한국소음진동공학회 '93추계학술대회논문집 pp. 77 ~ 83.

# 2자유도 Hamiltonian계의 Subharmonic Melnikov 해석과 혼돈양상에 대한 연구

박 철 희*, °이 근 수**

# On the Subharmonic Melnikov Analysis and Chaotic Behaviors in a 2-DOF Hamiltonian System

(Chol-Hui Pak, Keun-Soo Lee)

## 1. Introduction

In this paper, the dynamics of a 2-DOF not 1:1 resonant Hamiltonian system are studied. In the first part of the work, the behaviors of special periodic orbits called normal modes are examined by means of the harmonic balance mathod and their approximate stability are analyzed by using the Synge's concept named stability in the kinematico-statical sense[1,2]. Secondly, the global dynamics of the system for low and high energy are studied in terms of a perturbation analysis and Poincare' maps[3]. In this part, one can see that the unstable normal mode generates chaotic motions resulting from the transverse intersections of the stable and unstable manifolds. Although there exist analytic methods for proving the existence of infinitely many periodic orbits, chaos, they cannot be applied in our case and thus, the Poincare' maps constructed by direct numerical integrations are utilized fot detecting chaotic motions. In the last part of the work, the existence of arbitrarily many periodic orbits of the system are proved by using a subharmonic Melnikov's method[4]. We also study the possibility of the breakdown of invariant KAM tori[5] only when $h>h_0$ ($h_0$:bifurcating energy) and investigate the generality of the destruction phenomena of the rational tori in the systems perturbed by stiffness and inertial coupling.

## 2. Nonlinear Normal Mode Vibrations

A normal mode is periodic motion of the system which passes through the origin and which has two rest points. And the formal definition of normal mode was first introduced by Rosengerg[6] who utilize the concept "vibrations in unison". Consider the n-DOF conservative systems and the following equations of motion :

$$\ddot{x}_i + \frac{\partial V(x_i)}{\partial x_i} = 0 \quad , \; i = 1, 2, \cdots, n \qquad (1)$$

* 인하대학교 기계공학과, 정회원
** 인하대학교 기계공학과 대학원

Definition (Rosenberg, 1966)

*A normal mode is a any solution of the equation (1) which is a vibration in unison.*

If the modal curve is straight, then a normal mode is said to be *similar* and *nonsimilar* if curved. The concept of normal modes have very significant meaning since the resonance in forced vabrations occurs when the forcing frequency lies to the natural frequencies and the system vibrates in normal modes in the neighborhood of resonance[7]. In this section, the behaviors of approximate normal modes are investigated by using the harmonic balance method. This method is applicable to strongly nonlinear system and only first term approximation of harmonics in the Fourier series expansion gives a good result. Thus the normal modes will be approximated by considering just first term of harmonics in this section. Complete explanation can be found in [7]. Now consider the following equations of motion.

$$\begin{aligned} \ddot{x}_1 + x_1 + x_1^3 + \alpha_1(x_1-x_2) + \beta_3(x_1-x_3)^3 &= 0 \\ \ddot{x}_2 + x_2 + x_2^3 - \alpha_1(x_1-x_2) - \beta_3(x_1-x_2)^3 &= 0 \end{aligned} \qquad (2)$$

where the kinetic and potential energy is given by

$$\begin{aligned} T &= \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) \\ V &= \frac{1}{2}(x_1^2+x_2^2)+\frac{1}{4}(x_1^4+x_2^4)+\frac{\alpha_1}{2}(x_1-x_2)^2+\frac{\beta_3}{4}(x_1-x_2)^4 \end{aligned} \qquad (3)$$

Let us call the above equation system(I). By assuming that the solution is the first term in the Fourier series,

$$x = A\sin\omega t\,, \quad y = B\sin\omega t \qquad (4)$$

the following equation is obtained :

$$\alpha_1(A^2-B^2)+\frac{3}{4}(2\beta_3-1)(AB^3-A^3B)+\frac{3}{4}\beta_3(A^4-B^4)=0 \quad (5)$$

Because the modal curves are assumed to be straight , Eq.(5) can be transformed into polar coordinate through :

$$A = r\cos\theta\,, \; B = r\sin\theta \qquad (6)$$

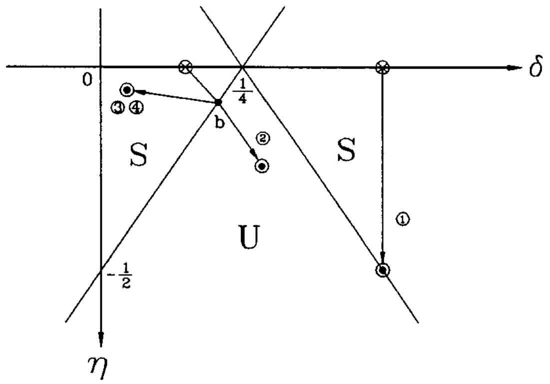


Fig.1 Strutt Chart ($\alpha_1$=0.5, $\beta_3$=0.1), b : bifurcation point

where $\theta$ is the angle between modal curve and $x$-axis and $r$ is the amplitude of normal mode. Then,

$$f(p) + g(p)r^2 = 0 \tag{7}$$

where

$$p = \tan\theta$$

$$f(p) = \alpha_1(1-p^4)$$

$$g(p) = \frac{3}{4}\{(1-2\beta_3)p(1-p^2) + \beta_3(1-p^4)\} \tag{8}$$

For sufficiently small energy, there are two similar normal modes.

$$p = +1 \quad : \quad \text{Symmetric } NM\ (x = y)$$

$$p = -1 \quad : \quad \text{Anti-symmetric } NM\ (x = -y) \tag{9}$$

And for large energy, from $g(p) = 0$, there may exist two other normal modes which are bifurcating out from $x = y$, or $x = -y$. From $g(p) = 0$, the values of $p$ corresponding to the bifurcating modes can be evaluated as follows :

$$p = \frac{-1 \pm 2\beta_3 + (1-4\beta_3)^{1/2}}{2\beta_3}, \quad \beta_3 < \frac{1}{4} \tag{10}$$

Thus, one can see that $\beta_3 = 1/4$ is a bifurcating point for sufficiently large energy. In fact, these bifurcating two modes are nonsimilar normal modes but it was shown that the bifurcating nonsimilar modes are close to straight line (similar mode) by Anand[8].

The approximate stability of the above normal modes can be analyzed via Synge's stability concept[1,2]. Synge, in a paper of 1926, has introduced the concept of stability called the stability in the kinematico-statical sense. This stability concept is equivalent to the orbital stability, as usual in the phase space. The governing equation for the stability can be oatained as follows :

$$\beta'' + (\delta + \eta\cos\tau)\beta = 0 \tag{11}$$

where $\delta = B_2/B_1$ , $\eta = B_3/B_1$ and where

$$B_1 = 4\omega^2(A^2+B^2)$$

$$B_2 = 2\alpha_1 AB + 3A^2B^2 + (1+\alpha_1)(A^2+B^2) + \frac{3}{2}\beta_3(A^4+B^4-2A^2B^2) \tag{12}$$

$$B_3 = -\left\{3A^2B^2 + \frac{3}{2}\beta_3(A^4+B^4-2A^2B^2)\right\}$$

The above equation is well known mathieu equation and the strutt chart for $\beta_3$=0.1 is plotted at Fig.1. Indeed, the anti-symmetric mode is unstable and both symmetric and bifurcating modes are stable for the case $\beta_3 < 1/4$ as the energy increases. The bifurcating energy $h_0$ in this case can be easily calculated and the value is $h_0 \fallingdotseq 1.73$.

## 3. Global Dynamics

### 3.1 Perturbation Analysis

In this section, we use the two variable expansion perturbation method[9] to study the dynamics of the system for low energies. In order for the perturbation method to be valid, one must assume that the nonlinear system neighbors a linear one. Thus the system under consideration is slightly modified of the following form :

$$\ddot{x}_1 + x_1 + \varepsilon x_1^3 + \varepsilon\alpha_1(x_1-x_2) + \varepsilon\beta_3(x_1-x_2)^3 = 0$$

$$\ddot{x}_2 + x_2 + \varepsilon x_2^3 - \varepsilon\alpha_1(x_1-x_2) - \varepsilon\beta_3(x_1-x_2)^3 = 0 \tag{13}$$

where $\varepsilon \ll 1$ is a small parameter.

Note that when $\varepsilon = 0$, the system is in 1:1 resonance and degenerates into two linear oscillators. Since the system have the two similar normal modes before bifurcation, one can replace $x_1$ and $x_2$ by new coordinate $x$, $y$ as follows :

$$x = x_1 + x_2, \qquad y = x_1 - x_2 \tag{14}$$

Then, Eq.(13) becomes

$$\ddot{x} + x + \varepsilon\left(\frac{1}{4}x^3 + \frac{3}{4}xy^2\right) = 0$$

$$\ddot{y} + (1+\varepsilon\Delta)y + \varepsilon\left\{(2\beta_3+\frac{1}{4})y^3 + \frac{3}{4}x^2y\right\} = 0 \tag{15}$$

and where $\Delta = 2\alpha_1$ playing a role of detuning parameter.

Next we replace t as independent variables :

$$\xi = t, \quad \eta = \varepsilon t \tag{16}$$

Then, the following averaged equations are obtained :

$$\frac{dR}{d\eta} = 0$$

$$\frac{d\psi}{d\eta} = -\frac{3}{64}R^2\sin 2\phi \sin 2\psi$$

$$\frac{d\phi}{d\eta} = -\frac{\Delta}{2} - \frac{3}{8}R^2[(2\beta_3+\frac{1}{2})\sin^2\psi \frac{1}{4}\cos^2\psi] - \frac{3}{32}R_2^2\cos\psi(\cos\phi+2) \tag{17}$$

The polar transformations are used for deriving the above equations. The first equation indicates that the total energy of the system is conserved during free vibration and this energy conservation holds only for this order of approximation.

$$R_1^2 + R_2^2 = R^2 = const. \tag{18}$$

The Eq.(17) can be integrated exactly[10] and the first integral of the motion is :

$$\begin{aligned} K(\psi,\phi) = & \left(-\frac{\Delta}{R^2}-\frac{3}{4}\beta_3\right)\cos 2\psi \\ & +\left(\frac{3}{16}\beta_3-\frac{3}{64}-\frac{3}{64}\cos 2\phi\right)\cos 4\psi \\ & +\frac{3}{32}\cos^2\phi = constant. \end{aligned} \tag{19}$$

Thus, for given parameters, the level curves of the integral (19) may be plotted on the $\psi$ - $\phi$ phase plane. The line $\psi$ = 0, $\pi/2$ are excluded since on these lines $R_2$ and $R_1$ vanishes and $\theta_1$, $\theta_2(\theta_2-\theta_1=\phi)$ are not defined. In fact, these lines correspond to the x-mode(y = 0), or y-mode(x = 0), and in the original system they indicate symmetric($x_1$ = $x_2$) and anti-symmetric($x_1$ = $-x_2$) modes. The level curves are plotted in Fig.2 for the specific parameter value. Fig.2 shows that the stability of symmetric mode is stable, since it appears to be surrounded by closed curves. And the two bifurcating modes are stable since they also appear as centers. Fixed points corresponding to the bifurcating modes are :

$$\begin{aligned} (\psi,\phi) = & \left\{\frac{1}{2}\cos^{-1}\left(\frac{4\beta_3 + 8\Delta/3R^2}{2\beta_3 - 1}\right), 0\right\}, \\ & \left\{\frac{1}{2}\cos^{-1}\left(\frac{4\beta_3 + 8\Delta/3R^2}{2\beta_3 - 1}\right), \pi\right\} \end{aligned} \tag{20}$$

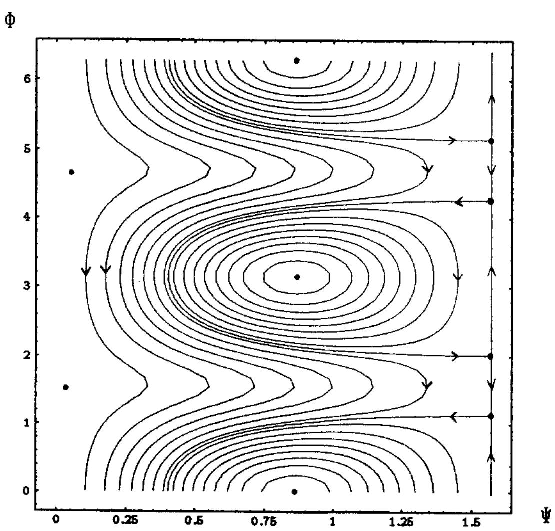

Fig.2 Phase Plot (R = 5.0, Δ = 1.0, $\beta_3$ = 0.02)

The stability mentioned above is indeed orbital stability (orbital stability is usually defined in the phase space). Evidently, the anti-symmetric mode become unstable after pitchfork bifurcation as shown in Fig.2. For more informations about this methodology, see Ref.[9,10].

## 3.2 Failure the Homoclinic Melnikov's Method

In the previous section, the dynamics of system(I) for low energies were investigated by means of the perturbation method and the homoclinic orbit was found in the averaged equations as the energy and nonlinear parameter vary. As the energy increases, the stable and unstable manifolds of the homoclinic orbit intesect transversely and the existence of an infinity of transverse homoclinic intersections can be proved by using the homoclinic Melnikov analysis[11]. Then the Smale-Birkoff homoclinic theorem can be applied to show the existence of Smale-horseshoes in its dynamics.

Theorem(Smale-Birkoff homoclinic theorem)
*Let $f: R^n \rightarrow R^n$ be a diffeomorphism such that p is a hyperbolic fixed point and there exists a point $q \neq p$ of transversal intersection between $W^s(p)$ and $W^u(p)$. Then f has a hyperbolic invariant set $\Lambda$ on which f is topologically equivalent to a subshift of finite type.*

In the above thoerem, the invariant set $\Lambda$ contains :
- A countable infinity of periodic orbits.
- An uncountable infinity of bounded nonperiodic motions.
- A dense orbits.

Moreover, the map $f|_\Lambda$ structurally stable.

This is the brief scenario for proving that the system posesses no global analytic second integral. But in the present case, one cannot prove analytically the existence of Smale-horseshoes and thus, chaotic motions since the unperturbed Hamiltonian of the system(I) has no such homoclinic orbits. Therefore, the only possible way for detecting the transverse intersections of invariant manifolds is by direct numerical integrations of the equations of motion. In Ref.[12], the transverse homoclinic intersections of stable and unstable manifolds were computed by using numerical integrations and as initial values, he used the linear eigenspace about each fixed point but computation of transverse intersections was not performed in this paper.

## 3.3 Poincare' Maps

In this section, a Poincare' map of a flow which is one of the powerful technique for studying flows in nonlinear dynamics was constructed and the standard method developed by Month and Rand[3] was followed.
The Poincare' section Σ can be defined by :

$$\Sigma = \{x_1 = 0, \dot{x}_1 > 0\} \cap \{H = h\} \tag{21}$$

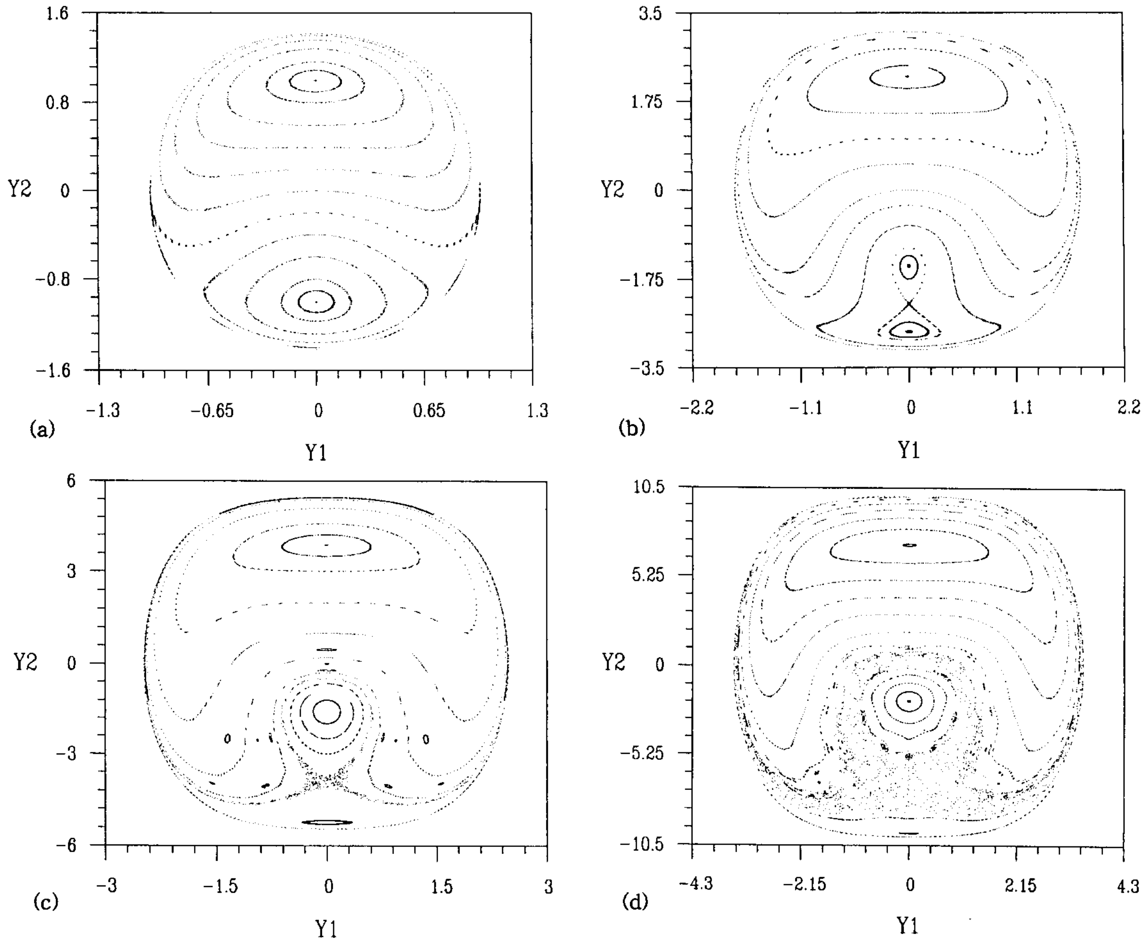


Fig.3 Poincare' Maps($\beta_3$ = 0.1 < 1/4), h : (a) 1.0 (b) 5.0 (c) 15.0 (d) 50.0

The choice of a particular $\Sigma$ is not too critical since any two such maps are topologically equivalent. The Poincare' map is illustrated at Fig.3 and for this, the equations of motion were numerically integrated with a 4th order Runge-Kutta algorithm. The nonlinear parameter $\beta_3$ = 0.1 was used for detecting a large and small scale chaotic motions as the energy increases. From the examination of the Poincare' maps, one can see that the unstable normal mode generates the global chaotic motions. And one can also detect the subharmonic motions generated from the breakdown of invariant KAM tori and the existence of subharmonic orbits will be proved in terms of analytic method.

## 4. Subharmonic Melnikov Analysis

### 4.1 Melinkov's Method : Global Perturbations

Consider 2-DOF Hamiltonan Systems of the form

$$H^{\varepsilon}(q,p) = H_0(q,p) + \varepsilon H^1(q,p) \tag{22}$$

where the unperturbed Hamiltonian $H_0 = F_1 + F_2$.
For $\varepsilon$ = 0, the system $H_0$ is completely integrable since it degenerates into two SDOF oscillators. It is possible to reduce the 2-dimensional Hamltonian system as follows :

$$\begin{aligned} \frac{dq_1}{d\theta_2} &= -\frac{\partial L^0}{\partial p_1} - \varepsilon\frac{\partial L^1}{\partial p_1} + O(\varepsilon^2) \\ \frac{dp_1}{d\theta_2} &= \frac{\partial L^0}{\partial q_1} + \varepsilon\frac{\partial L^1}{\partial q_1} + O(\varepsilon^2) \end{aligned} \tag{23}$$

where

$$\begin{aligned} L^0 &= F_2^{-1}(h - F_1(q_1,p_1)) \\ L^1 &= -\frac{H^1(q_1,p_1,\theta_2;L^0(q_1,p_1;h))}{\Omega(L^0(q_1,p_1;h))} \end{aligned} \tag{24}$$

Since $H^1$ is $2\pi$ periodic in $\theta_2$, so is $L^1$, and the reduced system (23) is in the form of a periodically perturbed system, one can apply the Melnikov theory for subharmonic motions[4]. And the subharmonic Melnikov function is given by :

$$M(\theta_0) = \int_0^{2\pi m} \{L^0, L^1\}(q_1(\theta_2), p_1(\theta_2), \theta_2+\theta_0 ; h)d\theta_2 \quad (25)$$

where $\{\bullet, \bullet\}$ is the poisson bracket.

A direct application of the Melnikov's method gives the following result.

Theorem (Veerman, 1985) :

*Fix $h>0$, $m$, $n$ relatively prime integer and choose $\varepsilon$ sufficiently small. Then, if $M(\theta_0, m, n, h)$ has simple zeros as a function of $\theta_0$ in $[0, 2\pi m/n)$ (or $M(t_0, m, n, h)$ as a function of $t_0$ in $[0, T_1]$), the resonant torus given by $(q_1, p_1, \theta_2) = (q_1(\theta_2-\theta_0), p_1(\theta_2-\theta_0), \theta_2)$ breaks into $2k=j/m$ distinct $2\pi m$-periodic orbits and there are no other $2\pi m$-periodic orbit in its neighberhood.*

It was proven that each $2\pi m$-periodic orbit pierces the Poincare' section at $\theta_2=0$ precisely $m$ times before closing up and for sufficiently small $\varepsilon$, precisely $k$ of pericdic orbit are hyperbolic and $k$ are elliptc[4]. For details, see Ref.[4,13].

## 4.2 Application of the Method for System(I)

Now the outlined theory reviewed in the previous section can be applied for the system(I). For this, let us assume that in the system(I) the linear and nonlinear coupling are weak. Then the system(I) can be written as follows :

$$\begin{aligned} \ddot{x}_1 + x_1 + x_1^3 + \varepsilon\alpha_1(x_1-x_2) + \varepsilon\beta_3(x_1-x_2)^3 &= 0 \\ \ddot{x}_2 + x_2 + x_2^3 - \varepsilon\alpha_1(x_1-x_2) - \varepsilon\beta_3(x_1-x_2)^3 &= 0 \end{aligned} \quad (26)$$

And corresponding Hamiltonian is

$$\begin{aligned} H^{\varepsilon}(q,p) &= \frac{1}{2}(p_1^2+p_2^2) + \frac{1}{2}(q_1^2+q_2^2) + \frac{1}{4}(q_1^4+q_2^4) \\ &\quad + \varepsilon\frac{\alpha_1}{2}(q_1-q_2)^2 + \varepsilon\frac{\beta_3}{4}(q_1-q_2)^4 \\ &\equiv H_0(q,p) + \varepsilon H^1(q,p) \end{aligned} \quad (27)$$

where $(q_1,q_2)=(x_1,x_2)$, $(p_1,p_2)=(\dot{x}_1,\dot{x}_2)$ are generalized coordinates and momenta and where

$$\begin{aligned} H_0(q,p) &\equiv F_1(q_1,p_2) + F_2(q_2,p_2) \\ &= \frac{p_i^2}{2} + \frac{q_i^2}{2} + \frac{q_i^4}{4}, \quad i=1,2 \end{aligned} \quad (28)$$

The periodic orbits of the unperturbed system can be evaluated in terms of Jacobi elliptic functions[14].

$$\begin{aligned} q_i &= X_i cn[(1+X_i^2)^{1/2}t, k_i], \\ k_i^2 &= \frac{X_i^2}{2(1+X_i^2)}, \quad i = 1,2 \end{aligned} \quad (29)$$

where initial conditions $q_i(0)=X_i$, $\dot{q}_i(0)=p_i(0)=0$ and where $cn[\bullet, \bullet]$ is the elliptic cosine and $k_i$ is the elliptic modulus. The resonance relationship $nT_1=mT_2$ or $\omega_2/\omega_1=m/n$ leads to :

$$\frac{nK\left(\frac{(H_1^{1/4}-1)^{1/2}}{\sqrt{2}H_1^{1/4}}\right)}{H_1^{1/4}} = \frac{mK\left(\frac{(\overline{H}_1^{1/4}-1)^{1/2}}{\sqrt{2}\,\overline{H}_1^{1/4}}\right)}{\overline{H}_1^{1/4}} \quad (30)$$

where $H_1=1+4h_1$, $\overline{H}_1=1+4(h-h_1)$ and $h=h_1+h_2$.

In this equation $K[\bullet]$ is the complete elliptic integral of the first kind. It was shown that for fixed $h$,$m$ and $n$, in a certain range, Eq.(30) gives a unique solution for $h_1$[4]. Thus, in the following range, there exists a unique $h_1$ for given m,n and $h$ :

$$\begin{aligned} mK\left(\frac{((1+4h)^{1/2}-1)^{1/2}}{\sqrt{2}(1+4h)^{1/4}}\right)/(1+4h)^{1/4} &\leq nK(0) \\ nK\left(\frac{((1+4h)^{1/2}-1)^{1/2}}{\sqrt{2}(1+4h)^{1/4}}\right)/(1+4h)^{1/4} &\leq mK(0) \end{aligned} \quad (31)$$

If the value of $m/n$ is out of the range, the unique $h_1$ may not exist in $[0,h)$ and the perturbation analysis is not valid. Now the Melnikov function can be evaluated.

$$\begin{aligned} M(t_0) &= \frac{T_2}{2\pi}\int_{-mT_2/2}^{mT_2/2}\{F_1, H_1\}(t+t_0)dt \\ &= \frac{T_2}{2\pi}\int_{-mT_2/2}^{mT_2/2} -p_1(t)\,[\alpha_1(q_1(t)-q_2(t+t_0))+\beta_3(q_1^3(t) \\ &\quad -3q_1^2(t)q_2(t+t_0)+3q_1(t)q_2^2(t+t_0)-q_2^3(t+t_0))]dt \end{aligned} \quad (32)$$

Since $p_1(t)$ is even and $q_1(t)$ is odd function of time, the integrals of the term $p_1q_1$ and $p_1q_1^3$ vanish.

The integral (32) is very difficult to evaluate but following arguments are possible. For each $t_0 = kT_2/2 = knT_1/2m$, $q_2(t+t_0)$ becomes an even function of $t$ as follows :

$$q_2(t+t_0) = q_2(t+t_0) = (-1)^k q_2(t) \quad (33)$$

Thus, the integral (32) vanishes for $t_0 = kT_2/2$ and the following is obtained :

$$M(t_0 = \frac{kT_2}{2}, m, n, h) = 0 \quad (34)$$

Next, one has to show that the zeros of the Melnikov function are simple in oder to satisfy the aforementioned theorem :

$$\frac{dM(t_0, m, n, h)}{dt_0}\Big|_{t_0=kT_2/2} \neq 0 \quad (35)$$

Here, only final results will be given.

$$\begin{aligned} &M'(t_0=\frac{kT_2}{2}) = \\ &\frac{(-1)^k mT_2^2}{4\pi}\sum_{j,l=0}^{\infty}\left[\alpha_1\Lambda_1^j\Lambda_2^l+\beta_3(\Gamma_1^j\Lambda_2^l-(-1)^k\frac{3}{2}\Delta_1^j\Delta_2^l+\Lambda_1^j\Gamma_2^l\right] \end{aligned} \quad (36)$$

After some manipulations, one can obtain the Melnikov function as a function of $t_0$[4] and the results are :

$$M(t_0) =$$

$$\frac{mT_1T_2^2}{8\pi^2}\sum_{\mu=0}^{\infty}(2\mu+1)\left[\alpha_1\Lambda_1^{\mu}\Lambda_2^{\mu}+\beta_3\left(\Gamma_1^{\mu}\Lambda_2^{\mu}-\frac{3}{2}\Delta_1^{\mu}\Delta_2^{\mu}+\Lambda_1^{\mu}\Gamma_2^{\mu}\right)\right] \quad (37)$$

$$\times \sin\left[\frac{(2\mu+1)2\pi m t_0}{T_1}\right]$$

The $M$ and $M'$ is not identically zero with the exception of the following case :

$$\alpha_1\Lambda_1^{\mu}\Lambda_2^{\mu} + \beta_3\left(\Gamma_1^{\mu}\Lambda_2^{\mu} - \frac{3}{2}\Delta_1^{\mu}\Delta_2^{\mu} + \Lambda_1^{\mu}\Gamma_2^{\mu}\right) = 0 \quad (38)$$

In the above expression, summation was omitted. Therefore as mentioned earlier, for fixed $m$, $n$ and $h$, the Melnikov function (37) has simple zeros at $t_0 = kT_2/2$ only excepting for the case (38) and from the aforementioned theorem in the previous section, one can see that the resonant torus breaks into two distinct, $2m$ periodic orbits for the fixed energy level $h$.

## 4.3 Application of the Method for S(II)

As a second example, consider the following 2-DOF Hamiltonian system :

$$\begin{aligned} H^{\varepsilon}(q_i,p_i) &= \frac{1}{2}(p_1^2+p_2^2)+\frac{1}{2}(q_1^2+aq_2^2)+\frac{1}{4}(q_1^4+bq_2^4) \\ &\quad +\varepsilon\frac{\alpha_2}{2}q_2^2p_1^2+\varepsilon\frac{\beta_2}{2}q_1^2q_2^2 \qquad (39) \\ &\equiv H_0(q,p) + \varepsilon H^1(q,p) \end{aligned}$$

where $H_0(q,p) \equiv F_1(q_1,p_1) + F_2(q_2,p_2)$

$$\begin{aligned} F_1(q_1,p_1) &= \frac{1}{2}p_1^2 + \frac{1}{2}q_1^2 + \frac{1}{4}q_1^4 \\ F_2(q_2,p_2) &= \frac{1}{2}p_2^2 + \frac{1}{2}aq_2^2 + \frac{1}{4}bp_2^4 \end{aligned} \qquad (40)$$

The above Hamiltonian system is somewhat generalized in the sense that the unperturbed system, $F_1$ and $F_2$, are not restricted by linearized natural frequencies and inertial coupling is considered unlike the system analyzed in the previous section. Let us call the Hamiltonian (39) system(II). A same methodology utilized in the section 4.2 will be applied to prove the existence of arbitrarily many subharmonic orbits of the system(II). The following resonance condition has to satisfy in order for the system $F_1$ and $F_2$ be integrably related :

$$\frac{\alpha^{1/2}}{K(0)}\frac{K(k_1(h))}{(1+4h)^{1/4}} \le \frac{m}{n} \le \frac{K(0)(\alpha^2+4h/b)^{1/4}}{K(k_2(h))} \quad (41)$$

where

$$\begin{aligned} k_1(h) &= \frac{((1+4h)^{1/2}-1)}{\sqrt{2}(1+4h)^{1/4}}, \\ k_2(h) &= \frac{((\alpha^2+4h/b)^{1/2}-1)}{\sqrt{2}(\alpha^2+4h/b)^{1/4}} \end{aligned} \qquad (42)$$

and where $\alpha=a/b$.

The subharmonic Melnikov function can be calculated as follows :

$$M(t_0) =$$

$$\frac{T_1T_2^2}{16\pi^2}(\alpha_2-\beta_2)\sum_{\mu=0}^{\infty}\frac{\Delta_1^{(\mu)}\Delta_2^{(\mu)}}{(2\mu+1)}\sin\left[\frac{(2\mu+1)2\pi m t_0}{T_1}\right] \quad (43)$$

Thus, for the fixed $m$, $n$, and $h$, the Melnikov function has $2m$ simple zeros at $t_0=kT_2/2$ with the exception of the following case :

$$\alpha_2 - \beta_2 = 0 \quad (44)$$

Now main result of this paper can be readily stated as follows.

Theorem

*Suppose that $m$, $n$ are relatively prime integers and for fixed $H^{\varepsilon} = h>0$, the inequalities (31),(41) are satisfied. Then, for $\varepsilon$ sufficiently small, the resonant torus of the system(I),(II) breaks into $2\pi m$-periodic orbits and this situation occurs in general with the exception of the degenerating case (38) and (44) on the every energy level $H^{\varepsilon} = h>0$.*

*Remarks.* 1.The values of $M'(t_0)$ can be obtained approximately considering only first term ($\mu = 0$) and this approximation gives a good result since the series converge rapidly. 2.Although there exist infinitely many periodic orbits, one cannot prove their existence because $M'(t_0)$(and also $M(t_0)$) vanishes as $m,n\to\infty$. Thus only finite number of periodic orbits in the neighborhood of resonant torus can be found.

# 5. Concluding remarks

The dynamics of a 2-DOF Hamiltonian system with 4th order nonlinear coupling were investigated and the main results of the work can be stated as follows.

- There exist two stable similar normal modes on the low energy level and as the energy increases, antisymmetric mode becomes unstable via pitchfork bifurcation.

- The global dynamics were analyzed by means of both analytical and numerical techniques. And all the aforementioned analytic results were verified via Poincare' maps.

- Subharmonic Melnikov analysis were performed and from this, it was shown that the destruction phenomena of invariant KAM tori occurs in general only excepting for the degenerating cases.

Finally, the following topics are possible as extensions of this work.

- Analytic prove the existence of secondary islands together with numerical verification.

- Destruction of irrational KAM tori.

- Forced and Damped vibrations : strange attractor.

- Quantitative analysis : Lyapunov exponents.

- Mode analysis for system(II).

## REFERENCES


1. Synge J. L., 1926. On the Geometry of Dynamics. Philosofical Transcations A, Vol 226, pp. 31-106.

2. Pak C. H., 1989. On the Stability Behavior of Bifurcated Normal Modes in Coupled Nonlinear Systems. J.Applied Mechanics, Vol. 56, pp. 155-161.

3. Month L. A., Rand R. H., 1980. An Application of The Poincare Map to The Stability of Nonlinear Normal Modes. J.Applied Mechanics, Vol 47, pp. 645-651.

4. Veerman P., Holmes P., 1985. The Existence of Arbitrarily Many Distinct Periodic Orbits in a Two Degree of FreedomHamiltonian System. Physica 14D, pp. 177-192.

5. Lichtenberg A. J., Libermann M. A., 1983. Regular and Stochastic Motion. Springer-Verlag, New York.

6. Rosenberg R. M., 1966. On Nonlinear Vibrations of Systems with Many Degrees of Freedom. Advances in Applied Mechanics, Vol. 9, pp. 155-242.

7. Yun Y. S., 1990. On the Stability Relation Between Free and Forced Vibrations in Nonlinear Coupled Oscillators. Ph.D Thesis, Inha University.

8. Anand G. A., 1972. Natural Modes of Coupled Nonlinear Systems. International J. Nonlinear Mechanics, Vol. 7, pp. 81-91.

9. Nayfeh A. H., Mook D. T., 1979. Nonlinear Oscillations. John Wiley & Sons, New York.

10. Rand R. H., Pak C. H., Vakakis A. F., 1992. Bifurcation of Nonlinear Normal Modes in a Class of Two Degree of Freedom Systems. Acta Mechanica 3, pp. 129-145.

11.Holmes P, Marsden J., 1982. Horseshoes in Perturbations of Hamiltonian Systems with Two Degrees of Freedom. Comm.Math.Phys. 82, pp. 523-544.

12. Vakakis A. F., 1991. Analysis and Identification of Linear and Nonlinear Normal Modes in Vibrating Systems. Ph.D Thesis, California Institute of Technology.

13. Guckenheimer .J, Holmes P., 1983. Nonlonear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer-Verlag, New York.

14. Byrd P. F., Friedman M. D., 1971. Handbook of Elliptic Integrals for Engineers and Scientists, 2nd ed. Springer-Verlag, New York.