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Design of an Effective Controller via Disturbance Accommodating Left Eigenstructure Assignment

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Abstract - The transient responses of a linear system having undesired disturbances are dominantly governed by the system's left eigenstructure(eigenvalues/left eigenvectors). In control system design problem for altering the transient response of the system, both the controllability and the disturbance suppressibility should be considered simultaneously to obtain a robust, effective controller. The controllability of the system may be degraded if the left eigenstructure is chosen to suppress the disturbance, or vice versa. In this paper, first, proposed are a modal disturbance suppressibility measure and an improved version of the modal controllability measure suggested by Hamdan and Nayfeh. Second, a simple and general left eigenstructure assignment scheme, considering both the proposed modal disturbance suppressibility measure and the improved version of modal controllability measure, is suggested. When the previous works are applied to assign the left eigenstructure, the achieved eigenvalues as well as the left eigenvectors may differ from the desired ones. But the proposed left eigenstructure assignment scheme makes it possible to achieve the desired closed-loop eigenvalues exactly, provided the desired left eigenvectors reside in the achievable subspace. In case the desired left eigenvectors do not reside in the achievable subspace, the closed-loop eigenvalues are achieved exactly and the left eigenvectors are assigned to the best possible set of eigenvectors in the least square sense. Finally, a numerical example is included to confirm and demonstrate the usefulness of our propositions and to illustrate the proposed design scheme.

I. Introduction

The problem of eigenstructure assignment via linear state feedback control in a linear multivariable system is of vital importance in control theory and applications. The specified effect of the controller is achieved by assigning a certain set of eigenvalues and an associated set of eigenvectors to the closed-loop system. In general terms, the speed of response is determined by the assigned eigenvalues whereas the shape of response is furnished by the assigned eigenvectors[8]. It is well known that, apart from the case of single-input systems, specification of closed-loop eigenvalues does not uniquely determine a closed-loop system. The source of nonuniqueness can be identified as that coming from the freedom offered by state feedback beyond eigenvalue assignment, in selecting the associated eigenvectors from an admissible class[6]. The right eigenstructure(eigenvalues/right eigenvectors) is widely used to solve mode decoupling problems, while transient responses of a linear system having undesired disturbances are dominantly governed by the system's left eigenstructure (eigenvalues/left eigenvectors).

Andry *et al.*[1, 15] used right eigenstructure assignment for flight control system design, using output feedback, to decouple the modes of the L-1011 aircraft. Innocenti and Stanziola[12] analyzed the performance-robustness properties of right eigenstructure assignment against the standard LQR in order to define a loop transfer recovery procedure similar to that of the LQG/LTR, and examined the sensitivity properties of LQR and eigenstructure assignment in their application to the synthesis of a flight control system using state feedback for the lateral dynamics of the CH-47B helicopter in hover. Sobel and Cloutier[28] utilized the right eigenstructure to decouple the modes of the extended medium range air-to-air missile. Besides, the authors in [5-7] also dealt with the problem of right eigenstructure assignment. Zhang *et al.*[33] used the left eigenstructure to suppress undesired inputs, through orthogonalizing left eigenvectors to the disturbance input matrix of the system of uniform flexible beam vibration control problem. Kim and Junkins[17] utilized the left eigenstructure to improve the controllability of a flexible structure system through placing actuators on optimal locations. However, Zhang *et al.* did not take into account the control problems; and Kim and Junkins did not consider the disturbance suppression problems. In control system design problem, both the controllability and the disturbance suppressibility should be considered simultaneously. Otherwise, the controllability of the system could be degraded if the left eigenstructure is chosen only to suppress the disturbance, or vice versa. In the left eigenstructure assignment problem, the number of assignable left eigenvectors satisfying the prescribed design specifications are severely restricted by the ranks of output, input matrices and the imposed conditions if output feedback scheme is used[3, 4, 9, 10, 18, 19]. In case of state feedback, the feedback gain matrix for left eigenstructure assignment has been given in the least square sense, thus the closed-loop eigenvalues as well as the associated eigenvectors may not coincide with the desired eigenvalues and the desired eigenvectors, respectively.

In this paper, a simple and general left eigenstructure assignment scheme, based on the biorthogonality condition between the right and the left modal matrices of a given system, is proposed. The proposed scheme enables designers to consider both the modal disturbance suppressibility measure and the modal controllability measure in designing control systems. The closed-loop eigenvalues are exactly achieved if the system is controllable and the desired left eigenvectors are also achieved exactly if the eigenvectors are achievable. Even in case the desired left eigenvectors do not lie in the achievable subspace, the eigenvalues are exactly achieved provided the system is controllable and the left eigenvectors are assigned to the best possible set of eigenvectors in the least square sense. For the suggested left eigenstructure assignment scheme, a modal disturbance suppressibility measure is proposed, and Hamdan and Nayfeh's[11] measure of modal controllability is improved to reflect the magnitude of each element of a control input matrix B, guaranteeing consistency with their gross measure of controllability.

Consider a linear time invariant multivariable controllable system

$$\dot{x}(t) = Ax(t) + Bu(t) + Ef(t) \tag{1}$$

$$u(t) = Kx(t) \tag{2}$$

where (i) $x \in R^N$, $u \in R^m$, and $f \in R^n$ denote the state, control, and disturbance vectors, respectively; (ii) A, B, E, and K are real constant matrices of appropriate dimensions; and (iii) rank $B = m \neq 0$. The response of the given system due to control input $u(t)$ and disturbance $f(t)$ with zero initial conditions is represented using the modal matrices by[13]

$$x(t) = \Phi \int_0^t e^{\Lambda(t-\tau)} \{\Psi^{\mathrm{T}} Bu(\tau) + \Psi^{\mathrm{T}} Ef(\tau)\} d\tau \tag{3}$$

where Φ and Ψ are the right and the left modal matrices of the given system, respectively. Note, from *Eq.*(3), that the response to disturbance can be eliminated if the columns of left modal matrix Ψ are orthogonal to the columns of the disturbance input matrix E. Note also that the control effort is effectively transferred(that is, the manipulation is achieved with small control effort), if the left eigenvectors are parallel to the columns of the control input matrix B. Therefore, for both effective control and disturbance suppression, it is desired that the left eigenvectors of the system are simultaneously orthogonal to the columns of the disturbance input matrix E and parallel to the columns of the control input matrix B. Then, the corresponding system can be manipulated with small effort without being disturbed by the disturbance input. But there are cases that some columns of control input matrix B are parallel to those of disturbance input matrix E, in which case it is impossible to assign left eigenvectors to be simultaneously parallel to the columns of control input matrix B and orthogonal to those of disturbance input matrix E. In these cases, trade-off should be made between the controllability and the disturbance suppressibility.

The information from the conventional criteria for controllability is a binary(Yes/No) type, it does not say how controllable a system is. A vibratory mode of a beam, for example, is controllable if an actuator is placed near to the nodes although large energy is required to control it. However, if the distance between the actuator and the node is slightly larger, then less energy is required to control the same mode. Thus we can expect that an actuator slightly farther from the node would have better controllability, and this configuration needs less energy. Therefore, we have an obvious need for a quantitative measure of controllability. Hamdan and Nayfeh[11] proposed a modal controllability measure from angles between the left eigenvectors of the system matrix A and the columns of the control input matrix B for the system described by the triple (A,B,C). The results of Hamdan and Nayfeh are extended by Kim and Junkins[17] by introducing a new controllability index that combines Hamdan and Nayfeh's ideas with Skelton's modal cost analysis[27]. However, the measure of modal controllability proposed by Hamdan and Nayfeh does not reflect the magnitude of each element of a control input matrix B. The higher norm of a column of the B indicates the more power injected with the same input, and thus the better controllability. Thus, we improve the measure of modal controllability proposed by Hamdan and Nayfeh to reflect the magnitude of each element of a control input matrix B, guaranteeing consistency with their gross measure of controllability.

Similarly, for a linear system with disturbances, a quantitative measure of modal suppressibility is also required to investigate how suppressible the system's undesired disturbances are. We propose a modal disturbance suppressibility measure by using generalized angles between the left eigenvectors of the system matrix A and the columns of the disturbance input matrix E of the system. The proposed disturbance suppressibility measure coincide with the intuitive result used by Zhang *et al.*[33], where the left eigenvectors were tried to be placed orthogonal to the disturbance input matrix E to minimize the disturbance effect on the system.

A numerical example is included to confirm the usefulness of our propositions and to illustrate the proposed left eigenstructure assignment scheme.

II. Problem of Eigenstructure Assignment with State Feedback

The design of linear multivariable control systems by state-space techniques has been developed considerably during the past decade[8, 16]. One of main techniques to emerge is the (right)eigenstructure assignment technique, which is widely used in flight control, flexible structure control, and so forth. In the multi-input case, the calculated feedback gain matrix K for a given set of desired eigenvalues is not unique. The excess freedom available in state feedback in addition to eigenvalue assignment has been utilized to assign some of the components of the closed-loop eigenvectors.

In 1976, Moore[25] identified the freedom offered by state feedback beyond specification of the closed-loop eigenvalues for the case in which the desired closed-loop eigenvalues are distinct, and derived necessary and sufficient conditions for the existence of gain matrix K which yields prescribed eigenvalues and eigenvectors. In the paper, the eigenstructure assignment was used for response shaping, and an *ad hoc* procedure was suggested for designing a controller. The assumption of controllability was not required in the theorem presented earlier and found in Refs.[24, 25]. However, the theorem still *worked* as long as the uncontrollable eigenvalues were members of the desired closed-loop set of eigenvalues. An example in Ref.[25] showed that even though there is no hope of relocating an uncontrollable eigenvalue, there is some flexibility in altering the eigenvector associated with the uncontrollable eigenvalue. Subsequently, different algorithms have been presented for generating the achievable eigenspace for the closed-loop eigenvector set using state feedback. Klein and Moore[21] extended the results of Ref.[25] to characterize the class of generalized eigenvector chains which can be obtained with a given set of nondistinct eigenvalues.

It is evident that a systematic eigenstructure assignment algorithm(using *best* eigenvectors chosen from the achievable eigenspace) is desired. Such a method will enable application of the eigenstructure assignment to practical systems, resulting in improvement of dynamic response.

Consider *Eq.*(1) in section I. If a constant real state feedback(*Eq.*(2)) is applied to *Eq.*(1), the closed-loop system becomes

$$\dot{x}(t) = (A + BK)x(t) + Ef(t) \tag{4}$$

A set of complex numbers Λ is called *symmetric* if every nonreal element of Λ is accompanied by its conjugate. Let $\Lambda = \{\lambda_1, \cdots, \lambda_s\}$ be a symmetric set of complex numbers and let $\{d_i | i = 1, \cdots, s;\ s \leq N\}$ be a set of positive integers satisfying $\sum_{i=1}^{s} d_i = N$. In Refs.[2, 14], it is shown that if the closed-loop system has s blocks of order $d_1, \cdots, d_s$, in its Jordan canonical form, there are s corresponding generalized right and left eigenvector chains defined by

$$(A + BK - \lambda_i I_N)\phi_{i1} = 0 \tag{5}$$

$$(A + BK - \lambda_i I_N)\phi_{ij} = \phi_{ij-1}, \quad j = 2, \cdots, d_i \tag{6}$$

$$\psi_{id_i}^{\mathrm{T}}(A + BK - \lambda_i I_N) = 0 \tag{7}$$

$$\psi_{ij}^{\mathrm{T}}(A + BK - \lambda_i I_N) = \psi_{ij+1}^{\mathrm{T}}, \quad j = 1, \cdots, d_i - 1 \tag{8}$$

where ϕ_{ij} and ψ_{ij} are the generalized right and left eigenvectors of the given system, respectively. The problem of eigenstructure assignment is then to choose the feedback gain matrix K such that the required conditions for the eigenvalues and eigenvectors are satisfied.

In the following, the superscript $(\cdot)^*$ denotes the conjugate of a given complex vector or scalar $(\cdot)$. A matrix Φ is defined by

$$\Phi = [\Phi_1, \Phi_2, \cdots, \Phi_i, \cdots, \Phi_s]$$

where Φ_i is an $(N \times d_i)$ submatrix of the form

$$\Phi_i = [\phi_{i1}, \phi_{i2}, \cdots, \phi_{id_i}]$$

and matrices Ψ, and W which appears in the next theorem, are defined similarly.

The following theorem gives necessary and sufficient conditions for the existence of K which yields prescribed eigenvalues and eigenvectors.

Theorem 2.1 [21, 22]
There exists a real matrix K such that for $i = 1, \cdots, s$,

$$(A + BK - \lambda_i I_N)\phi_{i1} = 0 \tag{9}$$

$$(A + BK - \lambda_i I_N)\phi_{ij} = \phi_{ij-1}, \quad j = 2, \cdots, d_i \tag{10}$$

if and only if the following conditions are satisfied.

1) The vectors in $\{\phi_{ij} | i = 1, \cdots, s;\ j = 1, \cdots, d_i\}$ are linearly independent in C^N and $\lambda_i = \lambda_k^*$ implies $\phi_{ij} = \phi_{kj}^*(j = 1, \cdots, d_i)$.

2) There exists a set of vectors $\{w_{ij} | i = 1, \cdots, s; j = 1, \cdots, d_i\}$ such that

$$\begin{bmatrix} A - \lambda_i I_N & | & B \end{bmatrix} \begin{bmatrix} \phi_{ij} \\ -- \\ w_{ij} \end{bmatrix} = \phi_{ij-1} \tag{11}$$

where $\phi_{i0} = 0$.

Thus, we can find a desirable feedback gain matrix K based on Theorem 2.1. Necessary and sufficient conditions show that the closed-loop eigenstructure assignment by state feedback is constrained by the requirement that the generalized right eigenvectors lie in certain subspace. The techniques presented in Theorem 2.1 are useful in the controller design using eigenstructure assignment. However, the theorem provides only the right eigenstructure assignment scheme.

If we try to assign the left eigenstructure based on Theorem 2.1, the equations(*Eqs*.(7), (8)) can be rewritten as follows:

$$(\lambda_i I_N - A^{\mathrm{T}} - K^{\mathrm{T}} B^{\mathrm{T}})\psi_{id_i} = 0 \tag{12}$$

$$(\lambda_i I_N - A^{\mathrm{T}} - K^{\mathrm{T}} B^{\mathrm{T}})\psi_{ij} = \psi_{ij+1}. \tag{13}$$

Consider the case with distinct eigenvalues only(that is, $d_i = 1$, for all $i = 1, \cdots, N$ in *Eq*.(12)) for simplicity. Then, the matrix form of the equation(*Eq*.(12)) can be rewritten as

$$\begin{bmatrix} \lambda_i I_N - A^{\mathrm{T}} & | & I_N \end{bmatrix} \begin{bmatrix} \psi_{i1} \\ ----- \\ -K^{\mathrm{T}} B^{\mathrm{T}} \psi_{i1} \end{bmatrix} = 0, \tag{14}$$

or

$$\begin{bmatrix} \lambda_i I_N - A^{\mathrm{T}} & | & B \end{bmatrix} \begin{bmatrix} \psi_{i1} \\ ------ \\ -B^{+} K^{\mathrm{T}} B^{\mathrm{T}} \psi_{i1} \end{bmatrix} = 0, \tag{15}$$

where the superscript $(\cdot)^+$ denotes the pseudoinverse of a given matrix $(\cdot)$. For all cases, the feedback gain matrix K is given in the least square sense. Thus it happens that the achieved closed-loop eigenvalues may not coincide with the desired eigenvalues. Thus, it can be said that the left eigenstructure assignment scheme by state feedback based only on Theorem 2.1 is of little use, because the achieved closed-loop eigenvalues may not coincide with the desired eigenvalues.

On the other hand, if we use output feedback scheme so as to assign the left eigenstructure to satisfy the design specifications, the number of assignable left eigenvectors satisfying the prescribed conditions are severely restricted by the rank of output matrix and the imposed conditions[22, *Theorem* 2]. Therefore, it can be said that the left eigenstructure assignment by output feedback is practically too restricted to be used for controller design problems.

In order to avoid the problems mentioned before, a novel left eigenstructure assignment scheme, using state feedback partially based on Theorem 2.1, is proposed in section IV.

III. Measures of Controllability and Disturbance Suppressibility

The measure of controllability is important because it says how easily the system can be manipulated with small energy. Similarly, the measure of disturbance suppressibility is also important because it says how much the disturbance affect the system performance. In subsection A, the measure of modal controllability suggested by Hamdan and Nayfeh is briefly explained. Then the improved version of modal controllability measure is proposed. In subsection B, a measure of modal disturbance suppressibility is proposed to investigate how suppressible the system's undesired disturbances are. The measures of modal controllability and disturbance suppressibility will be used to assign left eigenstructure of the closed-loop system in section IV.

A. Measures of Modal and Gross Controllability

Klein *et al*.[20], Longman and Alfriend[23], and Viswanathan *et al*. [30-32] have introduced several approaches to measure the degree of controllability of a linear dynamical system, and then developed numerical methods to generate approximate values of their controllability measures for any linear time-invariant system. They introduced a *recovery region* to define the degree of controllability, where the recovery region is defined as the set of initial states that can be returned to the origin in a given amount of time T using bounded controls(or bounded energy, or bounded fuel). The degree of controllability is a scalar measure of the size of the region, where the scalar is chosen as the shortest distance from the origin to an initial state which cannot be returned to the origin in time T. In Refs.[30-32], an approximate computation of the time optimal degree of controllability(based on the minimum distance to the boundary of the parallelepiped that approximates the recovery region) is developed. In Ref.[20], a lower bound is generated by discretizing the system equations, however, the upper bound approximation has been shown in Ref.[32], to be good and easy to generate for modal systems with sufficiently large T. The use of degree of controllability for optimizing actuator locations is studied in Ref.[23].

In this subsection, first, we review Hamdan and Nayfeh's measure of modal controllability which uses the generalized angles between the left eigenvectors of the system matrix A and the columns of the control input matrix B for the system described by the triple (A, B, C). It is shown in Ref.[11] that this measure of modal controllability has interesting connection with Viswanathan's degree of controllability, and it is also related to the singular values of Moore's balancing method[17, 26]. Next, we improve the measure of modal controllability suggested by Hamdan and Nayfeh to reflect the magnitude of each element of a control input matrix B, guaranteeing consistency with their gross measure of controllability.

The PBH eigenvector test specifies that any column vector b_j of a control input matrix B cannot be orthogonal to the i-th left eigenvector ψ_i of A, if the i-th mode of the system is controllable. A problem with the PBH eigenvector test arises when a vector b_j is not orthogonal to an i-th left eigenvector, but is very nearly so. It is obvious that modal controllability of the i-th mode from the j-th input is not zero, but it is nearly so. By considering the angles θ_{ij} between the two vectors as a measure of orthogonality, Hamdan and Nayfeh proposed $\cos\theta_{ij}$ as a measure of modal controllability. In other words, by introducing ideas based upon a geometrical interpretation of the PBH eigenvector test, Hamdan and Nayfeh proposed the following theorem.

Theorem 3.1 [11]
Assume that the right eigenvectors ϕ_i and a corresponding set of left eigenvectors ψ_i are normalized such that $\psi_i^T\phi_j = \delta_{ij}$, where δ_{ij} is the Kronecker delta function, then

1) A measure of controllability of the i-th mode from the j-th actuator input of the given system is $\cos\theta_{ij}$, where θ_{ij} is the angle between b_j and ψ_i, *i.e.*,

$$\cos\theta_{ij} = \frac{|\psi_i^T b_j|}{\| \psi_i \|_2 \| b_j \|_2}. \qquad (16)$$

2) A gross measure of controllability ρ_i of the i-th mode from all inputs is defined as follows:

$$\rho_i = \| g_i \|_2 \qquad (17)$$

where

$$g_i = \frac{\psi_i^T B}{\| \psi_i \|_2}.$$

Note that the magnitudes of each element of B is reflected on the gross measure of controllability. Since, in general, all the inputs are used to control a mode, the second measure of Theorem 3.1 can be taken as each mode's measure of controllability.

However, it seems that Hamdan and Nayfeh's measure of modal controllability for individual inputs is not consistent with the gross measure of controllability. In other words, the gross measure of controllability proposed by Hamdan and Nayfeh often becomes smaller than some modal controllability measures for individual inputs, especially in cases that all the elements of the matrix B are much smaller than 1. Thus, the measure of modal controllability for each individual input in Theorem 3.1 is modified to reflect the magnitude of each element of a control input matrix B, as well as to be consistent with the gross measure of controllability of a mode. The improved measure of modal controllability is stated in the following proposition.

Proposition 3.2 (Improved Measure of Modal Controllability)
A scalar measure of modal controllability μ_{ij} of the i-th mode from the j-th actuator input of the given system is defined as follows:

$$\mu_{ij} = (\cos\theta_{ij}) \| b_j \|_2 \qquad (18)$$

$$= \frac{|\psi_i^T b_j|}{\| \psi_i \|_2}, \quad i = 1, \cdots, N; \; j = 1, \cdots, m \qquad (19)$$

Remark: The improved measure of modal controllability reflects the magnitude of each element of a control input matrix B. When $\| b_j \| = 1$ $(j = 1, \cdots, m)$, as a special case, the improved measure of modal controllability coincides with the Hamdan and Nayfeh's described in Theorem 3.1.

B. Measures of Modal and Gross Disturbance Suppressibility

In this subsection, we propose a modal disturbance suppressibility measure and a gross disturbance suppressibility measure for a linear system having undesired disturbances to deal with the disturbance suppression problems.

Proposition 3.3 (Measure of Modal Disturbance Suppressibility)
A measure of modal disturbance suppressibility ν_{ik} of the i-th mode from the k-th disturbance input of the given system is defined as follows:

$$\nu_{ik} = (\cos\gamma_{ik}) \| e_k \|_2 \qquad (20)$$

$$= \frac{|\psi_i^T e_k|}{\| \psi_i \|_2}, \quad i = 1, \cdots, N; \; k = 1, \cdots, n \qquad (21)$$

where γ_{ik} is the angle between the k-th column vector e_k of the disturbance input matrix E and the i-th left eigenvector ψ_i of the left modal matrix Ψ of the given system.

Remark: The proposition 3.3 assigns a measure of disturbance suppressibility for the i-th mode from the k-th disturbance input, which is proportional to the magnitude of e_k as well as the angle between the subspaces spanned by e_k and ψ_i. When they are orthogonal, the measure of disturbance suppressibility is mapped to zero, this means that the k-th disturbance is completely suppressed and does not appear in the i-th mode. Note that the smaller value of the disturbance suppressibility measure means the better suppression of the disturbance. The proposed measure of modal disturbance suppressibility is a continuous function of the distance between the two spaces.

Let us define the following $(n \times n)$ diagonal matrix V:

$$V = \text{diag}\,[\| e_1 \|_2, \| e_2 \|_2, \cdots, \| e_n \|_2]$$

and let H, the matrix which is composed of disturbance suppressibility measures, be an $(N \times n)$ matrix defined by

$$H = (\cos\Gamma)V \qquad (22)$$

where the matrix $\cos\Gamma$ is composed of each $\cos\gamma_{ik}$. Using these definitions, we state the following propositions.

Proposition 3.4 (Gross Measure of Disturbance Suppressibility)
The Euclidean norm σ_i of the i-th row of H is a gross measure of disturbance suppressibility of the i-th mode from all disturbances.

The gross measure of disturbance suppressibility defined in Proposition 3.4 can be represented by the following proposition.

Proposition 3.5 The scalar gross measure of disturbance suppressibility σ_i of the *i-th* mode from all disturbances can be rewritten as follows:

$$\sigma_i = \| h_i \|_2 \tag{23}$$

where $$h_i = \frac{\psi_i^T E}{\| \psi_i \|_2}, \quad i = 1, \cdots, N.$$

Remark: Each entry of the vector h_i is the component of a column vector of E in the direction of ψ_i. Thus, the measure σ_i represents the gross degree of disturbance suppressibility of the *i-th* mode from all disturbances.

When we compute the measures of controllability and disturbance suppressibility, we should be careful in dealing with the coordinate transformation. Since these measures are invariant only under orthonormal coordinate transformations, the measures should be used consistently only after all transformations, including scaling, have been carried out[13].

IV. Algorithm for Designing an Effective Controller via Disturbance Accommodating Left Eigenstructure Assignment

In this section, we propose a novel left eigenstructure assignment scheme considering the measure of modal disturbance suppressibility as well as the improved measure of modal controllability described in section III.

The purpose of this section is as follows: (*i*) determine desired left modal matrix Ψ^d considering the measure of modal disturbance suppressibility as well as the improved measure of modal controllability, (*ii*) propose a new design procedure which makes it possible to achieve desired eigenvalues exactly, provided the system is controllable, and at the same time, achieve the desired left eigenvectors in the least square sense in case the eigenvectors do not reside in the achievable subspace.

A. Determination of a Desired Left Modal Matrix Ψ^d

In this subsection, we consider the problem of defining a family of a desired left modal matrix Ψ^d that has the specified modal controllability and disturbance suppressibility weightings.

Recall the system equation(*Eq.*(1)) and the relevant assumptions concerning matrices A, and B. Each column vector ψ_{ij}^d of the desired left modal matrix Ψ^d can be generated as follows to reflect the specified modal controllability and disturbance suppressibility weightings.

$$\psi_{ij}^d = \sum_{k=1}^{m} \alpha_k b_k^n + \sum_{l=1}^{\text{rank}(\ker(E))} \beta_l e_l^{\perp}, \quad i = 1, \cdots, s;\ j = 1, \cdots, d_i \tag{24}$$

where $0 \le \alpha_k \le 1$, $0 \le \beta_l \le 1$, $\sum_{k=1}^{m} \alpha_k + \sum_{l=1}^{\text{rank}(\ker(E))} \beta_l = 1$, the weighting factor α_k is the desired controllability weighting corresponding to the *k-th* column of the normalized control input matrix B(*i.e.*, b_k^n), $e_l^{\perp}$ is the *l-th* normalized column of the null space of the disturbance input matrix E, and β_l is the disturbance suppressibility weighting corresponding to $e_l^{\perp}$. Note, for the proposed method of weighting, that the signs of all the column vectors of Ψ^d, B, and ker(E) should be adjusted such that the greatest angle between the column vectors be not greater than 90 degrees. This prevents two or more column vectors from canceling one another when weightings are added. Now, the desired left modal matrix Ψ^d can be written as

$$\Psi^d = [\Psi_1^d,\ \Psi_2^d,\ \cdots,\ \Psi_i^d,\ \cdots,\ \Psi_s^d] \tag{25}$$

where Ψ_i^d is an $(N \times d_i)$ submatrix of the form

$$\Psi_i^d = [\psi_{i1}^d,\ \psi_{i2}^d,\ \cdots,\ \psi_{id_i}^d].$$

If the desired eigenvalues are complex, a slight alteration of the equation(*Eq.*(25)) is required to get better results. Assume that $\lambda_1 = \lambda_2^*$ and all other eigenvalues are real distinct, which implies $\phi_{11} = \phi_{21}^*$, and $\psi_{11} = \psi_{21}^*$. From these relations, the desired left eigenvectors ψ_{11}^d and ψ_{21}^d, corresponding to the desired complex conjugate eigenvalues λ_1 and λ_2, are reformulated to be complex conjugate to each other by multiplying $(1+j)$ and $(1-j)$ for convenience, respectively. The reformulated desired left modal matrix is given as follows:

$$\Psi^d = [\psi_{11}^d(1+j),\ \psi_{21}^d(1-j),\ \psi_{31}^d,\ \cdots,\ \psi_{N1}^d]. \tag{26}$$

B. Best Possible(Achievable) Left Modal Matrix Ψ^a

In this subsection, a new, simple, and general left eigenstructure assignment scheme, avoiding the difficulties in the existing methods, is derived from the biorthogonality condition of the right(Φ) and the left(Ψ) modal matrices of a given system, and the *best possible* left modal matrix Ψ^a is determined in the least square sense in case the desired left modal matrix Ψ^d does not reside in the achievable subspace, guaranteeing the desired eigenvalues to be achieved exactly.

For this, we define an $(N \times (N+m))$ matrix

$$Q_i \equiv [\ A - \lambda_i I_N \ \mid \ B\ ], \tag{27}$$

and a compatibly partitioned matrix

$$N_i \equiv \begin{bmatrix} N_{1i} \\ -- \\ N_{2i} \end{bmatrix}, \quad \text{for } i = 1, \cdots, s \tag{28}$$

where the columns of an $((N+m) \times m)$ matrix N_i form a basis for the null space of Q_i. Then, according to Theorem 2.1, the achievable right eigenvector ϕ_{ij}^a should be lie in the span of $\{N_{1i}\}$ for $j = 1, \cdots, d_i$. For rank $B = m$, it can be shown that the columns of N_{1i} are linearly independent[1]. An achievable generalized right modal matrix Φ^a can be obtained as follows:

$$\Phi^a = [\Phi_1^a,\ \Phi_2^a,\ \cdots,\ \Phi_i^a,\ \cdots,\ \Phi_s^a] \tag{29}$$

where Φ_i^a is an $(N \times d_i)$ submatrix of the form

$$\Phi_i^a = [\phi_{i1}^a,\ \phi_{i2}^a,\ \cdots,\ \phi_{id_i}^a]$$

and ϕ_{ij}^a is given as a linear combination of the columns of N_{1i}, that is,

$$\phi_{ij}^a = N_{1i} p_{ij}, \quad j = 1, \cdots, d_i. \tag{30}$$

In *Eq.*(30), the $(m \times 1)$ coefficient vector p_{ij} is chosen to minimize the performance index defined as follows:

$$J = \| (\Psi^d)^T \Phi_{aug}^a P - I_N \| \tag{31}$$

where the $(mN \times N)$ coefficient matrix P is formed as follows:

$$P = \text{block diag}\ [P_1,\ P_2,\ \cdots,\ P_i,\ \cdots,\ P_s]$$

with $$P_i = \text{block diag}\ [p_{i1},\ p_{i2},\ \cdots,\ p_{id_i}]$$

The $(N \times N)$ matrix Ψ^d is determined as in the previous subsection to reflect the specified modal controllability and disturbance suppressibility weightings, and the $(N \times mN)$ augmented achievable right modal matrix Φ^a_{aug} is formed as follows:

$$\Phi^a_{aug} = [N_{11},\ N_{12},\ \cdots\ ,N_{1i},\ \cdots\ ,N_{1s}], \quad (32)$$

The coefficient vector p_{ij} minimizing the performance index J is given by the following equation.

$$p_{ij} = (\Omega_{ij})^{+} n_k \quad (33)$$

where the $(N \times m)$ submatrix $\Omega_{ij}(i = 1,\ \cdots\ ,s;\ j = 1,\ \cdots\ ,d_i)$ is a component of the following matrix $\{(\Psi^d)^{\mathrm{T}}\Phi^a_{aug}\}$ of dimension $(N \times mN)$

$$\{(\Psi^d)^{\mathrm{T}}\Phi^a_{aug}\} = [\Omega_{11}, \Omega_{12}, \cdots, \Omega_{1d_1},\ \Omega_{21}, \Omega_{22}, \cdots,$$

$$\Omega_{2d_2}, \cdots, \Omega_{s1}, \Omega_{s2}\ \cdots, \Omega_{sd_s}], \quad (34)$$

and the vector n_k is the k-th column of an $(N \times N)$ identity matrix, and corresponds to the k-th submatrix of $\{(\Psi^d)^{\mathrm{T}}\Phi^a_{aug}\}$. If the case with distinct eigenvalues only is considered, *Eqs.*(32), (33), and (34) can be rewritten as follows, respectively:

$$\Phi^a_{aug} = [N_{11},\ N_{12},\ \cdots\ ,N_{1i},\ \cdots\ ,N_{1N}] \quad (35)$$

$$p_{i1} = (\Omega_{i1})^{+} n_i, \quad (36)$$

$$\{(\Psi^d)^{\mathrm{T}}\Phi^a_{aug}\} = [\Omega_{11},\ \Omega_{21},\ \cdots\ ,\Omega_{i1},\ \cdots\ ,\Omega_{N1}]. \quad (37)$$

Then, the achieved right modal matrix Φ^a is given by the following equation.

$$\Phi^a = \Phi^a_{aug} P. \quad (38)$$

Hence, the achieved generalized left modal matrix Ψ^a, satisfying the design specifications in the least square sense(in case the desired left modal matrix Ψ^d does not reside in the achievable subspace), is given, via biorthogonality condition of the right and the left modal matrices, by

$$\Psi^a = (\Phi^a)^{-\mathrm{T}}. \quad (39)$$

The normalized achieved left modal matrix(Ψ^a_{nor}) is used for calculations of the modal controllability, the disturbance suppressibility, and their gross measures described in section III.

Recall that the left eigenstructure assignment scheme by state feedback based on Theorem 2.1 can not be directly used to get desired left eigenstructure, and the assignable left eigenstructure is severely restricted if output feedback scheme is used. However, these difficulties are removed in our work, and the achieved left modal matrix satisfying the prescribed requirements is obtained easily in the least square sense when Ψ^d does not reside in the achievable subspace, assigning exact eigenvalues if the given system is controllable.

C. Algorithm

A design procedure based on the proposed ideas described in subsections A, and B, considering both the improved modal controllability and the disturbance suppressibility measures, is given in this subsection. The procedure for finding a desired feedback gain matrix K satisfying the design objectives is partially based on Theorem 2.1.

Step 1: Determine the desired eigenvalues and corresponding desired left eigenvectors to take account of the required modal controllability and disturbance suppressibility weightings such that

$$\psi^d_{ij} = \sum_{k=1}^{m} \alpha_k b^n_k\ +\ \sum_{l=1}^{\mathrm{rank}(\ker(E))} \beta_l e^{\perp}_l,\ \ i = 1, \cdots, s;\ j = 1, \cdots, d_i \quad (40)$$

where $0 \le \alpha_k \le 1$, $0 \le \beta_l \le 1$, $\sum_{k=1}^{m} \alpha_k + \sum_{l=1}^{\mathrm{rank}(\ker(E))} \beta_l = 1$. Note that in this step, the signs of all the column vectors of Ψ^d, B, and $\ker(E)$ should be adjusted such that the greatest angle between any two column vectors be not greater than 90 degrees. If some of the desired eigenvalues are complex conjugate pairs, a slight alteration is required according to the guideline described in subsection A of this section.

Step 2: Find maximal rank matrices N_i, and S_i such that

$$N_i = \begin{bmatrix} N_{1i} \\ -- \\ N_{2i} \end{bmatrix},\quad S_i = \begin{bmatrix} S_{1i} \\ -- \\ S_{2i} \end{bmatrix}$$

for $i = 1,\ \cdots\ ,s$ satisfying the following relation:

$$[\ A - \lambda_i I_N\ \mid\ B\ ]\,[\ S_i\ \mid\ N_i\ ] = [\ I_N\ \mid\ 0\ ] \quad (41)$$

where $N_i \in C^{(N+m)\times m}$, and $S_i \in C^{(N+m)\times N}$.

Step 3: Constitute the augmented achievable generalized right modal matrix Φ^a_{aug} given by *Eq.*(32).

Step 4: Select the coefficient vectors $p_{ij}(i = 1, \cdots, s;\ j = 1, \cdots, d_i)$ by *Eq.*(33), minimizing the performance index J in *Eq.*(31), using the determined desired left modal matrix Ψ^d given in Step 1 and the augmented achievable generalized right modal matrix Φ^a_{aug} given in Step 3.

Step 5: Form the achieved generalized right eigenvectors for $i = 1,\ \cdots\ ,s$ as follows:

$$\phi^a_{ij} = S_{1i}\phi^a_{ij-1} + N_{1i}p_{ij}, \qquad j = 1,\ \cdots\ ,d_i \quad (42)$$

where $\phi^a_{i0} = 0$, and construct the achieved generalized right modal matrix Φ^a.

Step 6: Calculate vector chains and construct the matrix W as follows:

$$w_{ij} = S_{2i}\phi^a_{ij-1} + N_{2i}p_{ij}, i = 1, \cdots, s; j = 1, \cdots, d_i \quad (43)$$

$$W = [w_{11},\ w_{12},\ \cdots\ ,\ w_{ij},\ \cdots\ ,\ w_{sd_s}]. \quad (44)$$

Step 7: Calculate the state feedback gain matrix

$$K = W(\Phi^a)^{-1}. \quad (45)$$

Step 8: Calculate the achieved generalized left modal matrix Ψ^a using *Eq.*(39), and normalize the matrix Ψ^a for convenience.

Step 9: Calculate the modal controllability, disturbance suppressibility, and their gross measures for the closed-loop system $(A + BK)$ using *Eqs.*(17), (19), (21), and (23) with the normalized achieved left modal matrix Ψ^a_{nor} given in Step 8.

V. A Numerical Example

In this example, the case that the number of unknown elements(N) of each left eigenvector is greater than the rank(m) of a control input matrix B is considered. Thus, the desired left eigenvectors will be assigned in the least square sense, assigning exact desired eigenvalues.

Consider a third-order two-input continuous controllable system with a disturbance,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ef(t) \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} f(t). \end{aligned}$$

The open-loop spectrum of A is given by $\Lambda^{open} = \{-1,\ 1,\ 2\}$. Let a set of the desired real distinct eigenvalues be $\Lambda^d = \{\lambda_1,\ \lambda_2,$

$\lambda_3\}=\{-1,\ -2,\ -3\}$ with $d_1 = 1$, $d_2 = 1$, and $d_3 = 1$. From the given system, N, m, and n are 3, 2, and 1, respectively.

Now, we determine each left eigenvector, to take account of the prescribed modal controllability and disturbance suppressibility weightings, as follows:

$$\begin{aligned}
\psi^d_{11} &= \alpha_1 b^n_1 + \alpha_2 b^n_2 + \beta_1 e^{\perp}_1 + \beta_2 e^{\perp}_2, \\
\psi^d_{21} &= \alpha'_1 b^n_1 + \alpha'_2 b^n_2 + \beta'_1 e^{\perp}_1 + \beta'_2 e^{\perp}_2, \\
\psi^d_{31} &= \alpha''_1 b^n_1 + \alpha''_2 b^n_2 + \beta''_1 e^{\perp}_1 + \beta''_2 e^{\perp}_2
\end{aligned}$$

where $b^n_1 = [0\ \ 0\ \ 1]^T$, $b^n_2 = [0\ \ 1\ \ 0]^T$, $e^{\perp}_1 = [0\ \ 1\ \ 0]^T$, and $e^{\perp}_2 = [1\ \ 0\ \ 0]^T$. Consider the following two cases.

Case 1: ($\alpha_1 = 0.6, \alpha_2 = 0.1, \beta_1 = 0.1, \beta_2 = 0.2$), ($\alpha'_1 = 0.7$, $\alpha'_2 = 0.1$, $\beta'_1 = 0.1$, $\beta'_2 = 0.1$), and ($\alpha''_1 = 0.65$, $\alpha''_2 = 0.1$, $\beta''_1 = 0.1$, $\beta''_2 = 0.15$).

In this case, the desired left modal matrix Ψ^d, and its normalized matrix Ψ^d_{nor} are given as follows:

$$\Psi^d = \begin{bmatrix} 0.2 & 0.1 & 0.15 \\ 0.2 & 0.2 & 0.2 \\ 0.6 & 0.7 & 0.65 \end{bmatrix},$$

$$\Psi^d_{nor} = \begin{bmatrix} 0.3015 & 0.1361 & 0.2154 \\ 0.3015 & 0.2722 & 0.2872 \\ 0.9045 & 0.9526 & 0.9333 \end{bmatrix}.$$

The coefficient vectors p_{ij} minimizing the performance index are given by

$$\begin{aligned}
p_{11} &= \left[\ -28.3399 \quad -21.4055\ \right]^T, \\
p_{21} &= \left[\ 40.1776 \quad 66.0873\ \right]^T, \\
p_{31} &= \left[\ -0.0261 \quad -14.4337\ \right]^T.
\end{aligned}$$

The normalized achieved right(Φ^a_{nor}) and left(Ψ^a_{nor}) modal matrices are given in the least square sense as follows:

$$\Phi^a_{nor} = \begin{bmatrix} 0.7045 & -0.4418 & 0.3015 \\ -0.7045 & 0.8836 & -0.9045 \\ 0.0859 & -0.1552 & 0.3015 \end{bmatrix},$$

$$\Psi^a_{nor} = \begin{bmatrix} 0.6216 & 0.2788 & 0.1041 \\ 0.4263 & 0.3860 & 0.2224 \\ 0.6572 & 0.8793 & 0.9694 \end{bmatrix}.$$

The matrix W in *Eq.*(44), and the state feedback gain matrix K are given by

$$W = \begin{bmatrix} 30 & -40 & 0 \\ 10 & -56.25 & 13.3333 \end{bmatrix},$$

$$K = \begin{bmatrix} 2.0189 & -1.3585 & -6.0943 \\ -1.4874 & -1.9057 & 3.7704 \end{bmatrix}.$$

The modal controllability measure μ_{ij}, disturbance suppressibility measure ν_{ik}, gross controllability measure ρ_i, and gross disturbance suppressibility measure σ_i are calculated and summarized in Table 1. The calculated measures of the modal controllability show that Input 1 is more effective than Input 2 for controlling all the modes of the given system. The table also shows that the first mode has better disturbance suppressibility than the other two modes.

Case 2: ($\alpha_1 = 0.2, \alpha_2 = 0.1, \beta_1 = 0.1, \beta_2 = 0.6$), ($\alpha'_1 = 0.1$, $\alpha'_2 = 0.1$, $\beta'_1 = 0.1$, $\beta'_2 = 0.7$), and ($\alpha''_1 = 0.15$, $\alpha''_2 = 0.1$, $\beta''_1 = 0.1$, $\beta''_2 = 0.65$).

The modal controllability weightings of this case are assigned to be smaller than those of the Case 1, consequently the modal disturbance suppressibility weightings are accounted more than in the Case 1. In this case, the normalized desired left modal matrix Ψ^d_{nor} is given by

$$\Psi^d_{nor} = \begin{bmatrix} 0.9045 & 0.9526 & 0.9333 \\ 0.3015 & 0.2722 & 0.2872 \\ 0.3015 & 0.1361 & 0.2154 \end{bmatrix}.$$

The modal controllability measure μ_{ij}, disturbance suppressibility measure ν_{ik}, gross controllability measure ρ_i, and gross disturbance suppressibility measure σ_i are calculated and summarized in Table 1. The calculated measures of the modal controllability show that Input 1 is more effective than Input 2 for controlling all the modes of the given system. The table also shows that the first mode has better disturbance suppressibility than the other two modes.

Table 1: Calculated Measures for the two cases

Cases	λ_i	μ_{ij}		ρ_i	ν_{ik}	σ_i
		Input 1	Input 2	In.1 & In.2	Dist. 1	Dist. 1
Case 1	λ_1	0.6572	0.4263	0.7833	0.6572	0.6572
	λ_2	0.8793	0.3860	0.9603	0.8793	0.8793
	λ_3	0.9694	0.2224	0.9946	0.9694	0.9694
Case 2	λ_1	0.1655	0.3611	0.3972	0.1655	0.1655
	λ_2	0.1378	0.3542	0.3801	0.1378	0.1378
	λ_3	0.1288	0.3815	0.4027	0.1288	0.1288

The normalized achieved right(Φ^a_{nor}) and left(Ψ^a_{nor}) modal matrices are given in the least square sense because the coefficient vectors p_{ij} are obtained in the least square sense, but the closed-loop eigenvalues are guaranteed to be achieved exactly even in this case. The matrices Φ^a_{nor} and Ψ^a_{nor} are given as follows:

$$\Phi^a_{nor} = \begin{bmatrix} -0.2284 & 0.3954 & 0.3015 \\ 0.2284 & -0.7908 & -0.9045 \\ 0.9464 & -0.4673 & 0.3015 \end{bmatrix},$$

$$\Psi^a_{nor} = \begin{bmatrix} 0.9177 & 0.9250 & -0.9153 \\ 0.3611 & 0.3542 & -0.3815 \\ 0.1655 & 0.1378 & -0.1288 \end{bmatrix}.$$

The calculated modal controllability measure μ_{ij}, disturbance suppressibility measure ν_{ik}, gross controllability measure ρ_i, and gross disturbance suppressibility measure σ_i in this case are also summarized in Table 1. In the table, the gross controllability measures of each mode for the Case 1 are greater than those of the Case 2, while the disturbance suppressibility measures of each mode for the Case 2 are smaller than those of the Case 1. These mean that the Case 1 has better controllability than the Case 2, on the other hand, the Case 2 has better disturbance suppressibility than the Case 1, which is consistent with the weightings given.

For the two cases in the example, the desired left eigenvectors were assigned in the least square sense, since the number of unknown elements(3) of each left eigenvector is greater than the rank(2) of B. However, the desired closed-loop eigenvalues were assigned exactly. The results of the two cases are also consistent with the assigned controllability and disturbance suppressibility weightings.

VI. Concluding Remarks

In this paper, the modal disturbance suppressibility measure and gross disturbance suppressibility measure of a given mode in all disturbances have been proposed. Also, Hamdan and Nayfeh's measure of modal controllability has been improved to be consistent with the gross measure of controllability, and to reflect the magnitude of each element of a control input matrix B. Finally, a simple and general left eigenstructure assignment scheme, considering modal disturbance suppressibility as well as the improved modal controllability, has been proposed. The proposed left eigenstructure assignment scheme makes it possible to achieve the desired closed-loop left eigenstructure exactly, provided the desired left eigenvectors reside in the achievable subspace. In case the desired left eigenvectors do not reside in the achievable subspace, the closed-loop eigenvalues are achieved exactly and the left eigenvectors are assigned to the best possible set of eigenvectors in the least square sense. A numerical example has confirmed the usefulness of our propositions and the proposed left eigenstructure assignment scheme.

References

[1] A. N. Andry, Jr, E. Y. Shapiro, and J. C. Chung, "Eigenstructure Assignment for Linear Systems," *IEEE Transactions on Aerospace and Electronic Systems*, vol. AES-19, no. 5, pp. 711-729, September 1983.

[2] C. T. Chen, *Linear System Theory and Design*. New York: Holt, Rinehart and Winston, 1984.

[3] J. W. Choi, J. G. Lee, Y. Kim, and T. Kang, "Eigenstructure assignment considering both modal controllability and suppressibility measures," *Proceedings of the 32nd SICE Annual Conference*, Kanazawa, Japan, pp. 1445-1451, August 1993.

[4] E. J. Davison, and R. Chatterjee, "A note on pole assignment in linear systems with incomplete state feedback," *IEEE Transactions on Automatic Control*, pp. 98-99, February 1971.

[5] G. Duan, "Solutions of the equation $AV + BW = VF$ and their application to eigenstructure assignment in linear systems," *IEEE Transactions on Automatic Control*, vol. AC-38, no. 2, pp. 276-280, February 1993.

[6] M. M. Fahmy, and J. O'reilly, "On eigenstructure assignment in linear multivariable systems," *IEEE Transactions on Automatic Control*, vol. AC-27, no. 3, pp. 690-693, June 1982.

[7] M. M. Fahmy, and J. O'reilly, "Eigenstructure assignment in linear multivariable systems - a parametric solution," *IEEE Transactions on Automatic Control*, vol. AC-28, no. 10, pp. 990-994, October 1983.

[8] M. M. Fahmy, and H. S. Tantawy, "Eigenstructure assignment via linear state-feedback control," *International Journal of Control*, vol. 40, no. 1, pp. 161-178, 1984.

[9] L. R. Fletcher, "An intermediate algorithm for pole placement by output feedback in linear multivariable control systems," *International Journal of Control*, vol. 31, no. 6, pp. 1121-1136, 1980.

[10] L. R. Fletcher, J. Kautsky, G. K. G. Kolka, and N. K. Nichols, "Some necessary and sufficient conditions for eigenstructure assignment," *International Journal of Control*, vol. 42, no. 6, pp. 1457-1468, 1985.

[11] A. M. A. Hamdan, and A. H. Nayfeh, "Measures of Modal Controllability and Observability for First- and Second-Order Linear Systems," *Journal of Guidance, Control, and Dynamics*, vol. 12, no. 3, pp. 421-428, May-June 1989.

[12] M. Innocenti, and C. Stanziola, "Performance - robustness trade off of eigenstructure assignment applied to rotorcraft," *Aeronautical Journal*, vol. 94, no. 934, pp. 124-131, 1990.

[13] J. L. Junkins, and Y. Kim, *Introduction to Dynamics and Control of Flexible Structures*. AIAA Education Series, Washington D.C.: American Institute of Aeronautics and Astronautics, Inc., 1993.

[14] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1980.

[15] T. Kang, and J. G. Lee, "Comment on "Eigenstructure Assignment for Linear Systems," " *IEEE Transactions on Acrospace and Electronic Systems*, vol. AES-28, no. 3, pp. 920-921, July 1992.

[16] M. A. Khan, and N. Sreenivasulu, "A systematic method for eigenstructure assignment and application to response shaping," *International Journal of Control*, vol. 43, no. 2, pp. 717-735, 1986.

[17] Y. Kim, and J. L. Junkins, "Measure of Controllability for Actuator Placement," *Journal of Guidance, Control, and Dynamics*, vol. 14, no. 5, pp. 895-902, September-October 1991.

[18] H. Kimura, "Pole assignment by gain output feedback," *IEEE Transactions on Automatic Control*, vol. AC-20, no. 8, pp. 509-516, 1975.

[19] H. Kimura, "A further result on the problem of pole assignment by output feedback," *IEEE Transactions on Automatic Control*, vol. AC-22, no. 6, pp. 458-463, 1977.

[20] G. Klein, R. E. Lindberg, and R. W. Longman, "Computation of a degree of controllability via system discretization," *Journal of Guidance, Control, and Dynamics*, vol. 5, no. 6, pp. 583-588, 1982.

[21] G. Klein, and B. C. Moore, "Eigenvalue-generalized eigenvector assignment with state feedback," *IEEE Transactions on Automatic Control*, vol. AC-22, no. 2, pp. 140-141, 1977.

[22] B. H. Kwon, and M. J. Youn, "Eigenvalue-Generalized Eigenvector Assignment by Output Feedback," *IEEE Transactions on Automatic Control*, vol. AC-32, no. 5, pp. 417-421, May 1987.

[23] R. W. Longman, and K. T. Alfriend, "Actuator placement from degree of controllability criteria for regular slewing of flexible spacecraft," *Acta Astronautica*, vol. 8, no. 7, pp. 703-718, 1981.

[24] J. Lu, H. Chiang, and J. S. Thorp, "Partial eigenstructure assignment and its application to large scale systems," *IEEE Transactions on Automatic Control*, vol. AC-36, no. 3, pp. 340-347, March 1991.

[25] B. C. Moore, "On the flexibility offered by state feedback in multivariable systems beyond closed loop eigenvalue assignment," *IEEE Transactions on Automatic Control*, pp. 659-692, October 1976.

[26] B. C. Moore, "Principal component analysis in linear systems: controllability, observability, and model reduction," *IEEE Transactions on Automatic Control*, vol. AC-26, no. 1, pp. 17-32, 1981.

[27] R. E. Skelton, *Dynamic System Control - Linear System Analysis and Synthesis*. New York: Wiley, 1988.

[28] K. Sobel and J. R. Cloutier, "Eigenstructure Assignment for the Extended Medium Range Air-to-Air Missile," *Journal of Guidance, Control, and Dynamics*, vol. 15, no. 2, pp. 529-531, March-April 1992.

[29] S. Srinathkumar, "Eigenvalue/Eigenvector Assignment Using Output Feedback," *IEEE Transactions on Automatic Control*, vol. AC-23, no. 1, pp. 79-81, February 1978.

[30] C. N. Viswanathan, R. W. Longman, and P. W. Likins, "A definition of the degree of controllability - a criterion for actuator placement," *Proceedings of the Second VPI&SU AIAA Symposium on Dynamics and Control of Large Flexible Spacecraft*, Blacksburg, VA., 1979.

[31] C. N. Viswanathan, and R. W. Longman, "The determination of the degree of controllability for dynamic systems with repeated eigenvalues," *Advances in the Astronautical Sciences*, vol. 50, 1983.

[32] C. N. Viswanathan, R. W. Longman, and P. W. Likins, "A degree of controllability definition: fundamental concepts and application to modal systems," *Journal of Guidance, Control, and Dynamics*, vol. 7, no. 2, pp. 222-230, 1984.

[33] Q. Zhang, G. L. Slater, and R. J. Allemang, "Suppression of Undesired Inputs of Linear Systems by Eigenspace Assignment," *Journal of Guidance, Control, and Dynamics*, vol. 13, no. 3, pp. 330-336, March-April 1990.