)
m : Mass of the pendulum (kg)
$\ell(t)$: Length of the pendulum (m)
g : Acceleration of gravity (m/s^2)

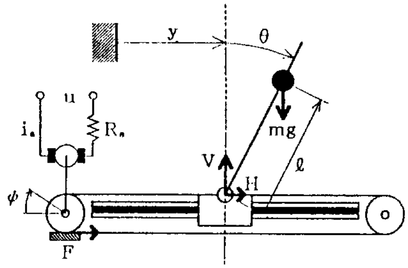

Fig. 2: Model of the inverted pendulum

$J_S(t)$	:	Moment of inertia around the gravity $= b'm\ell^2(t)$ $(kg\cdot m)$
f	:	Friction coefficient of the pendulum (N)
i_a	:	Armature current (A)
R_a	:	Armature resistance (Ω)
k_e	:	Armature inverse electromotive force (vs/rad)
k_T	:	Torque constant $(N\cdot m/A)$
r	:	Radius of the pulley (m)
J_L	:	Moment of inertia around the pulley $(kg\cdot m)$
F	:	Rotation resistance of the pulley (N)

Assume that the pendulum length $\ell(t)$ is a time function which converges to a constant value as $t\to\infty$, and that its velocity and acceleration are measured.

The nonlinear dynamic equation of the cart is given by

$$\begin{aligned}\ddot y(t) = & -\alpha\dot y(t) + \beta u(t) - \gamma\cdot sgn(\dot y(t)) \\ & - \delta\ell(t)\ddot\theta(t)\cos\theta(t) + \delta\ell(t)\dot\theta^2(t)\sin\theta(t) \\ & - \delta\ddot\ell(t)\sin\theta(t) - 2\delta\dot\ell(t)\dot\theta(t)\cos\theta(t) \qquad (42)\end{aligned}$$

where

$$\begin{aligned}\alpha &\overset{def}{=} \frac{k_e k_T}{R_a\{J_L + (M+m)r^2\}} \\ \beta &\overset{def}{=} \frac{k_T r}{R_a\{J_L + (M+m)r^2\}} \\ \gamma &\overset{def}{=} \frac{r^2}{J_L + (M+m)r^2}F \\ \delta &\overset{def}{=} \frac{mr^2}{J_L + (M+m)r^2}\end{aligned}$$

The rotation dynamic equation of the pendulum is given by

$$\ddot\theta(t) = -b\frac{\cos\theta(t)}{\ell(t)}\ddot y(t) + b\frac{g}{\ell(t)}\sin\theta(t) - a(t)\dot\theta(t) \qquad (43)$$

where

$$\begin{aligned}a(t) &\overset{def}{=} 2b\frac{\dot\ell(t)}{\ell(t)} + a_f\frac{1}{\ell^2(t)} \\ a_f &\overset{def}{=} \frac{f}{(1+b')m} \\ b &\overset{def}{=} \frac{1}{1+b'}\end{aligned}$$

Nonlinear equations (42) and (43) can be expressed as the following equations.

$$\begin{bmatrix} 1 & \delta\ell(t)\cos\theta(t) \\ \frac{\cos\theta(t)}{\ell(t)} & 1 \end{bmatrix}\begin{bmatrix}\ddot y(t) \\ \frac{\ddot\theta(t)}{b}\end{bmatrix} = \begin{bmatrix} F_1(\boldsymbol{x},t) + \beta u(t) \\ b\frac{g}{\ell(t)}\cdot\frac{\sin\theta(t)}{\theta(t)}\cdot\frac{\theta(t)}{b} - a(t)\frac{\dot\theta(t)}{b}\end{bmatrix} \qquad (44)$$

where

$$\begin{aligned}F_1(\boldsymbol{x},t) \overset{def}{=} & -\alpha\dot y(t) - \gamma\cdot sgn(\dot y(t)) + \delta\ell(t)\dot\theta^2(t)\sin\theta(t) \\ & - \delta\ddot\ell(t)\sin\theta(t) - 2\delta\dot\ell(t)\dot\theta(t)\cos\theta(t) \qquad (45)\end{aligned}$$

$$\boldsymbol{x}(t) = \left[y(t),\ \dot y(t),\ \frac{\theta(t)}{b},\ \frac{\dot\theta(t)}{b}\right]^T \qquad (46)$$

System (44) satisfies the single-input double-output system assumption discussed in Section 2. Thus, the proposed nonlinear control design is applied to the inverted pendulum system.

The parameters in equations (42) and (43) are identified by experiments and are given by

$$\alpha = 5.0,\quad \beta = 0.26,\quad \gamma = 0.27,\quad \delta = 0.1$$
$$a_f = 0.05,\quad b = 1.64,\quad g = 9.8$$

The nonlinear feedback gain and stepwise reference input discussed in Section 3 are as follows.

$$\begin{aligned}k_1(t) &= \frac{\ell(t)}{bg}\cdot\frac{\theta(t)}{\sin\theta(t)}P_{M0} \\ k_2(t) &= \frac{\ell(t)}{bg}\cdot\frac{\theta(t)}{\sin\theta(t)}(P_{M1} + a(t)k_1(t)) \\ k_3(t) &= \left(P_{M2} + a(t)k_2(t) + k_1(t) + \frac{bg}{\ell(t)}\frac{\sin\theta(t)}{\theta(t)}\right)\frac{\ell(t)}{\cos\theta(t)} \\ k_4(t) &= (P_{M3} + k_2(t) - a(t))\frac{\ell(t)}{\cos\theta(t)} \\ h(t) &= -\frac{\ell(t)}{bg}\cdot\frac{\theta(t)}{\sin\theta(t)}P_{M0}\end{aligned}$$

5.2 Simulation Results for the Stable Range

Figure 3 shows the $(\theta,\dot\theta)$ plane of the stable range obtained by the simulation. The simulation conditions are described in **Table 1**. The stable range using the proposed nonlinear controller is much larger than the range using a conventional linear controller. The linear controller is designed by the linearization of state-space equation (17) and the assignment of the poles.

5.3 Experimental Results

Table 2 shows the experimental conditions. The pendulum length $\ell(t)$ changes from 0.29 m to 0.46 m. **Figure 4** and **Figure 5** show the experimental results of the conventional linear controller and the proposed nonlinear controller, respectively. The linear controller is designed by the linearization so that the pendulum length is fixed at a constant 0.29 m. Using the proposed method, the pendulum angle $\theta(t)$ converges to zero when the gravity position moves up. On the other hand, using the linear one, it is impossible to stabilize the pendulum system.

Table 1: Simulation conditions

Initial value of $\theta(t)$	$-90 < \theta(0) < 90$ (deg)
Initial value of $\dot{\theta}(t)$	Arbitrary
Initial values of $y(t)$, $\dot{y}(t)$	$y(0) = \dot{y}(0) = 0$
Movable span of the cart	No limit
Input limit	No limit
Reference input	$v(t) = 0$
Desired eigenvalues	$-4, -4, -1, -1$

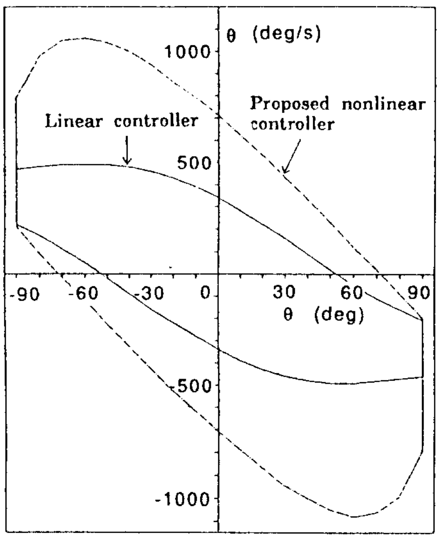

Fig. 3: Stable range

Table 2: Experimental conditions

Initial value of $\theta(t)$	$\theta(0) = 5$ (deg)
Initial value of $\dot{\theta}(t)$	$\dot{\theta}(0) = 0$ (deg/s)
Initial values of $y(t)$, $\dot{y}(t)$	$y(0) = \dot{y}(0) = 0$
Movable span of the cart	$\mid y(t) \mid \leq \pm 0.5$ (m)
Input limit	$\mid u(t) \mid \leq \pm 30$ (v)
Reference input	$v(t) = 0$
Desired eigenvalues	$-7, -8, -2, -2$

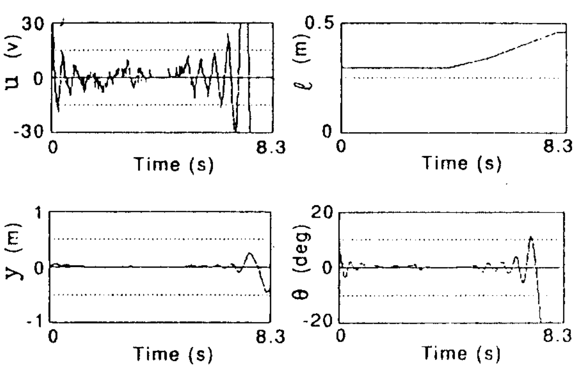

Fig. 4: Experimental results (Linear controller)

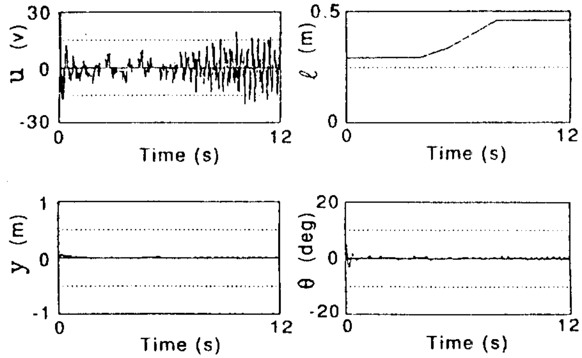

Fig. 5: Experimental results (Proposed method)

6 Conclusions

This paper discusses a nonlinear control design based on partial nonlinear feedback linearization, and the pole assignment approach for the single-input double-output nonlinear mechanical system. The nonlinear controller is applied to the inverted pendulum with the moving weight mechanism. Effective results were achieved for both simulations and experiments.

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