Robust Two Degree of Freedom H_∞ Control for Uncertain Systems

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ABSTRACT

This paper deals with the problem of robust TDF(Two Degree of Freedom) H_∞ control design for a linear system with parameter uncertainty in the state space model. The uncertain system considered here is with the time-invariant norm-bounded parameter uncertainty in the state matrix. A TDF H_∞ control design is presented which robustly stabilizes the plant, guarantees the robust H_∞ performance and improves the tracking performance for the closed-loop system in the face of parameter uncertainty. It is shown that a suitable stabilizing control law can be constructed in terms of a positive definite solution to a certain parameter-dependent algebraic Riccati equation and a good tracking performance can be constructed in terms of suitable feedforward control law.

1. Introduction

Over the past two decades there has been a great deal of interest in the problem of robust control design. H_∞ control theory has been developed for handling plants with unstructured uncertainty, i.e. exogenous signal uncertainty. The control law stabilizes the plant and also makes a selected closed loop transfer matrix as small as possible in the H_∞-norm sense. It is now known that a solution to this problem for linear time-invariant systems involves solving a pair of algebraic Riccati equations[1,2,3]. Very recently interest has been focused on the problem of robust H_∞ control for linear systems with parameter uncertainty in the state-space model[4,5]. Most H_∞ controllers are, however, based on ODF(One Degree of Freedom) structure so far. In ODF control systems it is well known that it is difficult or even impossible to optimize the design trade-offs between stability and performance which have to be made in the ODF control system design.

Here, the goal is to design a TDF(Two Degree of Freedom) H_∞ controller which robustly stabilizes an uncertain system while guaranteeing a prescribed level of disturbance attenuation in the H_∞ sense for the closed loop system for all admissible uncertainties and simultaneously improves the tracking performance for the closed loop system in the face of parameter uncertainty. Although Grimble[6] and Limebeer et al.[7] have proposed TDF H_∞ controllers, they are given under systems without parameter uncertainty. In the TDF H_∞ controller proposed here, the feedback controller is designed to meet robust stability and disturbance attenuation, while the feedforward controller is used to improve the tracking performance for the closed loop system in the face of parameter uncertainty. Here, necessary and sufficient conditions for quadratic stabilization with an H_∞ disturbance attenuation constraint of uncertain linear time-invariant system with norm-bounded uncertainties in the state matrix have been obtained in Reference [4].

In this paper we consider the problem of robust TDF H_∞ control for linear time-invariant systems with norm-bounded parameter uncertainty in the state matrix. The robust TDF H_∞ control problem is solved via the concepts of good tracking performance and quadratic stabilization with an H_∞-norm bound[4]. The concept of good tracking performance requires the existence of a feedforward controller to improves the tracking performance for the closed loop system in the face of parameter uncertainty. In addition, the quadratic stabilization requires a fixed feedback controller to robustly stabilize the plant and also guarantee an H_∞-norm bound constraint on disturbance attenuation for all admissible uncertainties. It is shown here that a suitable stabilizing control law can be constructed in terms of a positive definite solution to a certain parameter-dependent algebraic Riccati equation and a good tracking performance can be constructed in terms of suitable feedforward actions.

● *Notation* : The following notation will be used in this paper. $Re(\cdot)$ will refer to the real part of a complex number and I denotes the identity matrix. Also, $\lambda_{\min}[M]$ denotes the minimum eigenvalue of the matrix M. Also, the H_∞ norm of a stable transfer function $H(s)$ is defined by $\| H(s) \|_\infty = sup \{ \sigma_{\max}(H(j\omega)) : \omega \in R \}$ where $\sigma_{\max}(\cdot)$ stands for the maximum singular value of a matrix. $RH_\infty^{m\times n}$ denotes the set of real rational proper and stable transfer function matrices of dimension $m \times n$.

2. System Description and Definitions

In this section we will summarize the uncertain linear time-invariant systems and the theory for the feedback and feedforward control laws.

We consider the uncertain linear time-invariant systems described by state-space models of the form

$$\dot{x}(t) = [A + \Delta A(\sigma)]x(t) + B_1 w(t) + B_2 u(t) \qquad (2.1a)$$

$$z(t) = C_1 x(t) + D_1 u(t) \qquad (2.1b)$$

$$y(t) = C_2 x(t) \qquad (2.1c)$$

where $x(t) \in R^n$ is the *state*, $u(t) \in R^m$ is the *control input*, $w(t) \in R^p$ is the *disturbance input*, $y(t) \in R^r$ is the *measured output*, $z(t) \in R^q$ is the *controlled output*. A, B_1, B_2, C_1, D_1 and C_2 are constant real matrices of appropriate dimensions that describe the nominal system and $\Delta A(\sigma)$ is a matrix representing the real-valued parameter uncertainty in the state matrix, where σ is an uncertain parameter vector belonging to a compact set $\pi \subset R^l$. Furthermore, for a simplification we shall make the following assumption.

Assumption 2.1. $D_1^T[C_1, D_1] = [0, I]$.

Note that Assumption 2.1 causes no loss of generality. It is also a standard assumption in LQG optimal control and it amounts to the orthogonality of $C_1 x$ and $D_1 u$ in the cost function and to nonsingular control weighting.

The parameter uncertainty $\Delta A(\sigma)$ considered here is time-invariant and of the form $\Delta A(\sigma) = DF(\sigma)E$, where $F(\sigma) \in R^{i\times j}$ is an unknown matrix satisfying $F(\sigma)^T F(\sigma) \le I$ and D, E are known matrices of appropriate dimensions. It is also assumed that for each $F \in R^{i\times j}$ such that $F^T F \le I$, there exists a $\sigma \in \pi$ such that $F(\sigma) = F$.

In this section we will be concerned with the following notion of stabilizability for the uncertain system (2.1).

Definition 2.1. Let the constraint $\gamma > 0$ be given. The uncertain system (2.1) is said to be *stabilizable with an H_∞-norm bound* γ if there exists a fixed state feedback law $u_2 = -K_2(s)x$, where $K_2(s) \in RH_\infty^{m\times n}$ such that for all admissible parameter uncertainty $F(\sigma)$ the following conditions are satisfied :

(a) The closed loop system is asymptotically stable, i.e. $A_c = A + DF(\sigma)E - B_2K_2$ is asymptotically stable ;

(b) The closed loop transfer function from disturbance w to controlled output z, $T_{zw} = C_c(sI - A_c)^{-1}B_c$ satisfies the H_∞-norm bound $\| T_{zw}(s) \|_\infty \le \gamma$, where (A_c, B_c, C_c) is a state space model of the closed loop system.

In this paper we also use the concept of quadratic stabilization with an H_∞-norm bound[4].

Definition 2.2. Let the constraint $\gamma > 0$ be given. The uncertain system (2.1) is said to be *quadratically stabilizable with an H_∞-norm bound* γ if there exists a fixed linear state feedback law $u_2 = -K_2(s)x$ and a symmetric positive definite matrix $Q \in R^{n\times n}$ such that the inequality

$$A_c^T Q + QA_c + \frac{1}{\gamma^2} QB_cB_c^T Q + C_c^T C_c < 0$$

holds for any admissible uncertainty $F(\sigma)$, where $K_2(s) \in RH_\infty^{m\times n}$ and (A_c, B_c, C_c) is a state space model of the closed loop transfer function, $T_{zw}(s)$, from w to z.

3. Robust TDF H_∞ Control Law

This section considers the problem of robust TDF H_∞ control for the uncertain system (2.1). Assuming that perfect state information is available for feedback, we are concerned with designing a fixed feedback controller to stabilize the system with a given H_∞ constraint on disturbance attenuation for all admissible uncertainties which satisfy $F(\sigma)^T F(\sigma) \le I$ and a feedforward controller to improve the tracking performance for the closed-loop system in the face of parameter uncertainty. As in standard H_∞ control problems, we assume that the frequency weighting functions have been absorbed in the system description. Here attention will be restricted to a static state feedback control law, i.e. $u_2(t) = -K_2x(t)$, where $K_2(s) \in R^{m\times n}$. In this case we have that A_c, B_c and C_c are given by

$$A_c = A + DF(\sigma)E - B_2K_2 \quad (3.1a)$$

$$B_c = B_1 \quad (3.1b)$$

$$C_c = C_1 - D_1K_2 \quad (3.1c)$$

whereas

$$T_{zw}(s) = (C_1 - D_1K_2)(sI - A_c)^{-1}B_1. \quad (3.2)$$

3.1 Stabilizability for Uncertain Systems (State Feedback Control Law)

Xie and Souza[4,5] have shown that the quadratic stabilization problem with an H_∞-norm bound γ could be solved as follows :

Lemma 3.1. If the uncertain system (2.1) is *quadratically stabilizable with an H_∞-norm bound* $\gamma > 0$, then it is also stabilizable with the same H_∞-norm bound γ.

The approach adopted in [4,5] for solving the robust TDF H_∞ control problem involves solving a parameter- dependent algebraic Riccati equation associated with an H_∞-norm bound constraint γ and the uncertainty in the state matrix. Given the system (2.1) and any desired H_∞-norm bound constraint $\gamma > 0$, we define the following algebraic Riccati equation corresponding to the problem of quadratic stabilization with an H_∞-norm bound γ (referred to QSARE[4]) :

$$A^TQ + QA + Q\left[\frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T\right]Q + \varepsilon QDD^TQ + \frac{1}{\varepsilon}E^TE + C_1^TC_1 + I(\delta) = 0. \quad (3.3)$$

Note that when there is no uncertainty in the system, the matrices D and E can be set to zero. In this case the QSARE reduces to the well known algebraic Riccati equation corresponding to the problem of state feedback H_∞ control[3].

Theorem 3.1.[4] The uncertain system (2.1) is *quadratically stabilizable with an H_∞-norm bound* $\gamma > 0$ if and only if for a sufficiently small $\delta > 0$ there exists a constant $\varepsilon > 0$ such that the QSARE has a positive definite solution Q. Furthermore, a suitable feedback control law is given by

$$u_2(t) = -K_2x(t), \quad K_2 = B_2^TQ. \quad (3.4)$$

Theorem 3.1 provides necessary and sufficient conditions for quadratic stabilization with an H_∞-norm bound γ for the uncertain system (2.1). When there is no uncertainty in the system, i.e. $\Delta A(\sigma) = 0$, Theorem 3.1 reduces to a well known H_∞ control result for the nominal system[3].

3.2 Robust Tracking Performance for Uncertain Systems (Feedforward Control Law)

The aim of this subsection is concerned with the notion of robust tracking performance for the uncertain system (2.1). Most of the robust control design problems are solved through the state feedback control system based on the ODF scheme. However, when the ODF state feedback control system is faced with demanding tracking performance robustness, it is difficult or even impossible to make the trade-offs between stability and performance. Therefore, we will solve this problem by the TDF scheme including feedforward control law.

Lemma 3.2. If we can obtain K_2 from Theorem 3.1 and K_1 which is satisfied with below (3.5), K_1 guarantees a good tracking performance for uncertain system (2.1).

$$u_1(t) = K_1r(t) \quad (3.5a)$$

$$K_1 = -\left(C_2[(A + DE) - B_2K_2]^{-1}B_2 \right)^{-1} \quad (3.5b)$$

Proof : We assume first the number of control input u equals to that of output y. To select the feedforward controller K_1, we consider TFM(Transfer Function Matrix) from reference input $r(s)$ to output $y(s)$ with the largest uncertainty, i.e.

$$y(s) = G(s)r(s)$$

$$G(s) = C_2[I + \Phi(s)B_2K_2]^{-1}\Phi(s)B_2K_1$$

where $\Phi(s) = (sI - [A + DE])^{-1}$. Generally speaking, because reference input has energy in the low frequency regions, we select K_1 which satisfies the condition in DC $(s = 0)$

$$y(0) = Ir(0) \quad (3.6)$$

i.e., we select K_1 which makes $G(0)$ be identity matrix

$$C_2[I + \Phi(0)B_2K_2]^{-1}\Phi(0)B_2K_1 = I$$

or

$$K_1 = -\left(C_2[I - (A + DE)^{-1}B_2K_2]^{-1}(A + DE)^{-1}B_2 \right)^{-1}$$

$$= -\left(C_2[(A + DE) - B_2K_2]^{-1}B_2 \right)^{-1}.$$

Thus we have the required result (3.5).

□□□

Combining Theorem 3.1 with Lemma 3.2, we have the following result which is the solution of the robust TDF H_∞ problem :

Lemma 3.3. If the Theorem 3.1 and Lemma 3.2 are satisfied simultaneously, then the robust TDF H_∞ controller robustly stabilizes the plant, guarantees a robust H_∞ performance and improves the tracking performance for uncertain system (2.1) and any desired H_∞-norm bound constraint $\gamma > 0$. Furthermore, a suitable TDF control law is given by

$$u = u_1 + u_2 \tag{3.7}$$

where $u_1(t) = K_1 r(t)$ and $u_2(t) = -K_2 x(t)$ with

$K_1 = -(C_2[(A + DE) - B_2K_2]^{-1}B_2)^{-1}$ and $K_2 = B_2^T Q$, respectively.

Proof : Follows immediately from Theorem 3.1 and Lemma 3.2.

□□□

4. Numerical Example

Consider the uncertain system

$$\dot{x}(t) = \left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + Dfe\right)x(t) + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{4.1}$$

where

$$D = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \; e = 1, \; |f| \le 1, \; \gamma = 1, \; \varepsilon = 1.$$

We will apply the method described above to determine the range of values of a. Using above matrices, we now determine if the corresponding Riccati equation (2.4) has a positive definite solution. The TDF H_∞ controller proposed has been designed with the largest uncertainty, i.e., $f = 1$.

Fig. 1 and 2 show step responses of the closed-loop system for $a = 0.1$ and $a = 0.8$, respectively. Note that the real value of f has been taken as $f = 0.9$ and 0.1 in this simulation. The ODF H_∞ controller[4] is also designed for comparison. These curves indicate that the robust TDF H_∞ controller robustly stabilizes the plant, guarantees a robust H_∞ performance and improves the tracking performance for uncertain system (4.1) but that the ODF H_∞ controller does not show good tracking performances. Here, we can conclude that the TDF H_∞ controller is not only quadratically stabilizable but also improves a good tracking performance in the region of $0 < a \le 0.8$.

5. Conclusion

The aim of this paper is to deal with the problem of robust TDF H_∞ control design for a linear system with parameter uncertainty in the state matrix of the state-space model. We proposed the robust TDF H_∞ control design which robustly stabilizes the plant, guarantees a robust H_∞ performance and improves the tracking performance for uncertain system. Also, we showed via a simulation the robust stability and performance of the robust TDF H_∞ controller proposed here.

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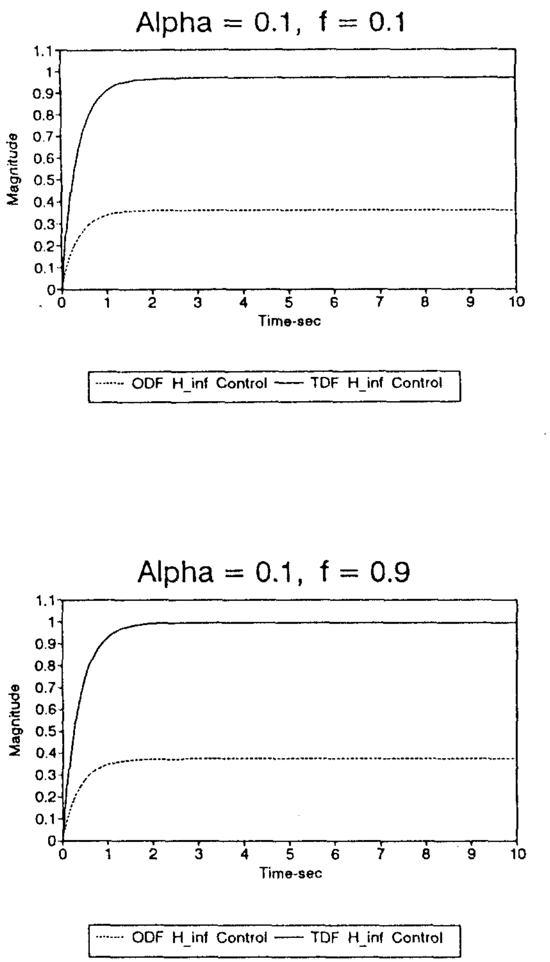

Fig. 1 : ODF and TDF closed-loop step response (α = 0.1).

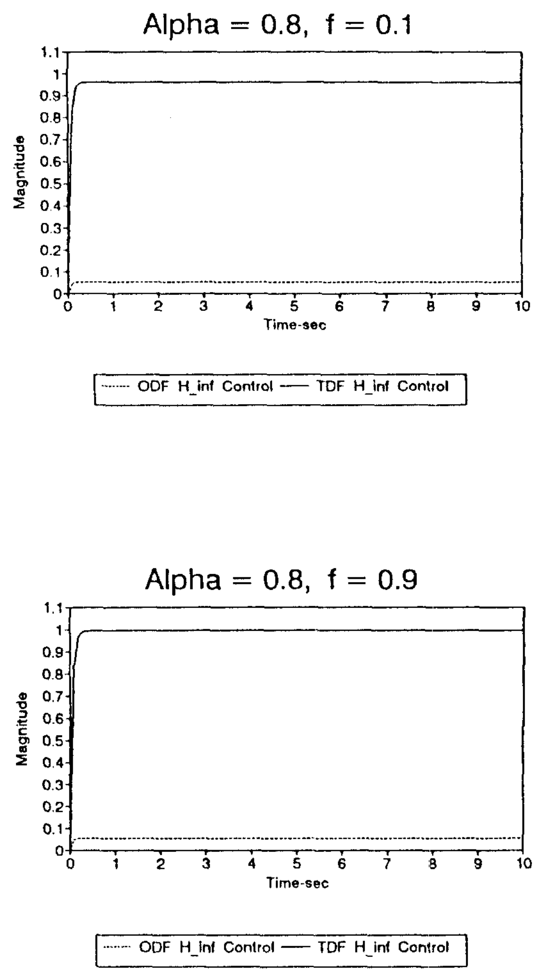

Fig. 2 : ODF and TDF closed-loop step response (α = 0.8).