

On the Properties of γ - ϵ for H_∞ Control by State Feedback and Computation of the Infimum of H_∞ Norm

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Abstract

It is well known that H_∞ control problem involves solving an algebraic Riccati equation which includes a pair of parameters (γ, ϵ) . Focusing on ϵ^* , the maximum of ϵ . We discuss in this paper about the properties between the H_∞ norm of a transfer function matrix and the parameters (γ, ϵ^*) . We can change the algebraic relation between γ and ϵ^* by the similarity transformation of a considered system. We can find a proper transformation to get a simple quadratic algebraic equation between γ and ϵ^* . This relation provide the H_∞ norm of a transfer function.

1. Introduction

The system considered here is given by the following equations:

$$\begin{aligned}\dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w \\ u &= Fx\end{aligned}\quad (1)$$

Where all the matrices in are constant real matrices of compatible dimensions. Assume that the pair (C_2, A) is detectable, and the pair (A, B_2) is stabilizable.

In this paper, we will restrict our attentions to such a system, that is

$$D_{11}=0, D_{12}^T C_1=0, D_{12}^T D_{12}=I$$

Let $G_F(s) = (C_1 + D_{12} F)(sI - A - B_2 F)^{-1} B_1$ be the transfer function matrix of system (1), then the following two statements are equivalent.⁽¹⁾

- [1] $\|G_F(s)\|_\infty < \gamma$ and $\text{Re } \lambda(G_F(s)) < 0$
where $\|G_F(s)\|_\infty =: \sup \sigma_{\max}[G_F(j\omega)] \quad \forall \omega$
[2] There exists an $\epsilon > 0$ such that Riccati equation:

$$A^T P + PA + \gamma^{-2} P B_1 B_1^T P - P B_2 B_2^T P + C_1^T C_1 + \epsilon I = 0 \quad (2)$$

has a positive definite solution $P > 0$.

It is well known that if there exists an $\epsilon_1 > 0$ such that (2) has a positive definite solution, then (2) has

a positive definite solution for any $\epsilon \in (0, \epsilon_1]$. In this paper, we replace system (1) with its similarity transformation, and discuss the properties among γ , ϵ^* and $\|G_F(s)\|_\infty$. With the proper choice of a similarity transformation, we can express the relation between γ and ϵ^* by a simple algebraic equation. The $\|G_F(s)\|_\infty$ can be computed by using this equation.

2. Preliminaries

In this section, we consider about a system (3) and introduce some lemmas as the preparations of the proofs of the main results in next the section.

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}\quad (3)$$

Lemma 1.⁽¹⁾ Let $G(s) = C(sI - A)^{-1} B + D$ be the transfer function matrix, then the following statements are equivalent.

- [3] $\text{Re } \lambda(A) < 0$ and $\|G(s)\|_\infty < \gamma$,
where $\|G(s)\|_\infty =: \sup \sigma_{\max}[G(j\omega)] \quad \forall \omega$
[4] $\text{Re } \lambda(A) < 0$ and $\|C_N(sI - A_N)^{-1} B_N\|_\infty < 1$
[5] $R = \gamma^2 I - D^T D > 0$ and there exists an $\epsilon > 0$ such that Riccati equation

$$P A_N + A_N^T P + P B_N B_N^T P + C_N^T C_N + \epsilon I = 0 \quad (4)$$

has a solution $P > 0$.

where $A_N = A + B R^{-1} D^T C$

$$B_N = B R^{-1/2}$$

$$C_N = (I + D R^{-1} D^T)^{-1/2} C$$

Lemma 2.⁽²⁾ Let $\text{Re } \lambda(A) < 0$, (A, B) is controllable and (C, A) is observable, then the Riccati equation

$$PA + A^T P + P B B^T P + C^T C = 0 \quad (5)$$

has a solution $P > 0$, if and only if the following

condition is satisfied.

$$I-B^T(-sI-A^T)^{-1}C^TC(sI-A)^{-1} \geq 0 \quad (6)$$

Lemma 3. Let matrix M,N be nonsingular, then

$$(M+UNV)^{-1} = M^{-1} - M^{-1}U(VM^{-1}U+N^{-1})^{-1}VM^{-1} \quad (7)$$

where matrix M,N,U,V have compatible dimensions

Proof : This lemma can be proven by simple matrix calculation with the use of inverse matrix lemma.

$$\begin{aligned} (M+UNV)^{-1} &= M^{-1} - M^{-1}U(I+NVM^{-1}U)^{-1}NVM^{-1} \\ &= M^{-1} - M^{-1}U(VM^{-1}U+N^{-1})^{-1}VM^{-1} \end{aligned} \quad (8)$$

Lemma 4. For the matrix A, B,C and D defined in the system (3), the following equation holds,

$$\begin{aligned} (sI-A-BR^{-1}D^TC)^{-1} \\ = (sI-A)^{-1} + (sI-A)^{-1}B(\gamma^2 I-D^TG(s))^{-1}D^TC(sI-A)^{-1} \end{aligned} \quad (9)$$

where $R = \gamma^2 I - D^TD$

Proof : We can easily give the proof of this lemma by using lemma 3.

$$\begin{aligned} (sI-A-BR^{-1}D^TC)^{-1} \\ = (sI-A)^{-1} + (sI-A)^{-1}B[-D^TC(sI-A)^{-1}B+R]^{-1}D^TC(sI-A)^{-1} \\ = (sI-A)^{-1} + (sI-A)^{-1}B(\gamma^2 I-D^TG(s))^{-1}D^TC(sI-A)^{-1} \end{aligned} \quad (10)$$

Before giving the proof of lemma 5, let us define two functions as

$$\begin{aligned} G1(s) &= C(sI-A)^{-1}B \\ G2(s) &= (sI-A)^{-1}B \end{aligned} \quad (11)$$

and $G^*(j\omega), G1^*(j\omega), G2^*(j\omega)$ is the complex conjugate transpose of $G(j\omega), G1(j\omega), G2(j\omega)$ respectively.

Lemma 5. For matrix A_N, B_N and C_N given in (4) and for a small positive number ϵ , the equation:

$$I-B_N^T(-sI-A_N^T)^{-1}(C_N^TC_N + \epsilon I)(sI-A_N)^{-1}B_N \geq 0 \quad (12)$$

holds if and only if

$$\gamma^2 I - G(s)^*G(s) - \epsilon G2(s)^*G2(s) \geq 0 \quad (13)$$

We only prove the necessity of the lemma 5:

Multiply $R^{1/2}$ on both sides of equation (12), we get equation (14).

$$R-B^T(-sI-A_N^T)^{-1}(C_N^TC_N + \epsilon I)(sI-A_N)^{-1}B \geq 0 \quad (14)$$

With some simple matrix computations, the following equation is obtained.

$$\begin{aligned} R-B^T(-sI-A_N^T)^{-1}C^TDR^{-1}D^TC(sI-A_N)^{-1}B \\ -B^T(-sI-A_N^T)^{-1}C^TC(sI-A_N)^{-1}B \\ -\epsilon B^T(-sI-A_N^T)^{-1}(sI-A_N)^{-1}B \geq 0 \end{aligned} \quad (15)$$

We can apply lemma 4 to obtain equation (16), (17) and (18)

$$\begin{aligned} B^T(-sI-A_N^T)^{-1}(sI-A_N)^{-1}B \\ = [G2(s)+G2(s)(\gamma^2 I-D^TG(s))^{-1}D^TG1(s)]^*[G2(s) \\ +G2(s)(\gamma^2 I-D^TG(s))^{-1}D^TG1(s)] \\ = [I+(\gamma^2 I-D^TG(s))^{-1}D^TG1(s)]^*G2(s)^*G2(s)[I \\ +(\gamma^2 I-D^TG(s))^{-1}D^TG1(s)] \end{aligned} \quad (16)$$

$$\begin{aligned} B^T(-sI-A_N^T)^{-1}C^TC(sI-A_N)^{-1}B \\ = [G1(s)+G1(s)(\gamma^2 I-D^TG(s))^{-1}D^TG1(s)]^*[G1(s) \\ +G1(s)(\gamma^2 I-D^TG(s))^{-1}D^TG1(s)] \\ = [I+(\gamma^2 I-D^TG(s))^{-1}D^TG1(s)]^*G1(s)^*G1(s)[I \\ +(\gamma^2 I-D^TG(s))^{-1}D^TG1(s)] \end{aligned} \quad (17)$$

$$\begin{aligned} B^T(-sI-A_N^T)^{-1}C^TDR^{-1}D^TC(sI-A_N)^{-1}B \\ = [G1(s)+G1(s)(\gamma^2 I-D^TG(s))^{-1}D^TG1(s)]^*DR^{-1}D^T \\ [G1(s)+G1(s)(\gamma^2 I-D^TG(s))^{-1}D^TG1(s)] \\ = [I+(\gamma^2 I-D^TG(s))^{-1}D^TG1(s)]^*G1(s)^*DR^{-1}D^TG1(s) \\ [I+(\gamma^2 I-D^TG(s))^{-1}D^TG1(s)] \end{aligned} \quad (18)$$

Putting equations (16), (17) and (18) into (15), and after some arrangements, we can obtain (19)

$$\begin{aligned} R-[I+(\gamma^2 I-D^TG(s))^{-1}D^TG1(s)]^*[G1(s)^*DR^{-1}D^TG1(s) \\ +G1(s)^*G1(s)+\epsilon G2(s)^*G2(s)][I \\ +(\gamma^2 I-D^TG(s))^{-1}D^TG1(s)] \geq 0 \end{aligned} \quad (19)$$

We multiply $\{[I+(\gamma^2 I-D^TG(s))^{-1}D^TG1(s)]^*\}^{-1}$ on the left side of (19), and $[I+(\gamma^2 I-D^TG(s))^{-1}D^TG1(s)]^{-1}$ on the right side of (19), and rearrange to obtain an identical equation to lemma 5:

$$\gamma^2 I - G(s)^*G(s) - \epsilon G2(s)^*G2(s) \geq 0 \quad (20)$$

□

3 Main Results

In this section, we consider systems (1) and (3), and give some properties about γ and ϵ^* in Riccati equation (2) and (4). At the end of this section, we propose the usage of these properties to compute the H_∞ norm of a transfer function matrix.

Theorem 1 Let $G(s) = C(sI-A)^{-1}B+D$ be the transfer function matrix of the stable system (3), $\|G\|_\infty < \gamma$, (A,B) controllable, and (C,A) observable, then the maximum ϵ^* that satisfies (4) has a solution $P > 0$, is a monotone increasing function of γ . especially, $\epsilon^* \rightarrow 0$, as $\gamma \rightarrow \|G\|_\infty$.

Proof : Form statements [3] and [4] of lemma 1,

$$I - B_N^T(-sI - A_N^T)^{-1}C_N^T C_N(sI - A_N)^{-1}B_N > 0 \quad (21)$$

We can choose a sufficiently small $\epsilon > 0$ such that

$$I - B_N^T(-sI - A_N^T)^{-1}(C_N^T C_N + \epsilon I)(sI - A_N)^{-1}B_N \geq 0 \quad (22)$$

Let us define $C_M = (C_N^T C_N + \epsilon I)^{1/2}$.

Because $\text{Re } \lambda(A_N) < 0$, (C_M, A_N) is observable, (A_N, B_N) is controllable, due to lemma 2 and lemma 5, the ϵ^* that satisfies the statement [5] is the same as the ϵ^* that satisfies the following equation.

$$\gamma^2 I - G(s)^* G(s) - \epsilon G_2(s)^* G_2(s) \geq 0 \quad (23)$$

At first, we prove the monotonicity of ϵ^* .

Let us assume $\gamma_2 > \gamma_1 > \|G\|_\infty$

$$\text{and } \epsilon^*(\gamma_1) \geq \epsilon^*(\gamma_2) \geq 0.$$

We show this assumption may induce a contradiction From (23),

$$\gamma_1^2 I - G(s)^* G(s) - \epsilon^*(\gamma_1) G_2(s)^* G_2(s) \geq 0 \quad (24)$$

$$\gamma_2^2 I - G(s)^* G(s) - \epsilon^*(\gamma_2) G_2(s)^* G_2(s) \geq 0 \quad (25)$$

By the above assumption,

$$\gamma_1^2 I - G(s)^* G(s) - \epsilon^*(\gamma_2) G_2(s)^* G_2(s) \geq 0$$

that is

$$(\gamma_1^2 - \gamma_2^2)I + \gamma_2^2 I - G(s)^* G(s) - \epsilon^*(\gamma_2) G_2(s)^* G_2(s) \geq 0 \quad (26)$$

By the definition of $\epsilon^*(\gamma_2)$, there exists a w_0, x_0 such that

$$\begin{aligned} & x_0^* [(\gamma_1^2 - \gamma_2^2)I + \gamma_2^2 I - G(jw_0)^* G(jw_0) \\ & - \epsilon^*(\gamma_2) G_2(jw_0)^* G_2(jw_0)] x_0 \\ & = (\gamma_1^2 - \gamma_2^2) \|x_0\|^2 < 0 \end{aligned} \quad (27)$$

This contradicts equation (26).

Next, we prove $\epsilon^* \rightarrow 0$, as $\gamma \rightarrow \|G\|_\infty$.

Assume there exist w_0, x_0 and $x_0^* x_0 = 1$ such that

$$x_0^* G(jw_0)^* G(jw_0) x_0 = \|G\|_\infty^2$$

Without lossing generality, we suppose $w_0 \neq \infty$

From (23)

$$\gamma^2 - \|G\|_\infty^2 - \epsilon^*(\gamma) x_0^* G_2(jw_0)^* G_2(jw_0) x_0 \geq 0 \quad (28)$$

It is clear that if $\gamma \rightarrow \|G\|_\infty$, then $\epsilon^*(\gamma) \rightarrow 0$ □

If we replace system (C,A,B,D) with its similar system (CT, T⁻¹AT, T⁻¹B,D), the monotony of $\epsilon^*(\gamma)$ does not change, but the shape of the function $\epsilon^*(\gamma)$ is changed by nonsingular matrix T. In the other words, we can choose a proper T such that the $\epsilon^*(\gamma)$ is expressed by a simple algebraic equation.

Corollary 1⁽³⁾ Consider system (3) with D=0.

Let $\text{Re } \lambda(A) < 0$ and $\|G\|_\infty < \gamma$, then

$$\|G\|_\infty = \frac{\gamma}{(1 + \epsilon^*(\gamma))^{1/2}} \quad (29)$$

where the nonsingular T is choosed as followings:

If rank (C) is full rank, then $T = C^{-1}$

If rank (C) is not full rank, then $T = (C^T C + \delta I)^{-1/2}$

Where δ is a sufficiently small positive number.

We consider system (1) with the following assumption
Assumption 1:

the pair (C_2, A) is detectable,

the pair (A, B_2) is stabilizable,

$$D_{11} = 0, D_{12}^T C_1 = 0, D_{12}^T D_{12} = I$$

Theorem 2. Consider system (1) under the assumption 1. We can obtain the following conclusions

(1) The maximum ϵ^* that satisfies (2) has a solution $P > 0$, is a monotone increasing function of γ .

(2) $\epsilon^* \rightarrow 0$ as $\gamma \rightarrow \|G_F(s)\|_\infty$

(3) Consider the similar system of system (1), by the proper choice of a nonsingular matrix T, we can get

$$\|G_F(s)\|_\infty = \frac{\gamma}{(1 + \epsilon^*(\gamma))^{1/2}} \quad (30)$$

Proof : The result follows by using Theorem 1 and Corollary 1 with some arrangements. □

4 Example

We consider an example for the open-loop system in which $C^T C$ is not full rank.

$$A = \begin{bmatrix} -0.08 & 0.83 & 0 & 0 \\ -0.83 & -0.08 & 0 & 0 \\ 0 & 0 & -0.7 & 9 \\ 0 & 0 & -9 & -0.7 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.4 & 0 & 0.4 & 0 \\ 0.6 & 0 & 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0.3 & 0 \\ 0 & -0.15 \end{bmatrix}$$

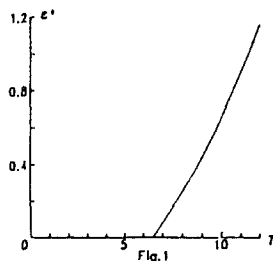
$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The relation between ϵ^* and γ is shown in Fig 1. This relation can be expressed in the following simple algebraic equation.

$$\epsilon^*(\gamma) = 1.03 \times 10^{-2} \gamma^2 + 1.91 \times 10^{-2} \gamma - 0.557 \quad (31)$$

Let $\epsilon^*(\gamma) = 0$, by the property (2) of theorem 2, we can compute $\|G\|_\infty = 6.45$

Here, an example is shown to use our method to compute the H_∞ norm of a transfer function matrix for an open-loop system. It is clear that this method can also be used to compute the H_∞ norm for a closed-loop system.



5. Conclusion

For a H_∞ problem, it is important to know the H_∞ norm of the transfer function matrix of a given system. There are a few methods to compute the H_∞ norm of a transfer function matrix.⁽⁴⁾⁻⁽⁶⁾ In this paper, we put our attention at the two parameters (γ, ϵ^*) of Riccati equation, and some properties are discussed. We found that the $\epsilon^*(\gamma)$ is monotony, and $\epsilon^*(\gamma) \rightarrow 0$ as $\gamma \rightarrow \|G\|_\infty$. The different choice of the similarity transformation of a given system can change the shape of $\epsilon^*(\gamma)$. By a proper choice of the similar system, we obtained a simple algebraic equation to express the relation between γ and ϵ^* , that is, the H_∞ norm of a transfer function can be easily computed by this equation.

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