Stabilizing Controllers for Plants with Perturbations

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Abstract This paper gives a convenient parametrization for the class of all stabilizing controllers in the presence of plant perturbation. The perturbations are constrained in such a way that plants are stabilizable by a nominal controller. By using the controller stabilizing a given plant with perturbation, we can obtain a parametrization of all stable closed-loop transfer functions, which are affine in the free parameter of the controller. It is easy to extend the controller to the case of a two-degree-of-freedom controller.

The parametrization for the case of perturbed plants can be also extended to a two-degree-of-freedom controller. The derived representations for stabilizing controllers are convenient for robust controller design and adaptive control problems.

In the following, \( R_v \) denotes the class of rational proper transfer functions, and \( RH^\infty \) denotes the class of stable transfer functions in \( R_v \).

2. Preliminaries

In this section, we first review the parametrization theory for the class of all stabilizing controllers for a nominal plant, and the dual class of plants stabilized by a nominal controller. Using convenient parametrizations of plants and controllers, we investigate closed-loop stability conditions for plants with perturbations parametrized in terms of a transfer function \( R \), and feedback controllers parametrized in terms of a transfer function \( Q \).

Consider a stabilizable and detectable nominal plant \( P_o \in R_v \) and proper stabilizing controllers for \( P_o \) as \( C \in R_v \), as in Fig.1, where the closed-loop system is well posed. That is

\[
\begin{bmatrix}
1 & -C \\
-P_o & 1
\end{bmatrix}^{-1}
\text{ exists and belongs to } RH^\infty
\]  

Consider also coprime factorizations for the plant \( P_o \) and some stabilizing controller \( C_o \) for \( P_o \) as

\[
P_o = N_o W_o^{-1} = \hat{N}_o \hat{W}_o^{-1} N_o; \quad N_o, W_o, \hat{N}_o, \hat{W}_o \in RH^\infty
\]

\[
C_o = U_o V_o^{-1} = \hat{U}_o \hat{V}_o^{-1} U_o; \quad U_o, V_o, \hat{U}_o, \hat{V}_o \in RH^\infty
\]

\[\text{Fig.1 Feedback control system.}\]
which satisfy the double Bezout identity as
\[
\begin{bmatrix}
\tilde{V}_n & \tilde{U}_n \\
-N_n & M_n
\end{bmatrix}
\begin{bmatrix}
M_n & U_n \\
N_n & V_n
\end{bmatrix}
= \begin{bmatrix}
\tilde{V}_n & \tilde{U}_n \\
-N_n & M_n
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\tag{4}
\]

The class of all stabilizing controllers for the nominal plant \( P_n \) can be characterized in terms of an arbitrary \( Q \in RH^- \) as (see [1,2])
\[
C(Q) = \{ (\bar{V}^{-1}, \bar{U}) \mid \bar{V} = V_n + N_n Q, \bar{U} = U_n + M_n Q \}
\tag{5}
\]
The controller \( C(Q) \) of (5) can be also written via (4) as
\[
C(Q) = C_n + \tilde{V}_n^{-1} \bar{Q} (1 + \tilde{V}_n^{-1} N_n Q)^{-1} \tilde{V}_n^{-1}
\tag{6}
\]
which can be reorganized as in Fig. 2 with
\[
J_n = \begin{bmatrix}
C_n & \tilde{V}_n^{-1} \\
V_n^{-1} & -V_n^{-1} N_n
\end{bmatrix} \in RH^- \tag{7, a)}
\]
\[
T_n = \begin{bmatrix}
1 & -C_n \\
-P_n & I
\end{bmatrix}
\begin{bmatrix}
M_n \\
N_n
\end{bmatrix} \in RH^- \tag{7, b)}
\]

It is known that the associated four closed-loop transfer functions between the \( u_i \) and \( e_i \) of Fig. 2 are affine in \( Q \) as follows: (see [4])
\[
\begin{bmatrix}
I & -C(Q) \\
-P_n & I
\end{bmatrix}
= \begin{bmatrix}
1 & -C_n \\
-P_n & I
\end{bmatrix}^{-1}
+ \begin{bmatrix}
M_n \\
N_n
\end{bmatrix} Q \begin{bmatrix}
\tilde{V}_n^{-1} -M_n \\
\tilde{N}_n^{-1} N_n
\end{bmatrix} \in RH^- \tag{8}
\]

The dual result is immediate on interchanging the role of controller and plant in the parametrization theory for the class of all stabilizing controllers. That is, under the above definitions, the class of all proper plants stabilizable by \( C_n \) is characterized by an arbitrary \( R \in RH^- \) as
\[
P(R) = N_n R + N_n V_n R, \quad \bar{M} = M_n R + U_n R
\]
\[
= M_n R + N_n V_n R, \quad \bar{M} = M_n R + U_n R
\tag{9}
\]

Then, the closed-loop transfer function matrix of \( (P(R), C_n) \) is given by
\[
\begin{bmatrix}
I & -C_n \\
-P(R) & I
\end{bmatrix}
= \begin{bmatrix}
I & -C_n \\
-P_n & I
\end{bmatrix}^{-1}
+ \begin{bmatrix}
U_n \\
V_n
\end{bmatrix} R \begin{bmatrix}
\tilde{V}_n^{-1} -M_n \\
\tilde{N}_n^{-1} N_n
\end{bmatrix} \in RH^- \tag{10}
\]

Concerning robust stabilization for plant \( P(R) \) with feedback controller \( C(Q) \), we can obtain the following theorem.

Theorem 1 (Tay et al. [7]).

Let \( (P_n, C_n) \) be a stabilizing nominal plant-controller pair. Let \( C(Q) \) be the class of controllers parametrized by \( Q \) as in (5) and \( P(R) \) be the class of plants parametrized by \( R \) as in (9). (Here there is no requirement that \( Q, R \in RH^- \).) Then, \( C(Q), C(n) \) forms a stabilizing pair if and only if \( Q \) stabilizes \( R \) in that
\[
\begin{bmatrix}
1 & -Q \\
-R & I
\end{bmatrix}^{-1} \in RH^- \tag{11}
\]

Moreover, the closed-loop transfer function matrix of \( (P(R), C(Q)) \) is given by
\[
\begin{bmatrix}
I & -C(Q) \\
-P(R) & I
\end{bmatrix}
= \begin{bmatrix}
1 & -C_n \\
-P_n & I
\end{bmatrix}^{-1}
+ \begin{bmatrix}
M_n \\
N_n
\end{bmatrix} Q \begin{bmatrix}
\tilde{V}_n^{-1} -M_n \\
\tilde{N}_n^{-1} N_n
\end{bmatrix} \in RH^- \tag{12}
\]

Remarks

1. Note that there is no requirement in this theorem for either \( R \) or \( Q \) to belong to \( RH^- \). This allows us to consider the case when the nominal controller does not stabilize the plant \( P_n(R) \).
2. For the simultaneous stabilization of \( P_n, P(R) \), there must be a stable controller \( Q \in RH^- \) stabilizing \( R \) (see [3]).
Fig. 3 Control system in the presence of plant perturbation.

3) The above results hold for the time-varying case (see [16]).
4) The closed-loop system \( (P,R,C(Q)) \) can be recognized as in Fig. 3 with \( P(R) \) parametrized by \( R \) as in (9), \( J_0 \) given by (7.a) and \( T_1 \) given by

\[
T_1 = \begin{bmatrix}
I & -C_0 \\
-P(R) & I \\
\tilde{N} & M
\end{bmatrix}^{-1} \begin{bmatrix}
M \\
N
\end{bmatrix} \quad (13)
\]

Thus information about the plant perturbation \( R \) can be deduced by observing the signals \( u \) and \( e \) (see [7]).

3. Stabilizing controllers for perturbed plants

In this section, we derive a convenient parametrization for the class of all stabilizing controllers in the presence of plant perturbation by using Theorem 1. The perturbations are constrained in such class as the plants are stabilizable by a nominal controller.

Theorem 2.

Suppose the pair \((P_0, C_0)\) is stable and \( P(R) \) is the class of plants parametrized by \( R \in RH^\infty \) as in (9). (Here we require that \( R \in RH^\infty \).) Then, under (2)–(4) and (9), the class of all stabilizing controllers for \( P(R) \) can be characterized in terms of an arbitrary \( Q(S) \) within the set

\[
Q(S) = \{ S \mid (1 + RS)^{-1} : S \in RH^\infty, \ I + RS \neq 0 \} \quad (14)
\]

368
Thus representation of (16) is immediate from the more familiar theory for the class of all stabilizing controllers.

5) The controller $C(Q(S))$ of (15) can be reorganized as in Fig.4(a) with $J_1$, given by (7.a). Note also that $C(Q(S))$ can be written via (19) as

$$C(Q(S)) = C_n \tilde{V}_n^{-1} S (1 + V_n^{-1} NS)^{-1} V_n^{-1}$$

which can be also reorganized as in Fig.4(b,c) with

$$J_1 = \begin{bmatrix} C_n & \tilde{V}_n^{-1} \\ V_n^{-1} & -V_n^{-1} N \end{bmatrix} R$$

$$T_2 = \begin{bmatrix} 1 & -C_n \\ -P(R) & I \end{bmatrix} M_n \tilde{N} \in RHI$$

Then, the associated four closed-loop transfer functions between the $u_i$ and $e_i$ of Fig.4 are affine in $S$ as in (17). Therefore, if we can identify $R$, then the coprime factorization for the perturbed plant can be given by (9), and the parameter $S$ can be utilized for adaptive tuning or robust controller design.

6) The dual result is immediate on interchanging the role of controller and plant in Theorem 2. Let $(P_n, C_n)$ is stable and $C(Q)$ is the class of controllers parametrized by $Q \in RHI^-$ as in (5). Then, under (2)-(5), the class of all proper plants stabilizable by $C(Q)$ is characterized in terms of an arbitrary $RHI$ within the set

$$R(H) = \{(1 + QI)^{-1} : H \in RHI^- , \|1 + QI\| \neq 1\}$$

as

$$P(R(H)) = N(R(H)) M(R(H))^{-1}$$

where

$$N(R(H)) = N_n + V_n R(H), \quad M(R(H)) = M_n + U_n R(H)$$

Moreover, by using the representation of the controller $C(Q)$ of (5), the plant $P(R(H))$ of (22) with (22) can be also parametrized in terms of an arbitrary $HI \in RHI^-$ as

$$P(R(H)) = N_1 M_1^{-1}, \quad N_1 = N_n + V_n I, \quad M_1 = M_n + U_n I$$

Then, the closed-loop transfer function matrix of $(P(R(H)), C(Q))$ is given by

$$\begin{bmatrix} 1 & -C(Q) \\ -P(R(H)) & I \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -C_n \\ -P_n & I \end{bmatrix}$$

$$+ \begin{bmatrix} M_n & 0 \\ N_n & V_n \end{bmatrix} \begin{bmatrix} O & Q \\ H & 0 \end{bmatrix} \begin{bmatrix} \tilde{V}_n & \tilde{N}_n \\ \tilde{N}_n & \tilde{M}_n \end{bmatrix} \in RHI^-$$

(25)

4. Two-degree-of-freedom controllers

In the previous section, we gave a parametrization for the class of all stabilizing one-degree-of-freedom controllers for perturbed plants stabilizable by a nominal controller. In this section, this result is extended in the case of a two-degree-of-freedom control system.

Consider a two-degree-of-freedom control system as shown in Fig.5. Where $P_n \in R$ is the nominal plant of (2),
\[ C = [C_1, C_2] \in \mathbb{R}^r \] is a two-degree-of-freedom controller, and \( u_1 \) is a reference signal. It is known that the class of all stabilizing two-degree-of-freedom controllers for the nominal plant \( P_n \) can be characterized in terms of arbitrary parameters \( Q, Q_2 \in \mathbb{R}^{r*} \), under (2) - (4), as (see [2])

\[ C(Q, Q_2) = (\hat{V}_n + Q_2 N_n)^{-1} \begin{bmatrix} (u + Q_1 M_n) & Q_2 \end{bmatrix} \]  

(26)

By using (4) and the following matrix identity:

\[ (A + BC)^{-1} = A^{-1} - A^{-1} B (I + CA^{-1} B)^{-1} CA^{-1} \]  

(27)

the controller \( C(Q_1, Q_2) \) of (26) can also be written as follows:

\[ C(Q_1, Q_2) = \begin{bmatrix} C_0 + D_0 \ & (\hat{V}_n^{-1} - D_0 N_n) Q_2 \end{bmatrix} \]  

(26.a)

where

\[ D_0 = \hat{V}_n^{-1} Q_1 (I + V_n^{-1} N_0 Q_1)^{-1} V_n^{-1} \]  

(23, b)

which can be reorganized as in Fig.6 with \( J_n \) and \( T_n \) given by (7), and \( Q = \{Q_1, Q_2\} \in \mathbb{R}^{r*} \). The associated six closed-loop transfer functions between the \( u_1 \) and \( e_1 \) of Fig.6 are affine in \( Q, Q_2 \in \mathbb{R}^{r*} \) as follows:

\[ W(P_n, C(Q_1, Q_2)) = \begin{bmatrix} 1 - P_n & -C_0 \ 1 & -P_n \end{bmatrix}^{-1} \begin{bmatrix} M_n & N_n \ 1 & N_n \end{bmatrix} Q_1 \begin{bmatrix} \hat{N}_n & M_n \ N_n & \hat{M}_n \end{bmatrix} \begin{bmatrix} M_n & N_n \ \end{bmatrix} Q_2 \]  

\[ \in \mathbb{R}^{r*} \]  

(29)

Moreover, by using the two-degree-of-freedom controller \( C(Q_1, Q_2) \), two specifications could be independently satisfied: one is closed-loop characteristics such as the disturbance rejection, and the other is transfer characteristics with respect to the reference command.

On the other hand, in the case of the presence of plant perturbation, by using Theorem 2 under (30), (9), and (19), the class of all stabilizing two-degree-of-freedom controllers for the perturbed plant \( P(R) \) of (9) characterized in terms of new arbitrary parameters \( S_1, S_2 \in \mathbb{R}^{r*} \) as

\[ C(Q(S_1, S_2)) = (\hat{V}_n s + S_3 N_n)^{-1} \begin{bmatrix} (u + S_1 M_n) & S_2 \end{bmatrix} \]  

(30)

Setting \( C(Q_1, Q_2) = C(Q(S_1, S_2)) \), we get a following relation between \( (Q_1, Q_2) \) and \( (S_1, S_2) \):

\[ [Q_1, Q_2] = [1 + P(R)]^{-1} [S_1, S_2] \]  

(31)

Note also that \( C(Q(S_1, S_2)) \) of (30) can be written via (19) as

\[ C(Q(S_1, S_2)) = \begin{bmatrix} C_0 + D \ & (\hat{V}_n^{-1} - D N) S_2 \end{bmatrix} \]  

(32)

where

\[ D = \hat{V}_n^{-1} S_1 (I + V_n^{-1} N S_1)^{-1} V_n^{-1} \]  

(33)

This class can be conveniently reorganized as in Fig.7 with \( J_n \) given by (7a), \( T_n \) given by (13), \( J_1 \) and \( T_2 \) given by (21), and \( S = [S_1, S_2] \in \mathbb{R}^{r*} \). The associated six closed-loop transfer functions between the \( u_1 \) and \( e_1 \) of Fig.7 are affine in \( S_1, S_2 \in \mathbb{R}^{r*} \) as follows:

\[ W(P(R), C(Q(S_1, S_2))) = \begin{bmatrix} 1 - P(R) & -C_0 \ 1 & -P(R) \end{bmatrix}^{-1} \begin{bmatrix} M_n & N_n \ 1 & N_n \end{bmatrix} S_1 \begin{bmatrix} \hat{N}_n & M_n \ N_n & \hat{M}_n \end{bmatrix} \begin{bmatrix} M_n & N_n \ \end{bmatrix} S_2 \]  

\[ \in \mathbb{R}^{r*} \]  

(31)

This result presents the same characteristics as the case for the nominal plant. That is, two specifications with respect to responses for reference commands and closed-loop characteristics could be independently satisfied.

5. Conclusions

By using a convenient representation of plants stabilizable by a nominal controller, the class of all stabilizing controllers for perturbed plants are characterized in terms of an arbitrary
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Fig. 7 Class of all stabilizing two-degree-of-freedom
controllers for perturbed plants.

stable proper transfer function. This class is conveniently
reorganized in accordance with the case of the nominal plant.
Moreover, we can obtain a parametrization of all stable
closed-loop transfer functions, which are affine in the free
parameter of the controller. The further problem to be done
is to investigate adaptive control problems.