System Model Reduction
by Weighted Component Cost Analysis

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ABSTRACT
Component Cost Analysis considers any given system driven by a white noise process as an interconnection of different components, and assigns a symmetric square matrix, called "component cost" to each component. These component costs measure the contribution of each component to a predefined quadratic cost function. One possible use of component costs is for model reduction by deleting those components that have the smallest component costs. The theory of Component Cost Analysis is extended to include finite-bandwidth colored noises. The results also apply when actuators have dynamics of their own. When the dynamics of this input are added to the plant, which is to be reduced by CCA, the algorithm for model reduction process will be called Weighted Component Cost Analysis (WCCA). Closed-form analytical expressions of component costs for continuous time case, are also derived for a mechanical system described by its modal data. This is very useful to compute the modal costs of very high order systems beyond Lyapunov solvable dimension. A numerical example for NASA's MINIMAST system is presented.

1. Introduction
There exist numerous schemes for model reduction. However, due to the requirement of many of these methods to solve Lyapunov equations, these schemes are not applicable to the model reduction of large flexible space structures due to the large dimension of these models. Modal Cost Analysis (MCA) is one method which has been developed especially for such large scale systems. The MCA is a special case of Component Cost Analysis (CCA) [1,2]. CCA considers any given system driven by a white noise process as an interconnection of different components. The definition of these components is up to analyst; they may have physical significance, or they may be defined for mathematical convenience. For example, in a multibody system, each body may be considered as a component and each component body may have several subcomponents. For any choice of components CCA assigns a matrix called "component cost" to each component. These component costs measure the contribution of each component to a predefined quadratic cost function. A reduced-order model of the given system may be obtained by deleting those components that have the smallest component costs, although only special coordinates can offer any guarantees by this reduction.

In the theory of CCA, the input is assumed to be a white noise process. However, such infinite-bandwidth white noise processes do not exist in the real world. In fact, any real actuator and sensor devices can only have finite bandwidth. Furthermore, the drawbacks of the infinite-bandwidth assumption for white noise processes are evident in infinite-dimensional systems, since the standard quadratic cost function is not finite in all cases (such as torque inputs and velocity outputs) [6].

To cope with this unrealistic situation we propose the practical approach of considering the dynamics of finite-bandwidth inputs. When the dynamics of this input are added to the plant, which is to be reduced by CCA, the algorithm for model reduction process will be called Weighted Component Cost Analysis (WCCA).

The purpose of this paper is to extend the theory of Component Cost Analysis when a linear system to be reduced is subjected to a finite-bandwidth colored noise which is modeled by linear dynamics. When a mechanical system is described by its modal data, each mode is considered as a component and analytical expressions of component costs (modal costs) will be derived for continuous time case. This analytical expression is very useful to compute the modal costs of very high order systems since Lyapunov equations need not be computed.

This paper is organized as follows: section 2 reviews the theory of CCA and section 3 provides analytical expressions of modal costs when a mechanical system is driven by white noises. Section 4 develops the theory of Weighted Modal Cost Analysis (WMC) for the system subjected to finite-bandwidth noises. A numerical example for NASA's MINIMAST system is presented in section 5.

2. Theory of Component Cost Analysis
Let a state space realization of a linear time-invariant system driven by zero mean white noise \( w \) with intensity \( W \), be given as

\[
\begin{align*}
\dot{x} &= Ax + Dw, \quad x \in \mathbb{R}^n, \quad w \in \mathbb{R}^m, \quad (21) \\
y &= Cx, \quad y \in \mathbb{R}^m
\end{align*}
\]

where \( x \) and \( y \) are, respectively, state and output vectors. The component form may be written as follows:

\[
\begin{align*}
\dot{x}_i &= \sum_{j=1}^{N} A_{ij} x_j + D_{iw}, \\
y &= \sum_{i=1}^{N} C_i x_i, \quad \sum_{i=1}^{N} n_i = n \\
x_i \in \mathbb{R}^{n_i}, \quad i = 1, 2, \ldots, N
\end{align*}
\]
where $N$ is the number of components and the state vector $x_i$ define the $i$-th component. Given the system (2.1) a simple quadratic cost function is defined by

$$V = E_W V(t), \quad V(t) \triangleq y(t)^T Q y(t)$$

(2.3)

where $E_W \triangleq \lim_{t \to \infty}$ is the expectation operator and $Q$ is a positive semi-definite output weighting matrix. Then, the component cost $V_i$ associated with each component $x_i$ is defined by

$$V_i = \frac{1}{2} E_W \left( \frac{\partial V(t)}{\partial x_i} x_i \right), \quad i = 1, 2, ..., N.$$  

(2.4)

It can be shown [4] that $V_i$ is calculated by the following formula:

$$V_i = \text{tr} \left[ X C_i^T Q C_i \right], \quad i = 1, 2, ..., N$$  

(2.5a)

where $\text{tr}$ is the matrix trace operator and the steady state covariance of the states $X$ satisfies the Lyapunov equation:

$$0 = AX + XA^T + D W D^T.$$  

(2.5b)

Clearly, since $V = \text{tr} \left[ X C_i^T Q C_i \right]$, the component costs $V_i$ satisfy the cost decomposition property:

$$V = \sum_{i=1}^N V_i.$$  

(2.6)

Because of the property (2.6) component costs $V_i$ in (2.5a) may be normalized as

$$V_i = \frac{V_i}{V}, \quad i = 1, 2, ..., N.$$  

(2.7)

Then a reduced-order model of the system (2.1) may be obtained by deleting those components that have the smallest $V_i$.

3. Analytical Expressions of Modal Costs

Usually the dynamics of large structures are modeled by their modal data extracted either by finite element analysis or by experiment. In this case components can be defined by natural frequencies and mode shapes, and hence each component has physical significance. If this is the case, it is possible to get an analytical expression for component costs $V_i$ in (2.5), which we shall call modal costs.

Let a mechanical structure be described as

$$\begin{aligned}
\dot{\eta}_i + 2 \zeta_i \omega_i \eta_i + \omega_i^2 \eta_i = & \frac{d_i^T W d_i}{\omega_i^2} \omega_i \eta_i + \frac{d_i^T W d_i}{\omega_i^2} \omega_i \eta_i = \frac{d_i^T W d_i}{\omega_i^2} \omega_i \eta_i + \frac{d_i^T W d_i}{\omega_i^2} \omega_i \eta_i
\end{aligned}$$  

(3.1)

$$y = \sum_{i=1}^N p_i \eta_i + \sum_{i=1}^N r_i \eta_i$$

where $\omega_i$ and $\zeta_i$ are, respectively, the natural frequency and damping ratio of mode $i$. Note that in (3.1) $w(t)$ represents a zero mean white noise with intensity $W$. For the system (3.1), the explicit solution of the Lyapunov equation (2.2) is known [2, 5] to be

$$X = \frac{d_i^T W d_i}{{\omega_i}^2} \left[ \begin{array}{ccc}
2 \zeta_i \omega_i & 2 \zeta_i \omega_i & (\omega_i^2 - \omega_j^2) \\
(\omega_i^2 - \omega_j^2) & \omega_i \omega_j (2 \zeta_i \omega_i + 2 \zeta_j \omega_j) & \omega_i \omega_j (2 \zeta_i \omega_i + 2 \zeta_j \omega_j)
\end{array} \right]$$  

(3.2)

where $X$ is the $(2 \times 2)$ block of $X$ in (2.5) with $X = [\eta_1, \eta_2, \ldots, \eta_N, \eta_N]^T$, and

$$A_i = w_i \eta_i (2 \zeta_i \omega_i + 2 \zeta_j \omega_j) (2 \zeta_i \omega_i + 2 \zeta_j \omega_j) + (\omega_i^2 - \omega_j^2)^2.$$  

Then the modal cost of the $i$-th mode can be obtained from (2.5a):

$$V_i = \text{tr} \left[ \sum_{j=1}^N X_j C_j^T Q C_j \right], \quad i = 1, 2, ..., N$$  

(3.3a)

where

$$C_i = [p_i, r_i]$$

and $A_i = \left[ \begin{array}{cc}
0 & 1 \\
-\omega_i^2 & -2 \zeta_i w_i
\end{array} \right].$  

(3.3b)

Note that for (3.4) the $i$-th component is defined by $x_i = [\eta_i, \eta_i]^T$, i.e., each component consists of only one mode shape. Although it is a formidable task to calculate by (3.4) all $V_i$ 's for a large scale system, it is certainly easier than trying to solve the Lyapunov equation (2.5) numerically.

For a lightly damped structure the modal cost $V_i$ of (3.4) can be approximated by setting $\zeta_i = 0$ for all $i$:

$$V_i \approx \frac{d_i^T W d_i}{4 \omega_i^2} (p_i^T Q p_i + \omega_i^2 T Q r_i) + \sum_{j=1}^N \frac{d_j^T W d_j}{4 \omega_j^2} (p_j^T Q r_i - p_j^T Q r_i).$$  

(3.5)

The approximate formula for MCA suggested by Skelton, et al [2] can be obtained by taking the first term from (3.5), or equivalently by assuming, for all $i$ and $j \neq i$, either $d_i^T W d_j = 0$ or $p_i^T Q r_i = 0$:

$$V_i \approx \frac{d_i^T W d_i}{4 \omega_i^2} (p_i^T Q p_i + \omega_i^2 T Q r_i).$$  

(3.6)

4. Weighted Modal Cost Analysis

In the previous sections, we assumed that the input noise $w$ is a white noise process. By considering the dynamics of finite-bandwidth actuators which drive the plant to be reduced, we will derive an MCA formula for more realistic cases. We shall call this Weighted Modal Cost Analysis (WMA).

Let the plant be given by

$$\begin{aligned}
\dot{\eta}_i + 2 \zeta_i \omega_i \eta_i + \omega_i^2 \eta_i &= b_i^T u + d_i^T w_p, \\
y &= \sum_{i=1}^N p_i \eta_i + \sum_{i=1}^N r_i \eta_i, \quad i = 1, 2, ..., N
\end{aligned}$$  

(4.1)

where $w_p$ is an additional plant noise with intensity $W_p$ and $u$ is the actuator output signal which is now colored by the actuator dynamics given by

$$\begin{aligned}
\dot{x}_a &= A_x x_a + D_x w_a, \quad x_a \in \mathbb{R}^m, \\
u &= C_x x_a + H_w w_a
\end{aligned}$$  

(4.2)

where $w_a$ is a zero mean white noise with intensity $W_a$. A state space description of the combined system (4.1) and (4.2) is obtained as

$$\begin{aligned}
\dot{x} &= A x + 5 u, \\
y &= C x
\end{aligned}$$  

(4.3a)
where

\[
A_p = \text{block diag} \left[ \begin{bmatrix} 0 & 0 \\ b^T C_p & 0 \end{bmatrix} \right], \\
D_p = \text{block diag} \left[ \begin{bmatrix} 0 & 0 \\ d^T & 0 \end{bmatrix} \right], \\
C_p = \{-[p, q]_{1,1}\}.
\]

(4.3a)

Note that by setting \( W_p = 0 \) we have only the colored noise input \( u \). Now that the system (4.3) is in the standard form of a linear-invariant system driven by white noise, its steady-state covariance matrix satisfies the following equation:

\[
0 = B E + E^T B^T + S \otimes S^T
\]

(4.3b)

where

\[
B = \begin{bmatrix} W_p & 0 \\ 0 & W_p \end{bmatrix}.
\]

(4.3c)

Let

\[
E = \begin{bmatrix} X_p & X_q \\ X_p^T & X_q^T \end{bmatrix}.
\]

(4.3d)

Then (4.4) can be partitioned into 3 equations:

\[
0 = A_p X_p + X_p A_p^T + D_p W_p D_p^T
\]

(4.6a)

\[
0 = A_q X_q + X_q A_q^T + D_q W_q D_q^T
\]

(4.6b)

\[
0 = A_p X_p + X_p A_p^T + A_q X_q + D_p W_p D_p^T + A_p X_p^T + D_q W_q D_q^T
\]

(4.6c)

Since the number of actuators is usually relatively small, the solution of (4.6a) can be easily obtained by any numerical method. However, for completeness, we assume here that actuator dynamics is described in second-order modal coordinates (instead of in state space form), and derive an analytical expression for \( X_a \) as follows: let actuators be represented by

\[
\hat{n}_a = 2 \alpha \omega_a \hat{a}_a, \quad \eta_a = \hat{d}_a W_a,
\]

(4.7)

For the state space form (4.2) we have

\[
X_a = \text{block diag} \left[ \begin{bmatrix} 0 & 1 \\ -\alpha^2 \omega_a & 0 \end{bmatrix} \right].
\]

(4.8a)

\[
A_a = \text{block diag} \left[ \begin{bmatrix} 0 & 1 \\ -\alpha^2 \omega_a & 0 \end{bmatrix} \right], \\
D_a = \text{block diag} \left[ \begin{bmatrix} 0 & 1 \\ -\alpha^2 \omega_a & 0 \end{bmatrix} \right], \\
C_a = \{-[p_a, q]_{1,1}\}.
\]

(4.8b)

In the same manner as in section 3, we get the Lyapunov solution for (4.6a):

\[
X_a = \frac{d^T W_a d_a}{\delta_a} \left[ \begin{bmatrix} (2 \alpha \omega_a \eta_a, 2 \alpha \omega_a \eta_a) \\ -\delta_a \end{bmatrix} \right]
\]

(4.9a)

where

\[
\delta_a = \omega_a \eta_a (2 \alpha \omega_a \eta_a + 2 \alpha \omega_a \eta_a) (2 \alpha \omega_a \eta_a + 2 \alpha \omega_a \eta_a)
\]

(4.9b)

\[
+ (\omega_a^2 - \omega^2) \eta_a^2.
\]

(4.9c)

Once \( X_a \) for (4.6a) is known, (4.6b) can be analytically solved due to the special structure of \( A_p, A_q, D_p, D_q \) of (4.4). Let

\[
X_a = \{-[a_i, b_i]_{1,1}\}.
\]

(4.10)

Then the solution of (4.6b) is given by

\[
a_i = \alpha \omega_a \eta_a (2 \alpha \omega_a \eta_a + 2 \alpha \omega_a \eta_a) (2 \alpha \omega_a \eta_a + 2 \alpha \omega_a \eta_a)
\]

(4.11a)

\[
b_i = -\alpha \omega_a \eta_a, \quad i = 1, 2, \ldots, N.
\]

(4.11b)

where \( I_a \) is an identity matrix of size of \( A_a \). Finally for (4.6c), consider the \((2 \times 2)\) block and let

\[
[X_a]_i = \begin{bmatrix} X_a^H & X_a^F \\ X_a^H & X_a^F \end{bmatrix}.
\]

(4.12a)

After some algebraic manipulation, we have the solution of (4.6c):

\[
X_a^H = -\frac{1}{\delta_a} \left[ \begin{bmatrix} (2 \alpha \omega_a \eta_a + 2 \alpha \omega_a \eta_a) (2 \alpha \omega_a \eta_a + 2 \alpha \omega_a \eta_a) \\ + (\omega_a^2 - \omega^2) \eta_a^2 \end{bmatrix} \right] (X_a C_a^T + D_a W_a H_a) b_i.
\]

(4.12b)

\[
X_a^F = \begin{bmatrix} \hat{a}_a (d_i^T W_a d_i + b_i^T H_a W_a H_b b_i) - b_i^T C_a \hat{a}_a - b_i^T C_a \hat{a}_a (d_i^T W_a d_i + b_i^T H_a W_a H_b b_i) - b_i^T C_a \hat{a}_a \end{bmatrix}.
\]

(4.12c)

\[
X_a^F = \begin{bmatrix} \hat{a}_a (d_i^T W_a d_i + b_i^T H_a W_a H_b b_i) - b_i^T C_a \hat{a}_a - b_i^T C_a \hat{a}_a (d_i^T W_a d_i + b_i^T H_a W_a H_b b_i) - b_i^T C_a \hat{a}_a \end{bmatrix}.
\]

(4.12d)

\[
X_a^F = \begin{bmatrix} \hat{a}_a (d_i^T W_a d_i + b_i^T H_a W_a H_b b_i) - b_i^T C_a \hat{a}_a - b_i^T C_a \hat{a}_a (d_i^T W_a d_i + b_i^T H_a W_a H_b b_i) - b_i^T C_a \hat{a}_a \end{bmatrix}.
\]

(4.12e)

where \( \delta_a \) and \( a_i \) are given by (3.3) and (4.11a), respectively. Now having the explicit solution given by (4.9), (4.11) and (4.12b) for (4.5), we define the cost function as given in (2.3) and get the analytical expression of modal costs for the plant:

\[
V_i = \text{tr} \left[ \sum_{i=1}^{N} X_p \hat{a}_a C_a \hat{a}_a \right],
\]

(4.13)

As we can see in (4.12), the \([X_p]_i\) are weighted by actuator parameters and so \( V_i \) in (4.13) is called Weighted Modal Costs. Notice that by setting \( b_i = 0 \) for all \( i \) and \( W = W_p, (4.12) \) leads to (3.2), which is for the standard white noise input case. As an approximation we take only the \( j = i \) term from (4.13) as we did for MCA (this is justified when all \( \hat{a}_a \) are small and
\[ d_i^T W d_i = 0, \text{ or } p_i^T Q r_i = 0 ; \]

\[ V_i = \frac{-d_i^T Q p_i + u_i^T Q r_i}{d_i^T w_i} \cdot (d_i^T W d_i + b_i^T H_i W_i H_i b_i - 2b_i^T C_A A_i^{-1} A_i^T u_i + v_i^T H_i b_i)^{-1} (X_i C_i^T + D_i W_i H_i) b_i \]

\[ + \frac{d_i^T Q p_i - u_i^T Q r_i}{u_i^T w_i} \cdot (X_i C_i^T + D_i W_i H_i) b_i. \]  

(4.14)

Notice again that by setting \( b_i = 0 \) for all \( i \) and \( W = W_w \), (4.14) leads to the standard formula for approximate MCA, \( V_i \) in (3.6).

5. Application: MINIMAST

The MINIMAST considered here is schematically represented by Figure 1. From a finite element model we have the following data:

\[ \eta_i + 2 \xi_i \omega_i \eta_i + \omega_i^2 \eta_i = b_i^T u + d_i^T w_i, \]

\[ y = \sum_{i=1}^{149} \eta_i, \ i = 1, 2, ..., 149, \]

(5.1)

where \( w_i \) is the noise input with intensity \( W_w = 1496.5 \)

12 (Newton)^2 from the shakers located at 3 corners of Bay 9. The natural frequencies and damping ratios of the MINIMAST structure are shown in Figure 2. The description of some global modes is given in Table 1. Damping ratios are obtained by the Rayleigh model:

\[ Z_i^w = a_i^w + \omega_i^w, \]

\[ i = 5, 6, ..., 149, \]

\[ \xi_i^w = 0.01194 \] 

and \( \xi_i^w = 0.05 \). There are three noisy Torque Wheel Actuators (TWA) on the Tip Plate at Bay 18. Each TWA is modeled as:

\[ x_s = A x_s + B u_s + D w_s, \]

\[ u = C x_s + H w_s, \]

(5.2)

where \( u_s \) is the command signal to TWA which shall set to zero, and \( w_s \) is a white noise with intensity

\[ W_s = 1838 (\text{Newton-Meter})^2. \] 

(For complete system data for MINIMAST, see [6,7].) Selected outputs are the translational displacements of 3 corners and the centroidal rotations at Bay 10, 14 and 18. For these selected outputs, \( r_i = 0 \) for all \( i \) (no velocity or acceleration outputs). The following modal costs are given:

\[ V_i = \sum_{i=1}^{149} X_i^T H_i Q p_i, \ i = 1, 2, ..., 149 \]

(5.3)

where \( X_i^T \) is the (1,1) element of (3.2) for MCA and is given in (4.12a) for the weighted MCA. Since \( r_i = 0 \) for all \( i \), the approximate modal costs (3.5) and (3.6) are exactly the same. The approximate (unweighted) modal costs are

\[ V_i \approx \frac{d_i^T W d_i (p_i^T Q p_i)}{d_i^T w_i}, \ i = 1, 2, ..., 149, \]

(5.4)

where we use \( W = W_w \) or \( W = W_u \) (= intensity of \( u \) as a white noise input). For the approximate weighted modal costs, (4.14) with \( r_i = 0 \) for all \( i \) will be used.

The normalized modal costs of the 50 highest-ranked modes are given in Figure 3. Similar plots are shown in Figure 4 when the actuator dynamics are included (hence \( W_u \)). Table 2 shows the corresponding rankings of modes. From Figures 3, 4 and Table 2, one should notice that the cost rankings obtained by the exact expressions (4.4) and (4.13) are quite different from those by the approximate expressions (4.5) and (4.14), even with fairly small damping (1 to 5%). One of the main reasons for this is that MINIMAST has a dense frequency spectrum (see Figure 2 and Equations (3.4), (3.5) and (4.13)). Notice also that the 5 highest-ranked modes give the same normalized modal costs with exact and approximate expressions. These 5 modes are 4 bending and 1 torsion modes (see Table 1). Based on these costs and rankings given in Figures 3, 4 and Table 2, six reduced-order models are obtained by retaining the highest-ranked modes in each case. Four cases are generated by only \( W_u \) inputs: the exact and the approximate of both the weighted and unweighted MCA. The remaining two cases are the exact and approximate MCA with only the \( W_w \) inputs. Observe, from Tables 1 and 2, that more global modes (e.g., modes 121, 122, 128 and 129) will be retained in a low-order reduced model when \( W_u \) is used as an input noise than when \( W_w \) is used.

The output covariance errors are calculated for each of the six cases to evaluate each reduced-order model. Figures 5 and 6 show the relative covariance errors of rotations at Bay 10. First of all, as expected in the cost analysis, the errors of those reduced-order models by exact analysis (either MCA or \( W_u \)) are quite smaller than those by approximation, except for low-order models (less than 8 modes). Figure 5 indicates that when the MINIMAST is subjected to the shaker noise, \( W_w \), we need more modes to get the "Relative Error" down to a small number. On the other hand, when the system is subjected to the TWA noise (\( W_s \)), we can get slightly improved reduced-order models if we use the weighted MCA instead of the MCA (see the first plots of Figures 5 and 6). In general, when different input sources are used for MCA, we shall have different reduced-order models, if there are different sets of input sources (e.g., actuator noises and shaker noises in MINIMAST), we recommend performing as many cost analyses as input sets and to take union of sets of the highest-ranked modes in order to get a reduced-order model which is "good" with respect to an overall performance.

7 Conclusion

This paper presents several new results. First, the expressions for modal costs are in explicit closed form. Secondly, frequency weighting has been added to include the case when the inputs are colored noises instead of white noises. These expressions are also in explicit closed form. The final contribution is to apply the theory to a large physical system, NASA's MINIMAST, with real (and therefore finite bandwidth) actuators. Our analysis was based upon a finite element model supplied by NASA. It is shown in [6,7] that these models are useful for control design. The advantage of the exact closed-form expressions is that previous approximate closed-form expressions were small-damping approximations. But the system damping might not be small and this section shows that large errors in the reduced-order models may arise from the use of the standard (small damping) modal cost analysis. Also previous theory could not treat the weighted case (e.g. with actuator dynamics), without having to resort to numerical approaches of component cost analysis, requiring the solution of Lyapunov equations. For large scale systems (such as the MINIMAST example in this paper) this would have been impossible with present day computers. Our closed-form results open up the
application of model reduction practice to large scale systems beyond "Lyapunov solvable" dimension. In fact these modal cost formulas can be applied to a structural system as large as the finite element code can compute modal data. In the future the inclusion of modal cost analysis into the finite element codes (e.g. NASTRAN) seems desirable.

References


Table 1. Description of Some Global Modes

<table>
<thead>
<tr>
<th>Mode</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>First Bending</td>
</tr>
<tr>
<td>2</td>
<td>First Bending</td>
</tr>
<tr>
<td>3</td>
<td>First Torsion</td>
</tr>
<tr>
<td>4</td>
<td>Second Bending</td>
</tr>
<tr>
<td>5</td>
<td>Second Bending</td>
</tr>
<tr>
<td>117</td>
<td>Tip Plate</td>
</tr>
<tr>
<td>118</td>
<td>Second Torsion</td>
</tr>
<tr>
<td>119</td>
<td>Tip Plate with 3rd Bending</td>
</tr>
<tr>
<td>120</td>
<td>Tip Plate with 3rd Bending</td>
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<td>121</td>
<td>Third Bending</td>
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<td>122</td>
<td>Third Torsion</td>
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<tr>
<td>123</td>
<td>Mid Plate</td>
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<tr>
<td>124</td>
<td>Tip Plate with 3rd Torsion</td>
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<tr>
<td>127</td>
<td>Third Torsion</td>
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<tr>
<td>128</td>
<td>Fourth Bending</td>
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<tr>
<td>129</td>
<td>Fourth Bending</td>
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<td>130</td>
<td>Tip Plate with 4th Torsion</td>
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<tr>
<td>131</td>
<td>Fourth Torsion</td>
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<td>132</td>
<td>Tip Plate with 4th Bending</td>
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<tr>
<td>140</td>
<td>Tip Plate with some Torsion</td>
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Table 2. Rankings of Modes

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Figure 1. MINIMAST Configuration

Figure 2. Natural Frequencies and Damping Ratio of MINIMAST

Figure 3. Ranked Modal Costs by MCA

Figure 4. Ranked Modal Costs by Weighted MCA (WMCA)

Figure 5. Output Covariance Error by MCA

Figure 6. Output Covariance Error by Weighted MCA