A Representation Theory of Linear Systems and Its Application to Simultaneous Stabilization

Hiroshi Inaba, Rixat Abdursul and Satoru Takahashi
Department of Information Sciences
Tokyo Denki University
Hatoyama-Machi, Hiki-Gun, Saitama 350-03, Japan

Abstract

This paper develops a representation theory of linear systems by means of doubly coprime factorizations, and applies the theory to the simultaneous stabilization problem for a given set of linear systems.

1. Introduction

This paper develops a representation theory of multivariable linear systems by means of doubly coprime factorizations. This development is based on the results of R. Saeks and J. Murray[1], and generalizes their results to multivariable linear systems[2]. Further, this representation theory is used to analyze the simultaneous stabilization problem for a set of linear systems.

First, it is shown that any linear systems can be represented by a unimodular matrix over the ring of proper stable rational functions, and various properties of such representations are presented. In particular, the set of all stabilizing compensators for a given system and the set of all linear systems which are stabilized by a given compensator is given in the frame work of this represetation theory. Further, applying these results to the problem of simultaneously stabilizing a given set of linear systems by a single compensator, necessary and sufficient conditions for the problem to be solvable are obtained.

2. A Representation Theory

First, let us introduce the following notations:

$\mathbf{R}(s)$	:=	the field of all real rational functions of s
$\mathbf{R}_p(s)$	:=	$\{ f \in \mathbf{R}(s) \mid f \text{ is proper} \}$
$\mathbf{S}$	:=	$\{ f \in \mathbf{R}_p(s) \mid f \text{ is stable} \}$
$\mathbf{M}^{p\times q}$	:=	the set of all $p \times q$ matries with elements in $\mathbf{M}$
I_q	:=	the $q \times q$ identity matrix

Now, notice that any $P \in \mathbf{R}_p(s)^{r\times m}$ has a *doubly coprime factorization* (*d.c.f.*) over $\mathbf{S}$, characterized as

$$P = ND^{-1} = \tilde{D}^{-1}\tilde{N} \tag{2.1}$$

$$\begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{D} \end{bmatrix} \begin{bmatrix} D & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix} = I_{m+r} \tag{2.2}$$

where $N, \tilde{N} \in \mathbf{S}^{r\times m}, D, Y \in \mathbf{S}^{m\times m}, \tilde{D}, \tilde{Y} \in \mathbf{S}^{r\times r}$ and $X, \tilde{X} \in \mathbf{S}^{m\times r}$(See, e.g., [3]). (N, D) is called a *right coprime factor*(*r.c.f.*) of P and $(\tilde{D}, \tilde{N})$ a *left coprime factor*(*l.c.f.*) of P.

Using a *d.c.f.* of P, we introduce two matrices R and $L \in \mathbf{S}^{(m+r)\times(m+r)}$ as follows:

$$R := \begin{bmatrix} D & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix}, \; L := \begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{D} \end{bmatrix}.$$

We call the matrix $R(L)$ a *doubly right*(*left*) *coprime representation* of P, abbreviated by *d.r.c.r.*(*d.l.c.r.*). Clearly, by the definition, R and L belong to $\mathbf{S}^{(m+r)\times(m+r)}$ and $R^{-1} = L$. In particular, if $P \in \mathbf{S}^{r\times m}$ then it has a *d.r.c.r.* of the form

$$R := \begin{bmatrix} I_m & 0 \\ P & I_r \end{bmatrix}. \tag{2.3}$$

It should be noticed that for linear system $P \in \mathbf{R}_p(s)^{r\times m}$ its *d.r.c.r.* $R \in \mathbf{S}^{(m+r)\times(m+r)}$ is not unique, and that R has its inverse matrix L in $\mathbf{S}^{(m+r)\times(m+r)}$. Therefore, the set of *d.r.c.r.*'s R of all $P \in \mathbf{R}_p(s)^{r\times m}$ constitutes a group, and so the following definition will be introduced.

Definition 2.1

(i) Let $\mathbf{GL_S}(k)$ denote the group consisting of all $k \times k$ unimodular matrices over $\mathbf{S}$.

(ii) Let $\mathbf{E}(p,q)$ denote the subgroup of $\mathbf{GL_S}(p+q)$ given by

$$\mathbf{E}(p,q) := \left\{ \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix} \middle| E_{11} \in \mathbf{GL_S}(p), E_{22} \in \mathbf{GL_S}(q), E_{12} \in \mathbf{S}^{p\times q} \right\}. \quad \square$$

Then, it is not difficult to prove the following theorem.

Theorem 2.2 Let $R, \tilde{R} \in \mathbf{GL_S}(m+r)$. Then, R and $\tilde{R}$ are *d.r.c.r.*'s of the same system $P \in \mathbf{R}_p(s)^{m\times r}$ if and only if there exists an $E \in \mathbf{E}(m,r)$ such that $R = \tilde{R}E$. $\square$

If linear system $P \in \mathbf{R}_p(s)^{r\times m}$ is stable (that is, $P \in \mathbf{S}^{r\times m}$) then a *d.r.c.r.* of P is given by (2.3). Therefore, it is meaningful to introduce the following subgroup $\mathbf{W}(m,r)$ of $\mathbf{GL_S}(m+r)$:

$$\mathbf{W}(m,r) := \left\{ \begin{bmatrix} I_m & 0 \\ P & I_r \end{bmatrix} \middle| P \in \mathbf{S}^{r\times m} \right\}.$$

Further, let us denote by $\mathbf{S}(m,r)$ the set of *d.r.c.r.*'s of all stable linear systems in $\mathbf{R}_p(s)^{r\times m}$. Then, it is easily seen that

$$\mathbf{S}(m,r) = \mathbf{W}(m,r)\mathbf{E}(m,r).$$

Note that $\mathbf{S}(m,r)$ does not form a group. However, since both $\mathbf{W}(m,r)$ and $\mathbf{E}(m,r)$ are groups, one obtains

$$\begin{aligned}\mathbf{S}(m,r)^{-1} &:= \{S^{-1} \in \mathbf{S}(m,r)\} \\ &= \mathbf{E}(m,r)\mathbf{W}(m,r).\end{aligned}$$

Now, the following theorem will be proved.

Theorem 2.3 Let $P \in \mathbf{R}_p(s)^{r\times m}$ and a *d.r.c.r.* of P be given as

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in \mathbf{GL_S}(m+r)$$

where $T_{11} \in \mathbf{S}^{m\times m}$, $T_{12} \in \mathbf{S}^{m\times r}$, $T_{21} \in \mathbf{S}^{r\times m}$ and $T_{22} \in \mathbf{S}^{r\times r}$. Then:

(i) $T \in \mathbf{S}(m,r)$ if and only if $T_{11} \in \mathbf{GL_S}(m)$.

(ii) $T \in \mathbf{S}^{-1}(m,r)$ if and only if $T_{22} \in \mathbf{GL_S}(r)$.

Proof. Only the statement (i) will be proved because the statement (ii) can be shown in a similar manner to (i).

(Necessary) Suppose that $T \in \mathbf{S}(m,r)$. Since $\mathbf{S}(m,r) = \mathbf{W}(m,r)\mathbf{E}(m,r)$, there exist $W \in \mathbf{W}(m,r)$ and $E \in \mathbf{E}(m,r)$ such that $T = WE$. In fact, T can be represented as

$$\begin{aligned}T &= \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ W_{21} & I_r \end{bmatrix}\begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix} \\ &= \begin{bmatrix} E_{11} & E_{12} \\ W_{21}E_{11} & W_{21}E_{12} + E_{22} \end{bmatrix}.\end{aligned}$$

Therefore, one obtains

$$T_{11} = E_{11} \in \mathbf{GL_S}(m).$$

(Sufficient) Suppose that $T_{11} \in \mathbf{GL_S}(m)$. Now, decompose T as

$$\begin{aligned}T &= \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \\ &= \begin{bmatrix} I_m & 0 \\ T_{21}T_{11}^{-1} & I_r \end{bmatrix}\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} - T_{21}T_{11}^{-1}T_{12} \end{bmatrix}.\end{aligned}$$

Since $T_{11}^{-1} \in \mathbf{GL_S}(m)$, one has $T_{21}T_{11}^{-1} \in \mathbf{S}^{r\times m}$ and hence the first matrix in the decomposition satisfies

$$W := \begin{bmatrix} I_m & 0 \\ T_{21}T_{11}^{-1} & I_r \end{bmatrix} \in \mathbf{W}(m,r).$$

Further, since W, $T \in \mathbf{GL_S}(m+r)$, the second matrix satisfies

$$\begin{aligned}E &:= \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} - T_{21}T_{11}^{-1}T_{12} \end{bmatrix} \\ &= W^{-1}T \in \mathbf{GL_S}(m+r)\end{aligned}$$

and

$$E^{-1} = T^{-1}W \in \mathbf{GL_S}(m+r).$$

Thus,

$$E \in \mathbf{E}(m,r),$$

which implies that

$$T = WE \in \mathbf{W}(m,r)\mathbf{E}(m,r) = \mathbf{S}(m,r).$$

This completes the proof of (i). □

3. Stabilization

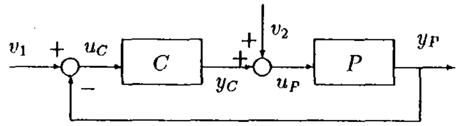

Fig.1 A Feedback System

The basic feedback system configuration is shown in Figure 1. Here, $P \in \mathbf{R}_p(s)^{r\times m}$ represents an m-input r-output linear system and $C \in \mathbf{R}_p(s)^{m\times r}$ a compensator for P. Then, the transfer matrix $F(C,P)$ from $\begin{bmatrix} v_2 \\ v_1 \end{bmatrix}$ to $\begin{bmatrix} u_P \\ u_C \end{bmatrix}$ is given by

$$\begin{aligned}F(C,P) &= \begin{bmatrix} I - C(I+PC)^{-1}P & C(I+PC)^{-1} \\ -(I+PC)^{-1}P & (I+PC)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} (I+CP)^{-1} & (I+CP)^{-1}C \\ -P(I+CP)^{-1} & I - P(I+CP)^{-1}C \end{bmatrix},\end{aligned}$$

and the following definition is given.

Definition 3.1 The feedback system in Figure 1 is said to be *stable* if

(i) $\det(I+PC) = \det(I+CP) \neq 0$, and

(ii) $F(C,P) \in \mathbf{S}$.

In this case, such a compensator C is said to *stabilize* P. □

Now, since

$$y_P = Pu_P = ND^{-1}u_P,$$

one obtains that

$$\begin{aligned}\begin{bmatrix} u_P \\ y_P \end{bmatrix} &= \begin{bmatrix} D & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix}\begin{bmatrix} \xi_P \\ 0 \end{bmatrix} \\ &= R_P\begin{bmatrix} \xi_P \\ 0 \end{bmatrix} \qquad (3.1)\end{aligned}$$

where $\xi_P := D^{-1}u_P$ and R_P is a *d.r.c.r.* of P. Similarly, one obtains that

$$\begin{bmatrix} u_C \\ y_C \end{bmatrix} = R_C\begin{bmatrix} \xi_C \\ 0 \end{bmatrix} \qquad (3.2)$$

where R_C is a *d.r.c.r.* of C.

Next, notice from Figure 1 that

$$u_P = v_2 + y_C, \quad u_C = v_1 - y_P. \qquad (3.3)$$

Then, introducing matrix Q by

$$Q := \begin{bmatrix} 0 & I_m \\ -I_r & 0 \end{bmatrix},$$

(3.3) can be represented as

$$\begin{bmatrix} u_P \\ y_P \end{bmatrix} = Q\begin{bmatrix} u_C \\ y_C \end{bmatrix} + \begin{bmatrix} v_2 \\ v_1 \end{bmatrix}. \qquad (3.4)$$

Now, substituting (3.1), (3.2) into (3.4) and arraying the resultant lead to

$$\begin{bmatrix} v_2 \\ v_1 \end{bmatrix} = [R_P P_1 + Q R_C Q^T P_2] \begin{bmatrix} \xi_P \\ \xi_C \end{bmatrix} \qquad (3.5)$$

where

$$P_1 = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix},$$
$$Q^T = \begin{bmatrix} 0 & -I_r \\ I_m & 0 \end{bmatrix}.$$

Thus, the following lemma holds.

Lemma 3.2 In Figure 1, C stabilize P if and only if

$$R_P P_1 + Q R_C Q^T P_2 \in \mathbf{GL_S}(m+r). \quad \square$$

Now, the following main theorem can be shown using Lemma 3.2, but the proof is ommited.

Theorem 3.3 Given a linear system $P \in \mathbf{R}_p(s)^{r\times m}$, let $\mathbf{R_C}(P)$ denote the set of representations of all compensators $C \in \mathbf{R}_p(s)^{m\times r}$ stabilizng P. Then,

$$\mathbf{R_C}(P) = Q^T R_P Q \mathbf{S}(m,r),$$

where R_P is a *d.r.c.r.* of P. $\square$

Further, one can obtain the following corollary.

Corollary 3.4 A linear system $P \in \mathbf{R}_p(s)^{r\times m}$ is strongly stabilizable if and only if

$$R_P \in \mathbf{S}(m,r)^{-1}\mathbf{S}(m,r)$$

where $R_P \in \mathbf{GL_S}(m+r)$ is a *d.r.c.r.* of P. $\square$

Remark 3.5 It is noted that Theorem 3.3 is equivalent to the parametrization thorem for stabilizing controllers in Youla et al.[4] and Desoer et al.[5]. Further, it is noted that Corollary 3.4 is equivalent to the result of Vidyasagar[3](pp.125(ii)),[6]. That is, $R_P \in \mathbf{S}(m,r)^{-1}$ $\mathbf{S}(m,r)$ if and only if there exists a $K \in \mathbf{S}^{m\times r}$ such that $D+KN \in \mathbf{GL_S}(m)$ where (D,N) is any *r.c.f.* of P. $\square$

4. Simultaneous Stabilization

The following theorem plays a key role in simultaneous stabilization.

Theorem 4.1 Let $C \in \mathbf{R}_p(s)^{m\times r}$ be given, and $\mathbf{R_P}(C)$ denote the set of representations of all linear systems $P \in \mathbf{R}_p(s)^{r\times m}$ which are stabilized by the compensator C. Then,

$$\mathbf{R_P}(C) = Q R_C Q^T \mathbf{S}(m,r)$$

where R_C is a *d.r.c.r.* of C. $\square$

Corollary 4.2 Let $\mathbf{P} \subset \mathbf{R}_p(s)^{r\times m}$ be a set of linear systems, and $\mathbf{R_P} \subset \mathbf{GL_S}(m+r)$ denote the set of *d.r.c.r.*'s of all $P \in \mathbf{P}$. Then, all linear systems $P \in \mathbf{P}$ are simultaneously stabilized by a single compensator in $\mathbf{R}_p(s)^{m\times r}$, that is, $\mathbf{P}$ is simultaneously stabilizable if and only if there exists a $T \in \mathbf{GL_S}(m+r)$ such that

$$\mathbf{R_P} \subset T\mathbf{S}(m,r). \quad \square$$

The following theorem will be proved.

Theorem 4.3 Let $\mathbf{P}$, $\mathbf{R_P}$ be those as in Corollary 4.2. Then, $\mathbf{P}$ is simultaneously stabilizable if and only if for any $P, P' \in \mathbf{P}$ there exist $T_P, T_{P'} \in \mathbf{S}(m,r)$ such that

$$T_P^{-1} T_{P'} = R_P^{-1} R_{P'} \qquad (4.1)$$

where $R_P, R_{P'}$ are *d.r.c.r.*'s of P, P'.

Proof. (Necessary) Suppose that $\mathbf{P}$ is simltaneously stabilizable, that is, there exists a compensator $C \in \mathbf{R}_p(s)^{m\times r}$ that stabilizes all $P \in \mathbf{P}$. Then, by Theorem 4.1, for each $R_P \in \mathbf{R_P}(C)$ there exists a $T_P \in \mathbf{S}(m,r)$ such that

$$R_P = Q R_C Q^T T_P.$$

Thus, for any $P, P' \in \mathbf{P}$

$$\begin{aligned} R_P^{-1} R_{P'} &= (Q R_C Q^T T_P)^{-1}(Q R_C Q^T T_{P'}) \\ &= T_P^{-1} T_{P'}, \end{aligned}$$

showing the necessary.

(Sufficiency) Suppose that (4.1) is satisfied, i.e., for any $P, P' \in \mathbf{P}$,

$$R_P T_P^{-1} = R_{P'} T_{P'}^{-1}. \qquad (4.2)$$

Now, take any $P' \in \mathbf{P}$ and define

$$R_C := Q^T R_{P'} T_{P'}^{-1} Q \qquad (4.3)$$

where C indicates a compensator whose *d.r.c.r.* is given to be the matrix $Q^T R_{P'} T_{P'}^{-1} Q$. Then, it is obvious that R_C is independent of the choice of P'.

Next, take any P. Then, noticing that Q is orthogonal, it follows from (4.2) and (4.3) that

$$\begin{aligned} R_P &= R_{P'} T_{P'}^{-1} T_P = Q(Q^T R_P T_P^{-1} Q) Q^T T_P \\ &= Q R_C Q^T T_P. \end{aligned}$$

Since $T_P \in \mathbf{S}(m,r)$, Theorem 4.1 implies that P is stabilized by C. Since $P \in \mathbf{P}$ was arbitrary this proves that $\mathbf{P}$ is simultaneously stabilizable. $\square$

Based on Theorem 4.3, the following corollary can be shown.

Corollary 4.4 Let $\mathbf{P}$, $\mathbf{R_P}$ be those as in Theorem 4.3. Then, $\mathbf{P}$ is simultaneously stabilizable if and only if for a fixed $P_0 \in \mathbf{P}$ there exists a $T_{P_0} \in \mathbf{S}(m,r)$ such that

$$T_{P_0}^{-1} T_P = R_{P_0}^{-1} R_P \quad \text{for all } P \in \mathbf{P}$$

where R_{P_0}, R_P are *d.r.c.r.*'s of P_0, P. $\square$

Remark 4.5 From Corollary 4.4, it follows that Theorem 4.3 is equivalent to Theorem 22 in Vidyasagar [3](pp.130). That is, there exists a set $\{ T_P \in \mathbf{S}(m,r) \mid P \in \mathbf{P} \}$ such that $T_P^{-1} T_{P'} = R_P^{-1} R_{P'}$ for any $P, P' \in \mathbf{P}$ if and only if there exists a $M \in \mathbf{S}^{m\times r}$ such that $A_i + M B_i \in \mathbf{GL_S}(m)$ for all i. $\square$

Next, we give a simple example.

Example 4.6 We consider the simultaneous stabilization problem for the following three linear systems:

$$P_1 = \begin{bmatrix} \frac{3(s-1)(s+2)}{s(s-2)} & \frac{3(s+2)}{s(s-2)} \\ \frac{2(s+1)}{s(s-2)} & \frac{2(s-1)(s+1)}{s(s-2)} \end{bmatrix}$$

$$P_2 = \begin{bmatrix} \frac{4(s-2)(s+2)}{(s-1)(s-3)} & \frac{4(s+2)}{(s-1)(s-3)} \\ \frac{3(s+1)}{(s-1)(s-3)} & \frac{3(s-2)(s+1)}{(s-1)(s-3)} \end{bmatrix}$$

$$P_3 = \begin{bmatrix} \frac{5(s-3)(s+2)}{(s-2)(s-4)} & \frac{5(s+2)}{(s-2)(s-4)} \\ \frac{4(s+1)}{(s-2)(s-4)} & \frac{4(s-3)(s+1)}{(s-2)(s-4)} \end{bmatrix}$$

Then, *d.r.c.r.*'s R_{P_i} of P_i $(i = 1,2,3)$ are given as

$$R_{P_1} = \begin{bmatrix} \frac{s-1}{s+1} & \frac{1}{s+2} & \frac{1}{s+2} & \frac{1}{(s+1)(s+2)} \\ \frac{1}{s+1} & \frac{s-1}{s+2} & \frac{-1}{(s+1)(s+2)} & \frac{-1}{s+1} \\ \frac{3(s+2)}{s+1} & 0 & 1 & 0 \\ 0 & \frac{2(s+1)}{s+2} & 0 & 1 \end{bmatrix}$$

$$R_{P_2} = \begin{bmatrix} \frac{s-2}{s+1} & \frac{1}{s+2} & \frac{1}{s+2} & \frac{1}{(s+1)(s+2)} \\ \frac{1}{s+1} & \frac{s-2}{s+2} & \frac{-1}{(s+1)(s+2)} & \frac{-1}{s+1} \\ \frac{4(s+2)}{s+1} & 0 & 1 & 0 \\ 0 & \frac{3(s+1)}{s+2} & 0 & 1 \end{bmatrix}$$

$$R_{P_3} = \begin{bmatrix} \frac{s-3}{s+1} & \frac{1}{s+2} & \frac{1}{s+2} & \frac{1}{(s+1)(s+2)} \\ \frac{1}{s+1} & \frac{s-3}{s+2} & \frac{-1}{(s+1)(s+2)} & \frac{-1}{s+1} \\ \frac{5(s+2)}{s+1} & 0 & 1 & 0 \\ 0 & \frac{4(s+1)}{s+2} & 0 & 1 \end{bmatrix}$$

Matrices $T_{P_i} (i = 1,2,3)$ are computed to be

$$T_{P_1} = \begin{bmatrix} \frac{s+2}{s+1} & \frac{s}{(s+2)^2} & 0 & 0 \\ \frac{s+4}{(s+1)^2} & \frac{s+1}{s+2} & 0 & 0 \\ \frac{3(s+2)}{s+1} & 0 & 1 & 0 \\ 0 & \frac{2(s+1)}{s+2} & 0 & 1 \end{bmatrix}$$

$$T_{P_2} = \begin{bmatrix} \frac{s+2}{s+1} & \frac{s-1}{(s+2)^2} & 0 & 0 \\ \frac{s+5}{(s+1)^2} & \frac{s+1}{s+2} & 0 & 0 \\ \frac{4(s+2)}{s+1} & 0 & 1 & 0 \\ 0 & \frac{3(s+1)}{s+2} & 0 & 1 \end{bmatrix}$$

$$T_{P_3} = \begin{bmatrix} \frac{s+2}{s+1} & \frac{s-1}{(s+2)^2} & 0 & 0 \\ \frac{s+6}{(s+1)^2} & \frac{s+1}{s+2} & 0 & 0 \\ \frac{5(s+2)}{s+1} & 0 & 1 & 0 \\ 0 & \frac{4(s+1)}{s+2} & 0 & 1 \end{bmatrix},$$

and these satisfy

$$T_{P_i}^{-1} T_{P_j} = R_{P_i}^{-1} R_{P_j} \quad (i,j = 1,2,3).$$

Thus, by Theorem 4.3 the three linear systems are simultaneously stabilizable. Now, following the proof of Theorem 4.3, a *d.r.c.r.* R_C of a simultaneously stabilizing compensator C is given by

$$R_C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{s+2} & \frac{-1}{(s+1)(s+2)} & 1 & 0 \\ \frac{-1}{(s+1)(s+2)} & \frac{1}{s+1} & 0 & 1 \end{bmatrix}.$$

Finally, such a simultaneously stabilizing compensator is obtained as

$$C = \begin{bmatrix} \frac{1}{s+2} & \frac{-1}{(s+1)(s+2)} \\ \frac{-1}{(s+1)(s+2)} & \frac{1}{s+1} \end{bmatrix}. \quad \square$$

5. Conclusions

This paper introduced and developed doubly right(left) coprime representations of linear systems. Then, using such representations, necessary and sufficient conditions for simultaneous stabilization were obtained.

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