Spectral Analysis of Random Process

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Abstract: *The spectrum estimation methods of random processes are expressed in this paper. Beginning with the basic theory, non-parametric and parametric methods are overviewed.*

As to non-parametric method, numerical calculation method is also discussed. As to parametric method, AR model is a very famous and effective model representing random process. Estimation methods of AR parameters which have been proposed are mentioned here.

Wavelet analysis is a recently interested technique in signal processing. An application of wavelet analysis is also shown.

1 Introduction

Spectral analysis is a very effective method to find the special features of the observed signal. So spectral analysis has been applied in many engineering fields since old times.

The observed signals in several fields are more or less random. This means that the wave form is different whenever we observe the same signal.

Mathmatically a random signal is expressed as a stochastic process. Spectral analysis is useful techinique for a weakly stationaly stochastic process. To find the power density spectrum is the spectral analysis for the process.

In practical analysis, it is necessary to calculate the spectrum numerically, and the calculation procedures were developed till now.

There is another problem in spectral analysis of random process, because statistical estimation is necessary for the analysis. From this, parametric method is proposed. Many estimation methods of AR parameters are developed. It is shown that the parametric method is especially useful for spectral analysis of a small number of data.

Wavelet analysis is a recently interested technique, an example of this method is shown in this paper.

2 Basic Theory of Spectral Analysis [1~4]

2.1 Power spectral density of stochastic process

Let $x(t)$ be an elgotic weakly stationary stochastic process with zero mean value.

That is,

$$E[x(t)] = 0 \quad , \quad E[x(t)x(s)] = \phi_x(t-s)$$

Where the correlation function is a function of time interval $(t-s)$. This means that the $x(t)$ is a weakly stationary stochastic process. And $E[\cdot]$ means averaging on the probability space.

Put $t - s = \tau$, *and the correlation function becomes a function of only* τ. *Performing the Fourier Transform to* $\phi_x(\tau)$, *we can get the power spectral density* $\Phi_x(\omega)$.

$$\begin{aligned}\Phi_x(\omega) &= F[\phi_x(\tau)] \\ &= \int_{-\infty}^{\infty} \phi_x(\tau) e^{-j\omega\tau} d\tau \end{aligned} \quad (1)$$

Power spectral density $\Phi_x(\omega)$ *shows the density of the power at an angular frequency* ω.

From the orthogonarity, the following Parseval's equality holds.

$$E[x^2(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_x(\omega) d\omega \quad (2)$$

To obtain the correlation function $\phi_x(\tau)$, *we calculate it in time average under the assumption of elgodicity.*

$$\phi_x(\tau) \approx \frac{1}{T} \int_0^T x(t)x(t+\tau) dt \quad (3)$$

Where time interval $(0, T)$ *is observing interval of the signal* $x(t)$, *and it is taken enough long. This procedure obtaining power spectral density is called Blackman-Tukey method.*

As well known, FFT(Fast Fourier Transform) was found as an effective method to calculate Fourier Transform. By applying FFT, the procedure obtaining power spectral density was changed from Blackman-Tukey method to FFT method.

At first, we calculate the Fourier Transform of $x(t)$.

$$\begin{aligned}X_T(j\omega) &= F[x_T(t)] \\ &= \int_{-\infty}^{\infty} x_T(t) e^{-j\omega t} dt \end{aligned} \quad (4)$$

where T *means the time interval of* $x(t)$.

From this, we can obtain the power spectral density of a stochastic process as follows:

$$\Phi_x(\omega) = E[\lim_{T\to\infty} \frac{1}{T} |X_T(j\omega)|^2] \quad (5)$$

The important point is that the averaging process is necessary even if we calculate Fourier Transform for long time interval (idealy infinite interval)[1,2].

These procedures are shown in Fig.1.

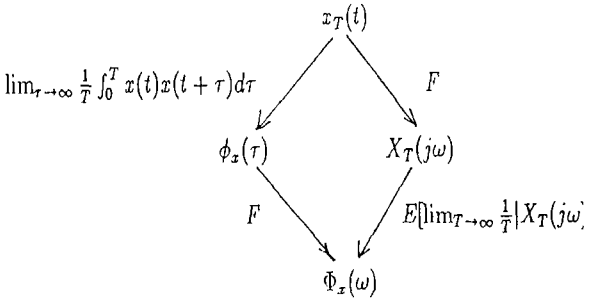


Figure 1: Two non-parametric procedures to obtain power spectral density

2.2 Numerical calculation to obtain power spectral density

To obtain the power spectral density, rather complex calculations would be necessary. So we must apply numerical calculation by digital computer after digitalizing the given stochastic process.

The stochastic process $x(t)$ *is transformed to a time series* $\{x_k\}$ *according to the well known Sampling Theorem.*

The Fourier Transform of $x(t)$ *is calculated numerically as follows:*

$$\hat{X}(j2\pi f) = \frac{\Delta}{2}[x_0 + 2\sum_{i=1}^{n-1} x_i e^{-j2i\pi f\Delta} + x_n e^{-j2n\pi f}$$

$$f = \frac{\omega}{2\pi}$$

In this calculation, the values of $\hat{X}$ are obtained for $f = k/n\Delta (k = 0, 1, 2, \cdots, n/2)$.

$$\hat{X}_k = \frac{\Delta}{2}[x_0 + 2\sum_{i=1}^{n-1} x_i e^{-j\frac{2\pi ik}{n}} + x_n] \quad (7)$$

To obtain the power spectral density, we must perform averaging process for $|\hat{X}_k|^2$.

We acquire the time series repeatedly, calculate the power spectral density for each time series and average them. Moreover it is also useful method to average in frequency domain .[1,2]

3 Non-parametric Method in Spectral Analysis [2~4]

3.1 AR(Auto-Regressive) model

In many cases, randomly changing time series $\{x_k\}(k = 1, 2, \cdots, N)$ can be represented by the following AR model:

$$x_n + a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_p x_{n-p} = e_n \quad (8)$$

where $\{e_n\}$ is white noise.

$$E[e_n] = 0 \quad , \quad E[e_n e_m] = D\delta_{nm}$$

In this model, the order p of the model and the parameters $(a_1, a_2, \cdots, a_p)$ are chosen as the model fits to the observed time series. When the order is defined, the parameters are obtained by the following Yule-Walker's equation.

$$\begin{pmatrix} 1 & \rho_1 & \cdots & \rho_{p-1} \\ \rho_1 & 1 & \cdots & \rho_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{p-1} & \rho_{p-2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{pmatrix} = \begin{pmatrix} -\rho_1 \\ -\rho_2 \\ \vdots \\ -\rho_p \end{pmatrix} \quad (9)$$

where $\rho = \phi_k/\phi_0$, $\phi_k = E[x_{n+k}x_n]$

The important point is that the coefficient matrix of Yule-Walker's equation is a Toepritz matrix. From this fact, many efficient calculation procedures are proposed as mentioned later in this paper.

Another method to find the parameters is to apply Least-Square Method.

Put

$$J = \sum_{n=p+1}^{N} (x_n + a_1 x_{n-1} + \cdots + a_p x_{n-p})^2 = \sum_{n=p+1}^{N} e_n^2 \quad (10)$$

Calculating the derivatives of J by the coefficient, we can obtain the following equation:

$$\hat{\phi}_k + a_1\hat{\phi}_{k-1} + \cdots + a_p\hat{\phi}_{k-p} = 0 \quad (11)$$

$$(k = 1, 2, \cdots, p)$$

where

$$\hat{\phi}_k = \frac{1}{N-k}\sum_{n=1}^{N-k} x_n x_{n+k} \quad (12)$$

Equation (11) is called Normal Equation.

When we get Yule-Walker's equation, we must find the coefficient matrix from the observed time series.

$$\rho_k \approx \frac{\hat{\phi}_k}{\hat{\phi}_0}$$

Then Yule-Walker's equation is the same as Normal equation.

As to choosing of the order of AR model, FPE and AIC proposed by AKAIKE are very famous .[3,4] *It is also practical method that we choose the order by watching the decreasing rate of J according to increasing of the order .*[2]

3.2 Recursive estimation of the AR parameters

We can derive the recursive estimation procedure of AR parameters, because the coefficient matrix of Yule-Walker's equation is Toepritz.

The algorithm is called Levinson-Durbin's algorithm which is recursive as to the order of the AR model.

The parameters of AR model with order (m+1) are obtained from the parameters of AR model with order m.

$$\begin{cases} a_j(m+1) = a_j(m) + a_{m+1}(m+1)a_{m-j+1}(m) \\ \qquad (j = 1, 2, \cdots, m) \\ a_{m+1}(m+1) = -\dfrac{\rho_{m+1} + \sum_{i=1}^{m} a_i(m)\rho_{m-i+1}}{1 + \sum_{i=1}^{m} a_i(m)\rho_i} \end{cases} \tag{13}$$

On the other hand, the estimation method recursive as to the data is also obtained. The basic method is recursive least square method.

Put

$$\mathrm{a} = [a_1, a_2, \cdots, a_p]^T$$
$$\mathrm{x} = [-x_{n-1}, -x_{n-2}, \cdots, -x_{n-p}]^T$$

then AR model is rewritten as follows:

$$x_n = \mathrm{a}^T \mathrm{x}_n + e_n \tag{14}$$

Let the parameters obtained by the data from x_{1-p} to x_N be a_N, then the recursive equation becomes as follows:

$$\begin{cases} \hat{\mathrm{a}}_N = \hat{\mathrm{a}}_{N-1} + \mathrm{K}_N \left(x_N - \mathrm{x}_N^T \hat{\mathrm{a}}_{N-1}\right) \\ \mathrm{k}_N = P_{N-1}\mathrm{x}_N \left(1 + \mathrm{x}_N^T P_{N-1} \mathrm{x}_N\right)^{-1} \\ P_N = \left(I - \mathrm{k}_N \mathrm{x}_N\right)^T P_{N-1} \end{cases} \tag{15}$$

3.3 Estimation of power spectral density by AR model

AR model is the output of the system shown in fig.2.

The power spectral density of the output of the system is given by the following equality:

$$\Phi_x(\omega) = \frac{D}{|1 + a_1 e^{-j\omega\Delta} + \cdots + a_p e^{-jp\omega\Delta}|^2} \tag{16}$$

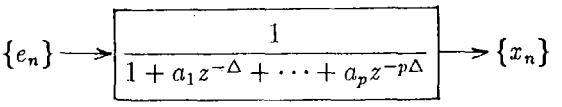


Figure 2: The system expressing AR model

where $\phi_e(\tau) = D\delta(\tau)$.

To obtain the power spectral density by this procedure, the system expressing AR model must be stable. In this case it is said that AR model is stable.

4 Estimation of Power Spectral Density from a small Number of Observed Data [5,6]

4.1 Estimation methods based on prediction error

One of the most important special features in obtaining power spectral density using AR model is to exterpolate the correlation function from the given one.

Let the correlation function be $\phi(\tau)$ ($\tau = 0, 1, \cdots, m$). In non-parametric method, we assume that $\phi(\tau) = 0 (\tau = m+1, m+2, \cdots)$. But AR model estimated from mth order Yule-Walker's equation has the non-zero correlation value at $\tau = m+1, m+2, \cdots$. From given correlation function, we estimate a correlation function as the linear combination of $e^{-\alpha\tau}\cos\omega\tau$ and/or $e^{-\alpha\tau}\sin\omega\tau$. So we can estimate the power spectral density beter than the non-parametric method for a small number of the observed data. This justification is also given from the viewpoint of MEM (Maximum Entropy Method).

4.2 Burg Method

The forward and backward prediction errors of AR model are given as follows:

Forward prediction error:

$$e_n^m = x_n + \sum_{k=1}^{m} a_k^{(m)} x_{n-k} \qquad (17)$$

Backward prediction error:

$$\gamma_n^m = x_{n-m} + \sum_{k=1}^{m} a_k^{(m)} x_{n-m+k} \qquad (18)$$

Put the cost function as J_m.

$$J_m = \sum_{n=m}^{N-1} \{(e_n^m)^2 + (\gamma_n^m)^2\} \qquad (19)$$

And we assume the following equality using in Levinson-Durbin's algorithm.

$$a_n^{(m)} = a_n^{(m+1)} + k_m a_{m-n}^{(m-1)} \quad (n = 1, 2, \cdots, m) \qquad (20)$$

The paramters minimizing J_m obtained easily by using this equality.

This algorithm is well known as Burg method.

By using Burg method, we can get a stable AR model. But the Levinson-Durbin's algorithm would not hold for the equation minimizing J_m. So Burg method is not theoretically correct. It is pointed out that the undesirable phenomena such as line spritting and bias can take place when we apply Burg method .[5]

4.3 Forward and backward covariance method

To avoid the theoretical inconsistency of Burg method, we consider to minimize J_m without using Levinson-Durvin's algorithm.

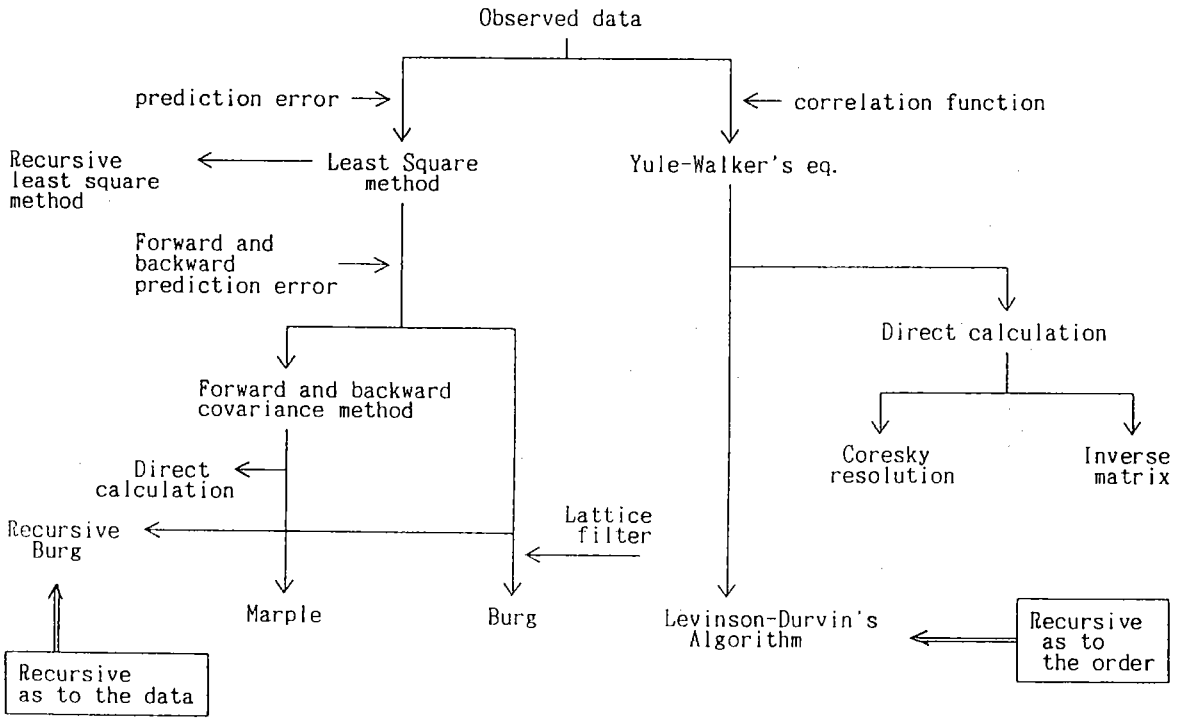


Figure 3: The method to obtain the coefficient of AR model

Put

$$\phi^{(m)}(k,j) = \sum_{i=0}^{N-m-1} (x_{i+m-j}x_{i+m-i} + x_{i+k}x_{i+j}) \tag{21}$$

$$k,j = 0,1,2,\cdots,m$$

The coefficients of AR model minimizing J_m satisfy the following equation.

$$\begin{pmatrix} \phi^{(m)}(0,0) & \cdots & \phi^{(m)}(0,m) \\ \vdots & \ddots & \vdots \\ \phi^{(m)}(m,0) & \cdots & \phi^{(m)}(m,m) \end{pmatrix} \begin{pmatrix} 1 \\ a_1^{(m)} \\ \vdots \\ a_{m-1}^{(m)} \\ a_m^{(m)} \end{pmatrix} = \begin{pmatrix} J_m \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{22}$$

In this equation, the coefficient matrix is not Toepritz. So, we cannot apply Levinson-Durvin's algorithm. The weakpoint of this method is that the stability of the AR model is not assured.

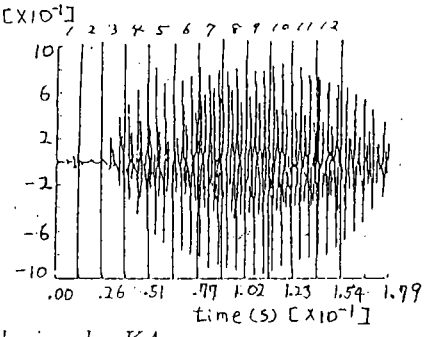
Marple derived an algorithm to solve the equation successively as to the order m.

The methods to obtain the coefficients of AR model are shown in figure.3.

4.4 An Example

Figure.4 shows a speach signal of KA. We divide the data in eleventh partial ones. The power spectral density is obtained for each interval by FFT and parametric method.

The result obtained by FFT seems to be different from the other results, because averaging process is not performed.



Speech signal : KA

the number of the data for each interval : 128

The order of AR model : 15

Figure 4: Speech signal KA

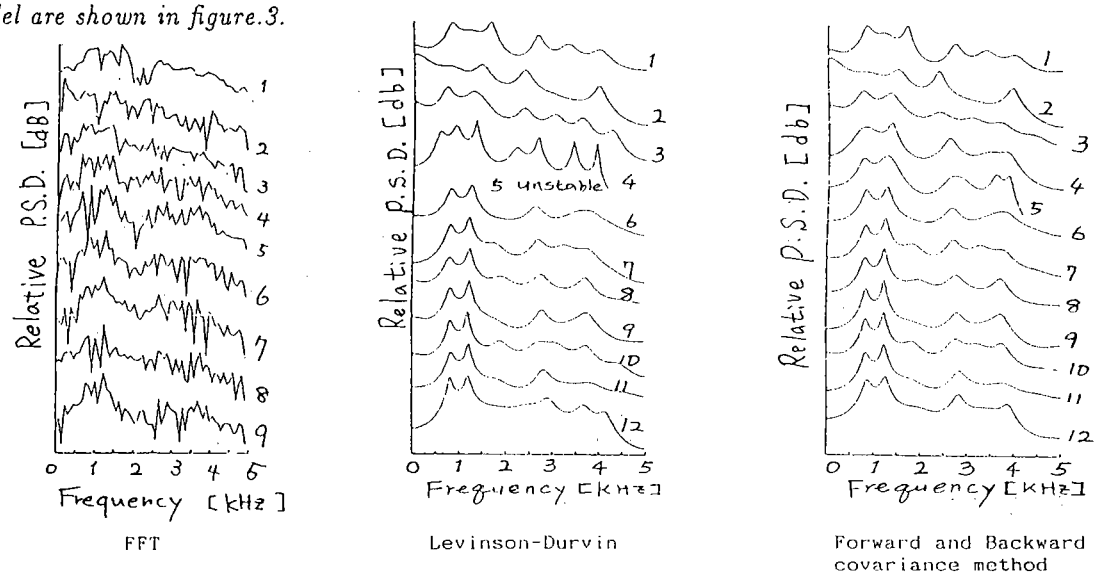


Figure 5: Power spectral density for each interval

5 Wavelet Analysis [7]

5.1 Wavelet transform

Wavelet transform of a signal $x(t)$ is defined as follows:

$$T_\phi(a,b) = \frac{1}{\sqrt{c}} \int_{-\infty}^{\infty} \Phi\left(\frac{t-b}{a}\right) x(t)dt \qquad (23)$$

where a and b are real number.

Function Φ is called analysing wavelet, and the following integral must be limited.

$$2\pi \int_{-\infty}^{\infty} \frac{|F[\phi(\omega)]|^2}{\omega} d\omega \qquad (24)$$

If the integral of $x(t)$ from $-\infty$ to ∞ is limited, the inverse wavelet transform is given as follows:

$$x(t) = \frac{1}{\sqrt{c}} \int_{-\infty}^{\infty} \int_{0}^{\infty} T_\phi(a,b)\Phi\left(\frac{t-b}{a}\right) \frac{1}{a^2} dadb \qquad (25)$$

The discrete wavelet transform is shown in the next section.

5.2 An efficient calculation method of wavelet transform

Wavelet transform requires more computaion time than Fourier transform. That leads to an obstacle of its engineering application. An efficient calculation method of wavelet transform was developed in my laboratory . [8]

The proposed algorithm is effective in case of Mayer's wavelet which is defined in frequency-domain. The algorithm is realized by deforming the analyzing wavelet in the part of time translation, and applying FFT algorithm to it. Actually this method is explained below.

The discrete wavelet transform of the observed signal $x(t)$ is given as follows:

$$\alpha_{j,k} = \int_{-\infty}^{\infty} x(t)\overline{\psi_{j,k}(t)}dt \qquad (26)$$

$$\psi_{j,k} = 2^{\frac{j}{2}}\psi(2^j t - k), \quad j,k \in Z$$

$$T : \text{length of data}$$

Applying Parseval's equality, we get the expression

$$\alpha_{j,k} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{x}(\omega)\overline{\hat{\psi}_{j,k}(\omega)}d\omega \qquad (27)$$

where $\hat{\psi}_{j,k}(\omega)$ is able to be rewritten as follows:

$$\hat{\psi}_{j,k}(\omega) = \exp\left(-i\frac{\omega T}{2^j}k\right)\hat{\psi}_{j,0}(\omega) \qquad (28)$$

Further, as integral range is limited to $[0Hz \sim 2f_nHz]$, where f_n is Nyquist frequency, the discrete wavelet transform on digital calculation is:

$$\alpha_{j,k} = \frac{\Delta\omega}{2\pi} \sum_{n=0}^{N-1} \exp\left(i\frac{2\pi}{2^j}n \cdot k\right) \overline{\hat{\psi}_{j,0}(\omega_n)}\hat{x}(\omega_n) \qquad (29)$$

$$N : \text{number of data}$$

$$\Delta\omega\left(=\frac{2\pi}{T}\right) : \text{frequency sampling width}$$

$$\omega_n = \Delta\omega \cdot n$$

Since $\exp\left(i\frac{2\pi}{2^j}n \cdot k\right)$ is periodical, we can calculate this expression by FFT algorithm. This method can decrease calculation time.

5.3 The extraction of an acoustic signal embedded in noise [8]

An extraction method of an acoustic signal in noise is proposed in this section. It is known that the frequency band width of the acoustic signal is located between $40Hz$ and $1.4KHz$.

Moreover the noise is white and the probability density function of its amplitude is Gaussian. Then we perform wavelet transform for these data. The frequency-band of the acoustic signal after transformation corresponds to the numbers of j. The average powers over short times of each frequency band are calculated from the wavelet coefficients as follows:

$$Average\ Power : \overline{E_{m,j}} = \sum_{k=m}^{n} \alpha_{j,k}{}^2 \qquad (30)$$

Our simulation shows that the average power in the part including an acoustic signal is much bigger than in the part not including it. Defining the threshold and cutting out coefficients which is smaller than the threshold, the essential signal is reconstructed by inverse wavelet transform. This is the extraction method by wavelet transform.

The author extracts an acoustic signal from the real data by two methods: wavelet transform and FIR band-pass filter. The results are shown in Fig.6.

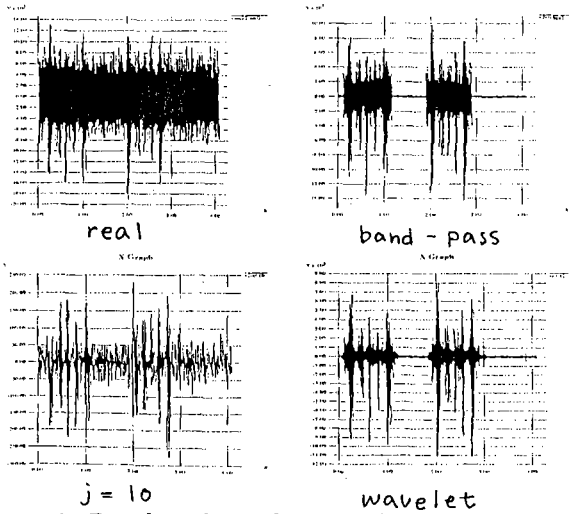


Figure 6: Results of wavelet transform and extraction

6 Conclusion

Spectral analysis of random process is not so easy, because we must estimate it from the observed data. So the many calculating methods were proposed till now. If we apply these methods carefully, the results are not so different. It will be important to use an appropriate method for the analysis, considering what features of the spectrum we want to find.

Recently it is wanted to find the partial nature in time domain. For this aim wavelet analysis is considered to be useful.

Bibliography

1. *Bendat,J.S and Piersol,A.G. Random Data: Analysis of Mesurement Procedures, John Wiley & Sons, 1971*
2. 秋月影雄, 松山泰男, 吉江修
   C言語 / ディジタル信号処理, 培風館, *1989*
3. 得丸英勝, 添田喬, 中溝高好, 秋月影雄
   工学基礎講座 *18* 計数・測定, 培風館, *1982*
4. 中溝高好
   信号処理とシステム同定, コロナ社, *1988*
5. 花崎泉, 秋月影雄
   スペクトル解析における *Burg* 法の改善, 計測自動制御学会論文集, *Vol.21,No.1,1985*
6. 秋月影雄
   短いデータの *AR* モデルの適合, 第 *16* 回確率システムシンポジウム, システム制御情報学会, *1984*
7. *C.K.Chui,An Introduction to Wavelets, Academic press*
8. 大庭正道, 浅野宜正, 秋月影雄
   Wavelet 変換による音声信号の抽出, 電気学会全国大会, *1994*