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Generalized Aspects of Riccati Equation Focused on the Roles of its Solution in Control Problem

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Abstract

It is well known that the Boyd's theorem states the relation between the imaginary eigenvalues of discriminant H of Riccati equation (A,R,Q) and the singular value of transfer function, but it is only available for $R \geqq 0$ and $Q \geqq 0$. In this paper, we extend Boyd's theorem for two case, that is, R is symmetric, Q is sign definite, and R is sign definite, Q is symmetric. We give under the condition that there is a real symmetric solution of Riccati equation the relation between H has imaginary eigenvalue and the maximum eigenvalue of transfer functoin. Finally, we give a necessary and sufficient condition to determine whether H has imaginary eigenvalue under some conditions.

1. Introduction

The Riccati equation considered in this paper has the form as follows.

$$A^TP+PA+PRP+Q=0 \qquad 1\text{-}1$$

Where A is real matrix, R , Q is real symmetric matrix. The discriminant H is

$$H=\begin{pmatrix} A & R \\ -Q & -A^T \end{pmatrix} \qquad 1\text{-}2$$

It is well known that the Riccati equation plays a great role in various control fields from the optimal regulator theory to H^∞ control theory. The feedback gain of control system is determined by the solution of Riccati equation, especially, the stable solution. But the solutions of (1-1) are determined by discriminant H, that is, the solution consists of eigenvectors or general eigenvectors of H, the position of eigenvalues of H in complex plane determine the stability of solution of (1-1). It is a well known result that if a solution is stable if and only if H has no imaginary eigenvalue. Therefore, if we want to have a stable solution, we hope to have a method to tell us the condition in which H has no imaginary eigenvalue. This is Boyd's theorem.[1]

For $R=BB^T$ and $Q=C^TC$, Boyd's theorem tells us that the transfer function $G(jw_0)=C(jw_0I-A)^{-1}B$ has singular value=1 if and only if H has a eigenvalue=jw_0.

In recent years, much attentions were paid on Riccati equation, because it plays an important role in the H^∞ control theory,we can say this equation in fact constitutes the bottleneckof all of linear system theory. It is necessary to know the constructure and the properties of solution of Riccati equation, and whether the Riccati equation determines all properties of linear system. Especially, the problem about stable solution.

In this paper, we started as an effort to extend the Boyd's theorem so that it can be available in the following cases.

That is, case 1, R is symmetric, Q is sign definite,
case 2, R is sign definite, Q is symmetric.

The cause we must have R or Q sign definite is that if not so, we can't have a transfer function corresponding to Boyd's theorem.

In section 2, there are two lemmas that are the preparation for the proof of theorems in section 3.

In section 3, we give some extensions to the Boyd's theorem, these results can be used to solve the H^∞ norm problem.

In section 4, we present some new results. Let R is sign definite and Q is symmetric, A has no imaginary eigenvalue , then under the condition that Riccati equation has real symmetric solution , a relation between H has imaginary eigenvalue and the maximum eigenvalue of transfer function is given. The Riccati equation in H^∞ control theory has the property that $Q>0$. In this case , we have a similar result .

The results mentioned above have a limitation that is A has no imaginary eigenvalue. Finally, we give a necessary and sufficient condition to determine whether H has imaginary eigenvalue.

2. Preparations

Before we give the proof of theorems in section 3, Let us give two lemmas that will play an important role in the proof of the theorems in the next section.

Lemma 1: Let H be a matrix of size $n \times n$, and H_{11}, H_{12}, H_{21}, H_{22} be matrices of size $n1 \times n1$, $n1 \times n2$, $n2 \times n1$ $n2 \times n2$. $n=n1+n2$.

$$H=\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \qquad 2\text{-}1$$

if $\det H_{11} \neq 0$, then

$$\det H=\det H_{11} \cdot \det(H_{22}-H_{21}H_{11}^{-1}H_{12}) \qquad 2\text{-}2$$

if $\det H_{22} \neq 0$, then

$$\det H=\det H_{22} \cdot \det(H_{11}-H_{12}H_{22}^{-1}H_{21}) \qquad 2\text{-}3$$

Proof. First, let us prove the case when $\det H_{11} \neq 0$.

I_{n1}, I_{n2} are unit matrices of size $n1 \times n1$, $n2 \times n2$.

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} I_{n1} & 0 \\ H_{21}H_{11}^{-1} & I_{n2} \end{pmatrix} \times \begin{pmatrix} H_{11} & 0 \\ 0 & H_{22}-H_{21}H_{11}^{-1}H_{12} \end{pmatrix} \begin{pmatrix} I_{n1} & H_{11}^{-1}H_{12} \\ 0 & I_{n2} \end{pmatrix} \qquad 2\text{-}4$$

From the fundamental properties of determinant, we can finish the proof.

$$\det H=\det H_{11} \cdot \det(H_{22}-H_{21}H_{11}^{-1}H_{12})$$

Next, let us prove the case when $\det H_{22} \neq 0$.

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} I_{n1} & H_{12}H_{22}^{-1} \\ 0 & I_{n2} \end{pmatrix} \times \begin{pmatrix} H_{11}-H_{12}H_{22}^{-1}H_{21} & 0 \\ 0 & H_{22} \end{pmatrix} \begin{pmatrix} I_{n1} & 0 \\ H_{22}^{-1}H_{21} & I_{n2} \end{pmatrix} \qquad 2\text{-}5$$

By the fundamental properties of determinant, we can finish the proof.

$$\det H=\det H_{22} \cdot \det(H_{11}-H_{12}H_{22}^{-1}H_{21})$$

□

Lemma2 is a simple consequence of lemma1.

Lemma2. Let A,B be matrices of size $n \times m$, $m \times n$. Then,

$$\det(I_n+AB)=\det(I_m+BA) \qquad 2\text{-}6$$

Proof.

$$\text{Let } A=\begin{pmatrix} I_n & A \\ -B & I_m \end{pmatrix} \qquad 2\text{-}7$$

From equation 2-2, we can obtain equation 2-8,

$$\det A=\det(I_m+BA) \qquad 2\text{-}8$$

From equation 2-3, we can obtain equation 2-9

$$\det A=\det(I_n+AB) \qquad 2\text{-}9$$

that is

$$\det(I_n+AB)=\det(I_m+BA)$$

□

With the preparations mentioned above, we can give a few theorems that are the extensions of Boyd's theorem.

3. The extension of Boyd's theorem

Let A, R, Q be the coefficient matrices of Riccati equation with size $n \times n$. H is the determinant of Riccati equation. throughout this section, we consider A has no imaginary eigenvalues.

$$H=\begin{pmatrix} A & R \\ -Q & -A^T \end{pmatrix} \qquad 3\text{-}1$$

It is important to know whether matrix H has imaginary eigenvalues if we want to have a stable solution for Riccati equation. Boyd's theorem answers this question, but it is only available for $R \geqq 0$ and $Q \geqq 0$. In this section, we extend Boyd's theorem so that it can be available for more sophisticated cases.

At first, let us give the Boyd's theorem for comparison.

Let $R=BB^T$, $Q=C^TC$, $G(jw)=B^T(jwI-A)^{-*}C^TC(jwI-A)^{-1}B$.

Body's theorem.[1] 1 is an eigenvalue of $G(jw_0)$ if and only if jw_0 is a eigenvalue of H.

Next, we give the extension of Boyd's theorem under different sign definite of R and Q.

Case1. $R=BB^T$, Q is symmetric.

$G1(jw)=B^T(jwI-A)^{-*}Q(jwI-A)^{-1}B$.

Theorem1. 1 is an eigenvalue of $G1(jw_0)$ if and only if jw_0 is an eigenvalue of H.

Proof.

$$jw_0I-H=\begin{pmatrix} jw_0I-A & -BB^T \\ Q & jw_0I+A^T \end{pmatrix} \qquad 3\text{-}2$$

By using equation 2-2 in lemma1, we can obtain 3-3

$$\det(jw_0I-H)=\det(jw_0I-A)\det(jw_0I+A^T+Q(jw_0I-A)^{-1}BB^T)$$

$$=-\det(jw_0I-A)\det(jw_0I-A)^*\det[I-(jw_0I-A)^{-*}Q(jw_0I-A)^{-1}BB^T] \qquad 3\text{-}3$$

By using equation 2-6 in lemma 2, we have equation 3-4

$$\det(jw_0I-H) = -\det(jw_0I-A)\det(jw_0I-A)^{*}\det[I-B^T(jw_0I-A)^{-*}Q(jw_0I-A)^{-1}B] \quad 3\text{-}4$$

Because $\det(jw_0I-A)\neq 0$, we can have a conclusion as follows.

$$\det(jw_0I-H)=0 \Leftrightarrow \det[I-B^T(jw_0I-A)^{-*}Q(jw_0I-A)^{-1}B]=0$$
$$\Leftrightarrow 1\in\Lambda(G1(jw_0)) \quad 3\text{-}5$$

$\Lambda(G1(jw_0))$ is the set of eigenvalues of $G1(jw_0)$. □

If R=-BBT, Q is symmetric, then we can have a similar result. Let $R=-BB^T$, Q is symmetric.
$G11(jw)=-B^T(jwI-A)^{-*}Q(jwI-A)^{-1}B$.

Mark 1. 1 is an eigenvalue of $G1(jw_0)$ if and only if jw_0 is an eigenvalue of H.

Let us discuss another case that can be found useful to explain the property in H^∞ theory.

Case2. $Q=C^TC$, R is symmetric.
$G2(jw)=C(jwI-A)^{-1}R(jwI-A)^{-*}C^T$.

Theorem2. 1 is an eigenvalue value of $G2(jw_0)$ if and only if jw_0 is an eigenvalue of H.
Proof.

$$jw_0I-H=\begin{pmatrix} jw_0I-A & -R \\ C^TC & jw_0I+A^T \end{pmatrix} \quad 3\text{-}6$$

By using equation 2-3 in lemma1, we can obtain 3-7

$$\det(jw_0I-H)=\det(jw_0I+A^T)\det(jw_0I-A+R(jw_0I+A^T)^{-1}C^TC)$$
$$=-\det(jw_0I-A)^{*}\det(jw_0I-A)\det[I-(jw_0I-A)^{-1}R(jw_0I-A)^{-*}C^TC] \quad 3\text{-}7$$

By using equation 2-6 in lemma 2, we have equation 3-8

$$\det(jw_0I-H) = -\det(jw_0I-A)^{*}\det(jw_0I-A)\det[I-C(jw_0I-A)^{-1}R(jw_0I-A)^{-*}C^T] \quad 3\text{-}8$$

Because $\det(jw_0I-A)\neq 0$, we can have a conclusion as follows.

$$\det(jw_0I-H)=0 \Leftrightarrow \det[I-C(jw_0I-A)^{-1}R(jw_0I-A)^{-*}C^T]=0$$
$$\Leftrightarrow 1\in\Lambda(G2(jw_0)) \quad 3\text{-}9$$

$\Lambda(G2(jw_0))$ is the set of eigenvalues of $G2(jw_0)$. □

If $Q=-C^TC$, R is symmetric, then we can have a similar result. Let $Q=-C^TC$, R is symmetric.
$G21(jw)=-C(jwI-A)^{-1}R(jwI-A)^{-*}C^T$.

Mark 2. 1 is an eigenvalue of $G2(jw_0)$ if and only if jw_0 is an eigenvalue of H.

We can obtain a conclusion from Boyd's theorem as follows.

Theorem3. $\lambda_{max}[B^T(jwI-A)^{-*}C^TC(jwI-A)^{-1}B]\geqq 1$ if and only if H has imaginary eigenvalues.

This theorem is proved by the fact that $\lambda_{max}(G(jw))$ is a continuous function of w and G(jw) is strictly proper. Similarly, we have the theorems as follows.

Case 3. $R=BB^T$, Q is symmetric.
Theorem 4. $\lambda_{max}[B^T(jwI-A)^{-*}Q(jwI-A)^{-1}B]\geqq 1$ if and only if H has imaginary eigenvalues.

Case 4. $R=-BB^T$, Q is symmetric.
Theorem 5. $\lambda_{max}[-B^T(jwI-A)^{-*}Q(jwI-A)^{-1}B]\geqq 1$ if and only if H has imaginary eigenvalues.

Case 5. $Q=C^TC$, R is symmetric
Theorem 5. $\lambda_{max}[C(jwI-A)^{-1}R(jwI-A)^{-*}C^T]\geqq 1$ if and only if H has imaginary eigenvalues.

Case 6. $Q=-C^TC$, R is symmetric
Theorem 6. $\lambda_{max}[-C(jwI-A)^{-1}R(jwI-A)^{-*}C^T]\geqq 1$ if and only if H has imaginary eigenvalues.

4. Main theorem

As mentioned in section 1, Riccati equation plays a great role in the control theory. The stable solution of Riccati equation can be used to establish the optimal control system. Therefore, it is important to know whether the Riccati equation exists a stable solution, that is, whether H has no imaginary eigenvalues. This question is answered by the following theorem.
The Riccati equation we consider is given as follows.

$$A^TP+PA+PRP+Q=0 \quad 4\text{-}1$$

$R=BB^T$, Q is symmetric.

The lemma stated as follows is useful in the proof of theorem 7.

Lemma 3.[2] Let P be a real symmetric solution to the equation 4-1, then P satisfies the unequality.

$$I-B^T(jwI-A)^{-*}Q(jwI-A)^{-1}B\geqq 0 \quad 4\text{-}2$$

Theorem 7. Assume $Re\,\lambda(A)\neq 0$. if Riccati equation 4-1 has a real symmetric solution, then H has imaginary eigenvalues, if and only if $\lambda_{max}[B^T(jwI-A)^{-*}Q(jwI-A)^{-1}B]=1$.
Proof. Because 4-1 has a real symmetric solution, then

$$\lambda_{max}[B^T(jwI-A)^{-*}Q(jwI-A)^{-1}B]\leqq 1$$

if H has an imaginary eigenvalue, according to theorem 4,

$$\lambda_{max}[B^T(jwI-A)^{-*}Q(jwI-A)^{-1}B]\geqq 1$$

that is

$$\lambda_{max}[B^T(jwI-A)^{-*}Q(jwI-A)^{-1}B]=1 \quad \text{for any } w$$

Therefore we finished the proof.

If $R=-BB^T$, and Q is symmetric, then we have the same result as that mentioned above.

Lemma 4. Let P be a real symmetric solution to the equation 4-3, then P satisfies the unequality.

$$I+B^T(jwI-A)^{-*}Q(jwI-A)^{-1}\geqq 0 \qquad 4\text{-}3$$

Theorem 8. Assume $Re\,\lambda(A)\neq 0$. If Riccati equation 4-1 has a real symmetric solution, then H has imaginary eigenvalues, if and only if $\lambda_{max}[-B^T(jwI-A)^{-*}Q(jwI-A)^{-1}B]=1$.
Proof. The proof is the same as the proof of theorem 7 if we consider lemma 4 and theorem 5. □

The Riccati equation considered in the H^∞ control theory has the feature that is the coefficient matrices R symmetric and $Q=C^TC>0$. In this case, we have the following theorem.

Theorem 9. Assume $Re\,\lambda(A)\neq 0$. If Riccati equation 4-1 has a real symmetric solution, then H has imaginary eigenvalues, if and only if $\lambda_{max}[C(jwI-A)^{-1}R(jwI-A)^{-*}C^T]=1$.
Proof. Let P be a real symmetric solution of the following Riccati equation.

$$A^TP+PA+PRP+C^TC=0 \qquad 4\text{-}4$$

Because of $C^TC>0$, we have $detP\neq 0$.
Let $P1=P^{-1}$, $P1$ is symmetric solution of equation 4-5.

$$AP1+P1A^T+P1C^TCP1+R=0 \qquad 4\text{-}5$$

By using lemma 3, we have the following result.

$$\lambda_{max}[C(jwI-A)^{-1}R(jwI-A)^{-*}C^T]\leqq 1 \qquad 4\text{-}6$$

If H has a imaginary eigenvalue, according to theorem 5

$$\lambda_{max}[C(jwI-A)^{-1}R(jwI-A)^{-*}C^T]\geqq 1 \qquad 4\text{-}7$$

that is, $$\lambda_{max}[C(jwI-A)^{-1}R(jwI-A)^{-*}C^T]=1 \qquad 4\text{-}8$$

Therefore we finished the proof. □

The theorems stated above have the same condition that $Re\,\lambda(A)\neq 0$. Since the solutions of Riccati equation may be written in the form $P=YX^{-1}$, where $Y=[y_1\, y_2 \cdots y_n]$, and $X=[x_1\, x_2 \cdots x_n]$. $z_i=(x_i^T\, y_i^T)^T$ is the eigenvector or general eigenvector of H. If we put a limitation on the eigenvector of H, a theorem can be obtained as follows.
At first, let us determine a set E.

$$E=\{(x^T\, y^T)^T / x^*Qx=y^*Ry,\ x^*y\neq 0,\ \forall x,y\in \mathbf{C}^{n\times 1}\}$$

Theorem 10. Assume the eigenvector $(x^T\, y^T)^T$ has the propersity $x^*y\neq 0$. H has no imaginary eigenvalues if and only if the member in set E is not the eigenvector of H.
Proof. At first, we prove the necessity.

$$\exists\, \lambda_0\in \mathbf{C},\ z_0=(x_0^T\, y_0^T)^T\in E,\ y_0^*Ry_0=x_0^*Qx_0.$$

$$\begin{pmatrix} A & R \\ -Q & -A^T \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \lambda_0 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \qquad 4\text{-}9$$

From equation 4-9, we can obtain two equation.

$$Ax_0+Ry_0=\lambda_0 x_0 \qquad 4\text{-}10$$
$$-Qx_0-A^Ty_0=\lambda_0 y_0 \qquad 4\text{-}11$$

Let $y_0^*(4\text{-}10)+x_0^*(4\text{-}11)$,

$$y_0^*Ry_0-x_0^*Qx_0+y_0^*Ax_0-x_0^*A^Ty_0=\lambda_0(y_0^*x_0+x_0^*y_0) \qquad 4\text{-}12$$

Equating the real part and imaginary part of 4-12.

$$y_0^*Ry_0-x_0^*Qx_0=Re\,\lambda_0\cdot 2Re(x_0^*y_0) \qquad 4\text{-}13$$
$$Im(y_0^*Ax_0)=Im\,\lambda_0\cdot Re(x_0^*y_0) \qquad 4\text{-}14$$

accoding to the fact that $z_0\in E$, from 4-13 we have
Case1: $Re\,\lambda_0=0$
Case2: $Re(x_0^*y_0)=0$
Consider case 2 as follows.
let $y_0^*\times(4\text{-}10)$,

$$y_0^*Ax_0+y_0^*Ry_0=\lambda_0 y_0^*x_0 \qquad 4\text{-}15$$

Because $Im(y_0^*Ax_0)=0$,the left side of (4-15) is real number
We have $Re\,\lambda_0=0$ since $Re(y_0^*x_0)=0$.
That is, if $z_0\in E$ is eigenvector of H, then H has imaginary eigenvalue.It contradicts to the condition that H has no imaginary eigenvalue.

Next, we prove the sufficiency.
Supposing that H has imaginary eigenvalue.
that is, $\exists w_0\in \mathbf{R},\ z_0=(x_0^T\, y_0^T)^T\in \mathbf{C}^{n\times 1}$
Let jw_0 is eigenvalue of H, z_0 is eigenvector corresponding to jw_0.

$$Hz_0=jw_0z_0 \qquad 4\text{-}16$$

It is the same as the proof of necessity, finally we have equation 4-13,
that is, $y_0^*Ry_0=x_0^*Qx_0$, $z_0\in E$.
It is a contradiction to the condition that the member of E is not eigenvalue of H. □

References

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