Pursuit-Evasion as a Dynamic Game

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Abstract

A study about two-dimensional pursuit-evasion dynamic games is presented and discussed. A pursuer tries to intercept an evader by a strategy based on proportional navigation guidance, while the evader tries to maximize a miss distance by the optimal control. The study is applied to a ball game and an air-combat game. The results show the same features exist in both games, therefore the study will be able to apply for general two dimensional dynamic games. In the ball game, the study is extended to cases where a goal exists, while in the air-combat game, some three-dimensional problems are solved and the results are also shown.

Introduction

A two-player game is the problem where there is a common performance index and one player strives to maximize it, while the other, to minimize it. If the whole process is dynamic, and expressed by a set of ordinary differential equations, we call the problem, as a "differential game". The kind of problems have attracted considerable interest in recent years and many studies have appeared in the literature. However, their results still seem to be difficult to apply to actual dynamic games. That is, most studies have been devoted to obtaining precise mini-max solutions for very simplified problems.

The early studies of Isaacs[1] and Merz[2] showed the existence of a large number of different kinds of solutions for a very simple problem called the "homicidal chauffeur". Actually, an enormous number of solutions may exist for multidimensional nonlinear differential games to know all the different kinds of solutions. It may be more helpful to us to find some features of the solutions under some assumptions, even if they are not precisely solved by a differential game approach.

This paper studies the pursuit-evasion problem between two players, which may be ball game players or a missile and an aircraft in an air-combat game. The pursuer is assumed to employ PNG(proportional navigation guidance) which is obtained as the optimal control against a nonmaneuvering target. The evader tries to maximize miss distance, however in a ball game where a goal exists, it has to reach the goal at the terminal time. The optimal controls of the evader are its lateral and longitudinal acceleration commands. The problems are reduced to nonlinear two-point boundary value problems, and are solved by the steepest ascent method.[3~4]

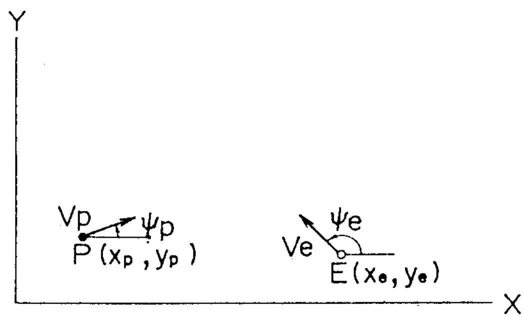

Fig.1 Pursuer and evader symbols

Mathematical Model

Figure 1 shows some symbols of two players in a plane. The equations of motion are given by

$$\dot{v}_e = a_{ve} \quad (1)$$
$$\dot{x}_e = v_e \cos\psi_e \quad (2)$$
$$\dot{y}_e = v_e \sin\psi_e \quad (3)$$
$$\dot{\psi}_e = a_e / v_e \quad (4)$$
$$\dot{x}_p = v_p \cos\psi_p \quad (5)$$
$$\dot{y}_p = v_p \sin\psi_p \quad (6)$$
$$\dot{\psi}_p = a_p / v_p \quad (7)$$
$$\dot{a}_p = (a_{pc} - a_p)/\tau_p \quad (8)$$

where subscripts "p" and "e" shows those of pursuer P and evader E, respectively. x and y are cartesian coordinates, v is velocity, ψ is azimuth of the velocity vector, a is lateral acceleration, a_{ve} is longitudinal acceleration of the evader. The pursuer lateral acceleration is approximated by a first

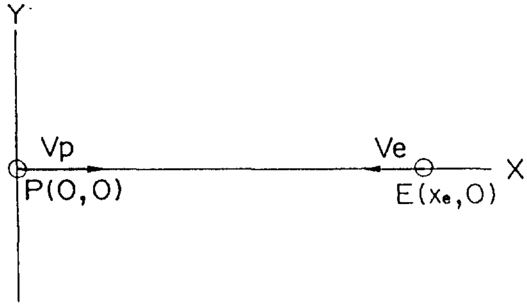

Fig.2 Initial condition of pursuer and evader

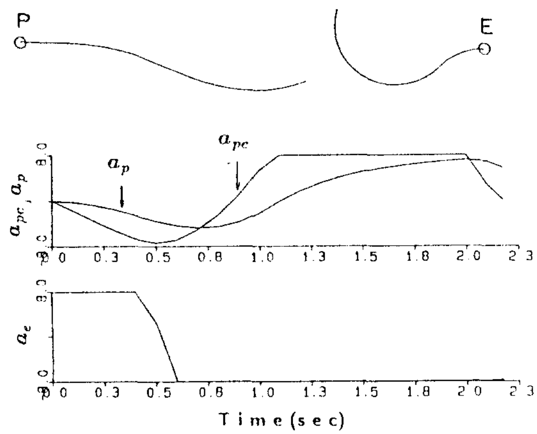

Fig.3 Optimal solution No.1 (ball game, without velocity control)

order lag τ_p to a lateral acceleration command a_{pc}, which is given by

$$a_{pc} = \begin{cases} N_p\, v_c\, \dot{\sigma} & for \quad |N_p\, v_c\, \dot{\sigma}| \leq a_{pcmax} \\ a_{pcmax}\ Sign(\dot{\sigma}) & for \quad |N_p\, v_c\, \dot{\sigma}| > a_{pcmax} \end{cases} \tag{9}$$

where the signum function $Sign$ takes the value ± 1 according to the sign of $\dot{\sigma}$.

In Eq.(9), N_p is the effective navigation ratio, v_c the closing velocity, and $\dot{\sigma}$ the line-of-sight turning rate given by

$$v_c = -(r_x \dot{r}_x + r_y \dot{r}_y)/r \tag{10}$$

$$\dot{\sigma} = (\dot{r}_y r_x - r_y \dot{r}_x)/r^2 \tag{11}$$

where r is relative range, r_x and r_y are its cartesian components,

$$r_x = x_e - x_p, \qquad r_y = y_e - y_p,$$
$$r = (r_x^2 + r_y^2)^{\frac{1}{2}} \tag{12}$$

The performance index J is r_f, which is the value of r at the terminal time t_f,

$$J \equiv r_f = r(t_f) \tag{13}$$

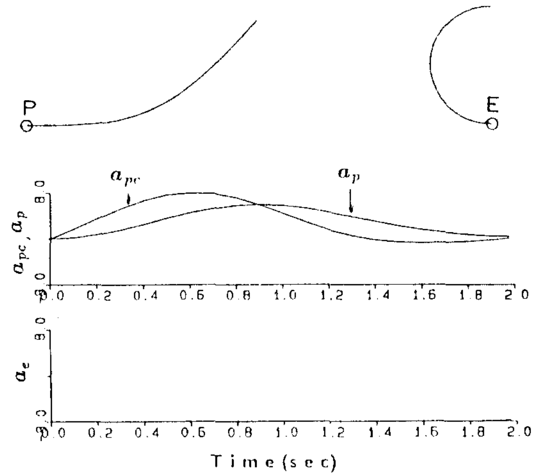

Fig.4 Optimal solution No.2 (ball game, without velocity control)

where t_f is the closest point of approach time determined from the next stopping condition Ω.

$$\Omega \equiv d/dt(r^2) = 0 \tag{14}$$

The J in Eq.(13) is maximized by the steepest ascent method, in relation to two control parameters, *viz.* the evader's longitudinal and lateral accelerations a_{ve} and a_e under the following constraints.

$$|a_{ve}| \leq a_{vemax} \tag{15}$$

$$|a_e| \leq a_{emax} \tag{16}$$

$$v_{emin} \leq v_e \leq v_{emax} \tag{17}$$

Ball Game Application (Without a Goal, Without Velocity Control)

In order to show a feature of optimal evasion, simple cases are shown where the evader velocity is constant, and without a goal. Figure 2 shows the initial condition of P and E, where nominal values $v_p = 7m/s$, $v_e = 5m/s$, $a_{pcmax} = a_{ecmax} = 8m/s^2$, $N_p = 4$, $\tau_p = 0.4s$ are selected. In the case initial distance $x_e = 24m$, we obtain two kinds of local optimal solutions shown in Figs 3 and 4. In the figures, the pursuer and evader trajectories, histories of a_{pc}, a_p and a_e are shown. In Fig.3, E takes first positive maximum a_e, then after $0.5s$, reverses its sign. The early behavior is considered as a kind of feint action. In Fig.4, E takes negative maximum a_e throughout the game. The r_f values resulted in Figs 3 and 4 are $4.13m$ and $12.16m$, respectively. Both are local optimal (neighboring extremal) solutions, however, the latter is the globally optimal solution. There are solutions symmetrical to Figs 3 and 4 in the sign of a_e, therefore four local optimal solutions exist in this case.

Figure 5 shows the case where initial x_e is $46m$. E

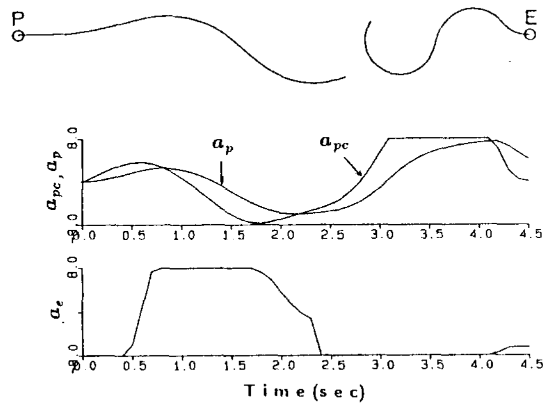

Fig.5 Optimal solution with a large initial distance (ball game, without velocity control)

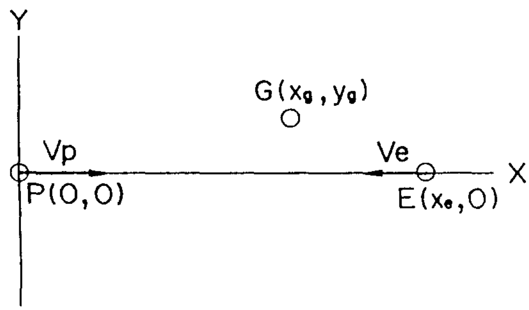

Fig.6 A ball game with a goal

takes negative maximum a_e first, then at $0.5s$, takes the positive maximum a_e, and at about $2s$, again changes its sign to the negative maximum. The first behavior is considered as the action to obtain an "optimal initial geometry" preceding to the following action which is the same kind as that of Fig.3,

Ball Game Application (With a Goal, Without Velocity Control)

Next we treat the case shown in Fig.6 where a goal G to the evader exist, and E tries to maximize the distance r between P at the time t_f when E reaches to G. Mathematically the problem is defind by substituting next stopping condition (18), instead of (14).

$$\Omega \equiv (x_g - x_e)^2 + (y_g - y_e)^2 - \varepsilon = 0 \qquad (18)$$

where ε is a small enough value. Figures 7 and 8 show the two kinds of local optimal solutions where G is located at $(10.5m, 1m)$, that is, $1m$ upwards ($+y$ direction) from the x axis. The corresponding r_f values are $4.61m$ and $2.84m$ respectivery, therefore the former is globally optimal. That is, when the goal is located at rightward to E, E should

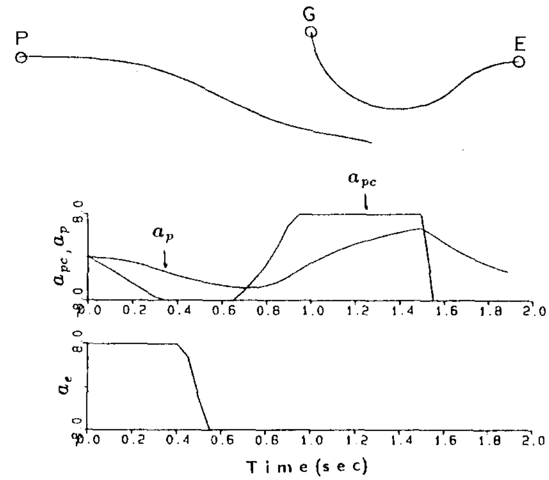

Fig.7 Optimal solution No.1 with a goal

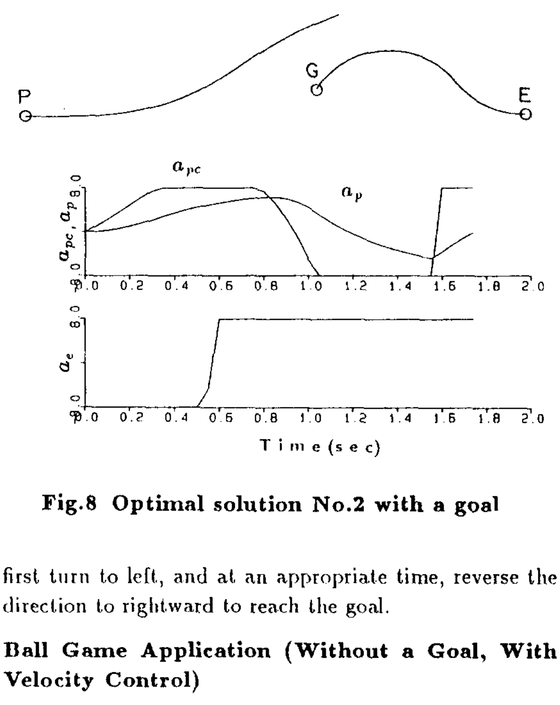

Fig.8 Optimal solution No.2 with a goal

first turn to left, and at an appropriate time, reverse the direction to rightward to reach the goal.

Ball Game Application (Without a Goal, With Velocity Control)

When P's velocity is low, E tries to avoid P with a high speed, while P's velocity is high and its turn radius is large, E tries to reduce the velocity and avoids P with a small turn radius. These are basic tactics in a pursuit-evasion game, therefore the velocity control plays an important roll in the game. Figures 9 and 10 show the optimal solutions with velocity control, which are corresponding to without velocity control cases of Figs 3 and 4. Constraint parameters for the evaders velocity are selected as $a_{vemax} = 8m/s^2$, $v_{emin} = 2m/s$, and $v_{emax} = 8m/s$. Other parameters are same as that of without velocity control cases. In Fig.9, E reduces its velocity and turns right

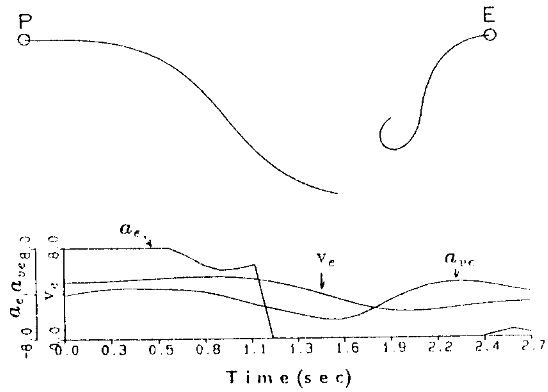

Fig.9 Optimal solution No.1 (ball game, with velocity control)

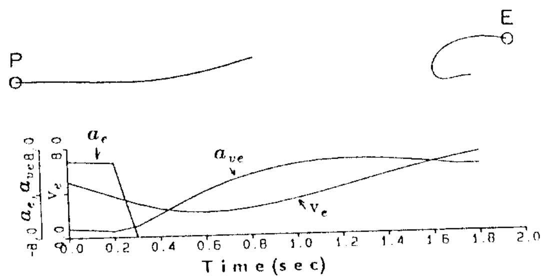

Fig.10 Optimal solution No.2 (ball game, with velocity control)

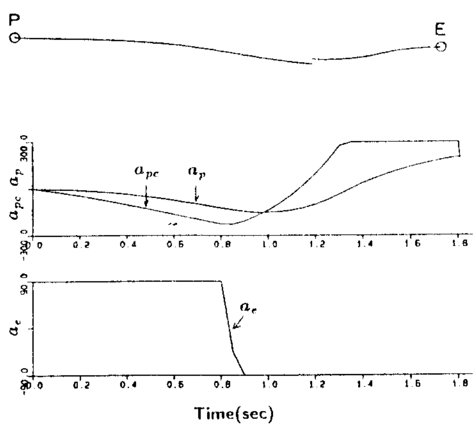

Fig.11 Optimal solution No.1 (air-combat, without velocity control)

to avoid P, while in Fig.10, E first reduces its velocity for a quick turn and after 180° turns, it increases the velocity to draw away from P. The r_f values in Figs 9 and 10 are 4.92m and 13.49m, which are larger than that of corresponding without velocity control cases, 4.13m and

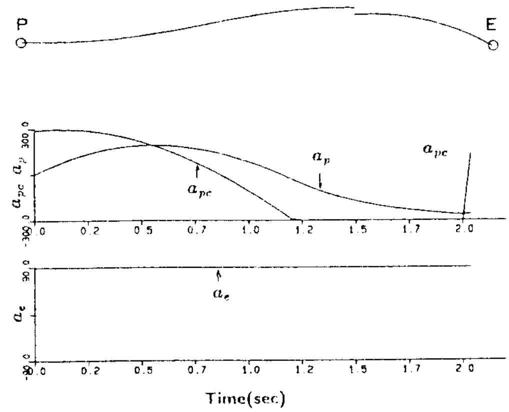

Fig.12 Optimal solution No.2 (air-combat, without velocity control, with an initial heading error)

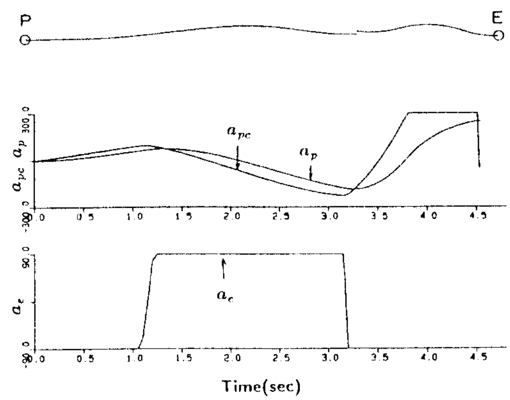

Fig.13 Optimal solution with a large initial distance (air-combat, without velocity control)

12.16m in Figs 3 and 4, therefore the effect of velocity control is clearly shown.

Air-Combat Application (Two-Dimensional)

The same mathematical process is applied to an air-combat problem of a tactical missile and an aircraft, just by accomodating the parameter values of two players to that of a missile (P) and an aircraft (E). In the problem, parameter values of $v_p = 700m/s$, $v_e = 300m/s^2$, $a_{pcmax} = 300m/s^2$, $a_{emax} = 90m/s^2$, $a_{vemax} = 14m/s^2$, $\tau_p = 0.4s$, and $N_p = 4$ are selected. Figure 11 shows the case without velocity control, and initial distance $x_e = 1800m$. This solution corresponds to that of Fig.3, the resulting r_f is 20.6m. The solution corresponds to Fig.4 does not appear because in this case v_p is far larger than v_e, therefore that

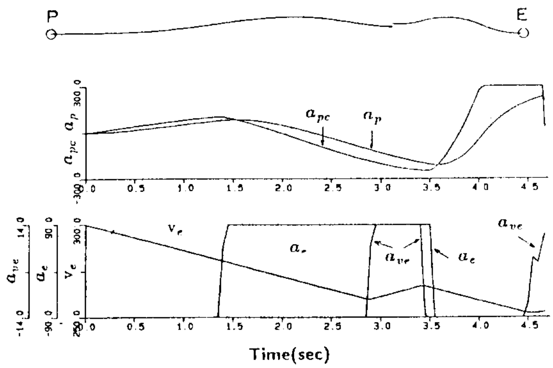

Fig.14 Optimal solution with a large initial distance (air-combat, with velocity control)

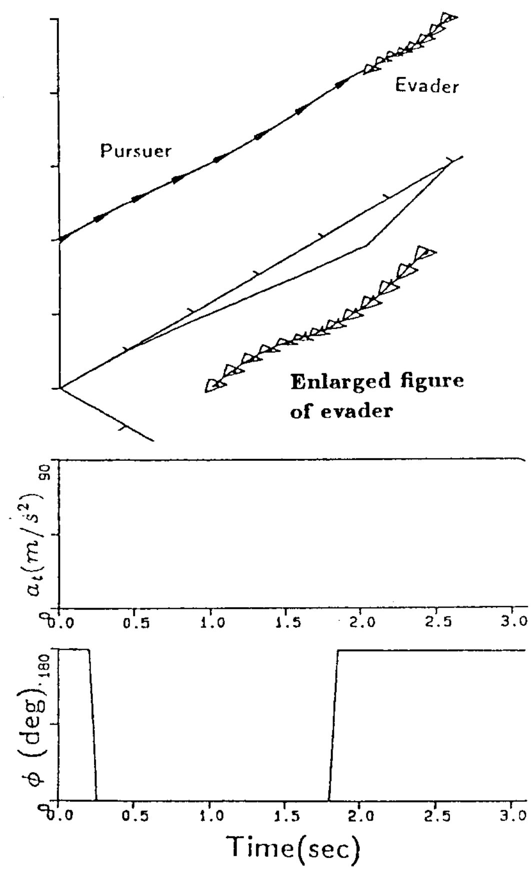

Fig.15 Optimal solution ex.1 in a three-dimensional air-combat game (vertical-S type)

kind of avoidance is impossible. However, when a large initial missile heading error exists, the kind of solution appears. For example, Fig.12 shows the case where initial x_e and ψ_e are 2000m and 150°, respectivery. The aircraft

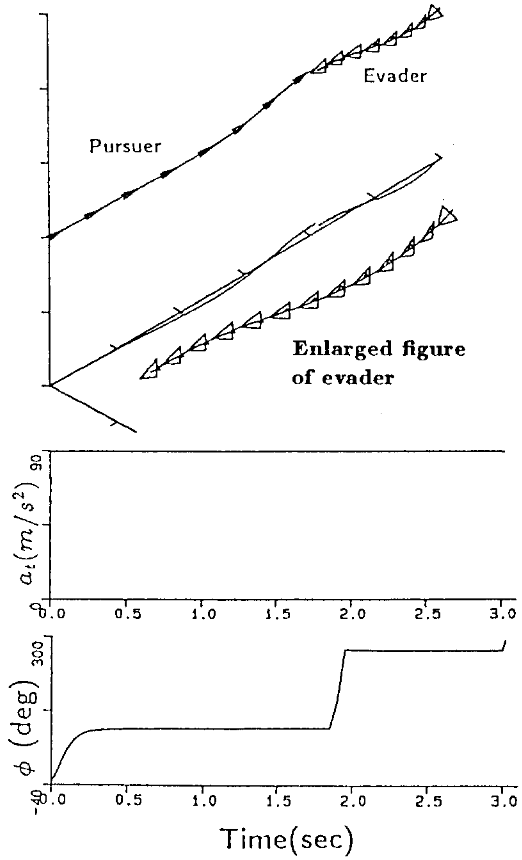

Fig.16 Optimal solution ex.2 in a three-dimensional air-combat game (horizontal-S type)

(E) takes the positive maximum a_e throughout the history. The resulting r_f is 30.0m in the case. Figure 13 shows the case where initial x_e is 4500m. The same a_e pattern as Fig.5 appears, and the resulting r_f is 30.3m. Figure 14 shows the case with velocity control, which corresponds to the without case Fig.13. From these figures the effect is not clearly shown, but the resulting r_f is 31.9m, which shows a slight improvement by employing the velocity control.

Air-Combat Application (Three-Dimensional)

Although it is not the purpose of this paper to discuss about the three-dimensional air-combat game, but two examples are shown to exhibit the usefulness of the above mentioned two-dimensional analysis. In the three-dimensional studies, the aircraft lateral acceleration a_e and bank angle ϕ are used as control variables. For a positive a_e value, ϕ = 0°, 90°, 180°, and 270° mean upward, rightward, downward, and leftward lateral accelerations, respectively. The equations of motion are abreviated here. The performance index is miss distance, which is the slant range r_f at the closest point of aproach between the missile and aircraft. Figures 15 and 16 show typical two optimal solutions. In Fig.15, E changes ϕ from 180° to 0° at $0.2s$ to take the maximum upward acceleration, then changes from 0° to 180° at $1.7s$ to take the maximum downward acceleration. The feature is the same as that of Fig.5, where the aircraft (E) maneuver is taken in a vertical plane, therefore called as a vertical-S. In the same manner, in Fig.16, the maneuver is taken in a horizontal plane, where the aircraft changes ϕ from 90° to 270°. This maneuver is called as a horizontal-S. The maneuver corresponding to Fig.12 is also obtained in the three-dimensional study, (the figure is abreviated here) which is called as a split-S or a sustained maximum g turn. As these maneuvers obtained from a three-dimensional pursuit-evasion study is essentially two-dimensional, therefore we may consider the preceding two-dimensional study results are applicable to three-dimensional cases (at least under some conditions).

Conclusions

A pursuit-evasion game between two players in a two-dimensional plane is studied and applied to a ball game and an air-combat game. In the problem, the pursuer employ proportional navigation to chase the evader, while the evader's optimal control is obtained by solving a nonlinear two-point boundary value problem. Generally, two kinds of local optimal solutions are obtained. In one solution, the evader first takes a maximum positive or negative lateral acceleration, then reverses its sign a short time before interception. In another solution, the evader takes a constant maximum lateral acceleration throughout the game. The advantage of the evader's velocity control is also shown. In the game where a goal exists to the evader, the evader first should take a lateral acceleration to the opposite side of the goal, then reverse its acceleration to reach the goal.

The same study is aplied to an air-combat game of a missile and an aircraft, the results show almost the same feature as that of the ball game. However, in this game, as the pursuer's velocity is far larger than evader's, the second type solution (constant maximum lateral acceleration) only appear when an initial heading error exists. Two examples of the three-dimensional study are also shown, which reduced to maneuvers in a plane called as a vertical-S and a horizontal-S. The second type solution is also obtained in a three-dimensional study, which is called as a split-S or a sustained maximum g turn. Therefore, the approach employed in this paper is useful for analyzing both ball games and air-combat games, and the results is applicable to a three-dimensional air-combat game.

Acknowledgment

The authors are indebted to Dr. S. Uehara, Director General for Research and Development of the Japan Defense Agency for giving us useful advice and discussing the contents of this paper.

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