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A Dual Approach To Input/Output Variance Constrained Control Problem

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ABSTRACT

An optimal controller, e.g. LQG controller, may not be realistic in the sense that the required control power may not be achieved by existing actuators, and the measured output is not satisfactory. To be realistic, the controller should meet such constraints as sensor or actuator limitation, performance limit, etc.

In this paper, the Input/Output Variance Constrained (IOVC) control problem will be considered from the viewpoint of mathematical programming. A dual version shall be developed to solve the IOVC control problem, whose objective is to find a stabilizing control law attaining a minimum value of a quadratic cost function subject to the inequality constraint on each input and output variance for a stabilizable and detectable plant.

One approach to the constrained optimization problem is to use the Kuhn-Tucker necessary conditions for the optimality and to seek an optimal point by an iterative algorithm. However, since the algorithm uses only the necessary conditions, the convergent point may not be optimal solution. Our algorithm will guarantee a sufficiency.

1. Problem Statement

Consider the following stabilizable and detectable plant

$$\left.\begin{aligned} \dot{x} &= A_p x + B_p u + D_p w \\ y &= C_p x \ , \end{aligned}\right\} \qquad (1.1a)$$

where · ,u and y are the vectors of state, input and output, respectively. A white noise, w, is assumed to have intensity W_p ;

$$\left.\begin{aligned} x &= [x_1,\ x_2,\ \cdots,\ x_{n_x}]^T \\ u &= [u_1,\ u_2,\ \cdots,\ u_{n_u}]^T \\ w &= [w_1,\ w_2,\ \cdots,\ w_{n_w}]^T \\ y &= [y_1,\ y_2,\ \cdots,\ y_{n_y}]^T \ . \end{aligned}\right\} \qquad (1.1b)$$

The control problem under consideration is following;

IOVC Control Problem

Find a stabilizing linear control law, u, attaining a minimum value of a quadratic objective function subject to the inequality constraint on each input and output variance for the given system (1.1), i.e.,

$$\left.\begin{aligned} &\min_{u\in\Omega} \ J(u) \\ &\text{subject to} \begin{cases} E_\infty\, y_i^2 \le \sigma_i^2, & i = 1, \cdots, n_y \\ E_\infty\, u_j^2 \le \mu_j^2, & j = 1, \cdots, n_u \end{cases} \end{aligned}\right| \qquad (1.2a)$$

where

$$J(u) \equiv E_\infty \left(y^T Q_o y + u^T R_o u \right) \ , \qquad (1.2b)$$

$$\Omega = \{ u \mid u = \text{stabilizing control law} \} \ , \qquad (1.2c)$$

$$E_\infty(\bullet) \equiv \lim_{t\to\infty} E\{\bullet(t)\} \qquad (1.2d)$$

and Q_o is a given positive semi-definite output weighting matrix, R_o a given positive definite input weighting matrix, σ_i^2 the given upper bound of the i-th output variance, μ_j^2 the given upper bound of the j-th input variance and $E(\bullet)$ indicates the expectation of $(\bullet)$.

■

A similar control problem was considered in [1]. When only the output variance constraints are imposed with $Q_o = 0$, the problem is called Output Variance Constrained (OVC) control problem and has been extensively investigated by several authors [2-6]. A companion problem called the Input Variance Constrained (IVC) control problem is also considered with $R_o = 0$ and only the input variance constraints [2,7].

Problems of the type (1.2) have also been solved by [8,9] by using a deep cut ellipsoidal algorithm [10] that tests for feasibility and provides both upper and lower bounds at each iteration. This allows one to computer and answer within a specified degree of accuracy. One approach [2-7] to the above problem may be to use the well-known Kuhn-Tucker necessary conditions for the optimality and to seek an optimal point by an iterative algorithm. However, since the algorithm uses only the necessary conditions, the convergent point obtained by the iterative algorithm is not, in general, guaranteed to be an optimal solution to the problem. We shall provide a dual version of IOVC control problem (1.2), whose solution is also a solution to the above problem. Hence a sufficiency is guaranteed.

2. Dual Problem

In an effort to get a dual version, we first consider the following problem intimately related to the problem (1.2);

Lagrangian Problem

$$\min_{u\in\Omega} L(u,q,r) \tag{2.1a}$$

where

$$L(u,q,r) \equiv J(u) + \sum_{i=1}^{n_y} q_i (E_\infty y_i^2 - \sigma_i^2) + \sum_{j=1}^{n_u} r_j (E_\infty u_j^2 - \mu_j^2) , \tag{2.1b}$$

the parameters q and r are fixed vectors whose elements are nonnegative, i.e.,

$$q \equiv [q_1, q_2, \cdots, q_{n_y}]^T, \quad q_i \ge 0, \quad i = 1, \cdots, n_y , \tag{2.1c}$$

$$r \equiv [r_1, r_2, \cdots, r_{n_u}]^T, \quad r_j \ge 0, \quad j = 1, \cdots, n_u . \tag{2.1d}$$

■

The Lagrangian L(u,q,r) can be rewritten as follows;

$$L(u,q,r) = E_\infty [y^T Q(q) y + u^T R(r) u] - q^T\sigma^2 - r^T\mu^2 \tag{2.2a}$$

where

$$Q(q) \equiv Q_o + \text{diag}[q] , \tag{2.2b}$$

$$R(r) \equiv R_o + \text{diag}[r] , \tag{2.2c}$$

$$\sigma^2 \equiv [\sigma_1^2, \sigma_2^2, \cdots, \sigma_{n_y}^2]^T , \tag{2.2d}$$

$$\mu^2 \equiv [\mu_1^2, \mu_2^2, \cdots, \mu_{n_u}^2]^T , \tag{2.2e}$$

where diag[•] is the diagonal matrix with the i-th diagonal element equal to the i-th element of vector •. One may immediately observe that the above Lagrangian (2.2a) is simply the objective function with the weighting matrices Q(q) and R(r) in a Linear Quadratic (LQ) or Linear Quadratic Gaussian (LQG) optimal control problem [11], save the terms $q^T\sigma^2$ and $r^T\mu^2$ which are constant. Since we are looking for an *optimal* (not suboptimal) control, we exclude the static output feedback control (whose solution is not guaranteed to be unique) so that the uniqueness of the solution to the above Lagrangian problem is ensured. It will be seen later that the uniqueness is critical in our development.

It is said that $(\bar{u}, \bar{q}, \bar{r})$ is a *saddle point* for the Lagrangian function L(u,q,r) if

$$L(\bar{u}, \bar{q}, \bar{r}) \le L(u, \bar{q}, \bar{r}) \quad \text{for all } u\in\Omega \tag{2.3a}$$

and

$$L(\bar{u}, \bar{q}, \bar{r}) \ge L(\bar{u}, q, r) \text{ for all } q\ge 0 \text{ and } r\ge 0. \tag{2.3b}$$

Then the following theorem provides a sufficient condition for $\bar{u}$ to be a solution to the IOVC control problem (1.2);

Theorem 1 [12]

Let $\bar{u} \in\Omega$, $\bar{q} \ge 0$ and $\bar{r} \ge 0$. Then $(\bar{u}, \bar{q}, \bar{r})$ is a saddle point for L(u,q,r) if and only if the following conditions are satisfied;

(a) $\bar{u}$ solves the Lagrangian problem (2.1) with $q = \bar{q}$ and $r = \bar{r}$.

(b) $\begin{cases} E_\infty \ \bar{y}_i^2 \le \sigma_i^2, & i = 1, \cdots, n_y \text{ and} \\ E_\infty \ \bar{u}_j^2 \le \mu_j^2, & j = 1, \cdots, n_u . \end{cases}$

(c) $\begin{cases} \bar{q}_i(E_\infty \ \bar{y}_i^2 - \sigma_i^2) = 0, & i = 1, \cdots, n_y \text{ and} \\ \bar{r}_j(E_\infty \ \bar{u}_j^2 - \mu_j^2) = 0, & j = 1, \cdots, n_u . \end{cases}$

Here $E_\infty \ \bar{y}_i^2$ is the i-th output variance when the control law, $\bar{u}$, is applied to the system (1.1). Furthermore, if $(\bar{u}, \bar{q}, \bar{r})$ is a saddle point for L(u,q,r), then $\bar{u}$ solves the IOVC control problem (1.2).

■

Notice that the first saddle point condition (a) in Theorem 1 is a minimization rather than a stationarity, as opposed to the Kuhn-Tucker necessary conditions for optimality. In fact, only the first condition is different from the Kuhn-Tucker conditions. The exploitation of the minimization of the saddle point conditions will be a very important step for developing a dual version of the IOVC control problem (1.2). The following corollary provides the uniqueness of the solution of the Lagrangian problem (2.1);

Corollary 1 [11]

Suppose that the parameters $\bar{q} \geq 0$ and $\bar{r} \geq 0$ are given, and that the plant (1.1) is both stabilizable and detectable. Then the solution in the condition (a) of Theorem 1 or the solution to the Lagrangian problem is unique, and the resulting closed-loop system is guaranteed to be asymtotically stable. ■

In order to ensure the sufficiency of the solution, the parameter $\bar{q}$ and $\bar{r}$ to be used in the Lagrangian problem (2.1) must be chosen so that other conditions (b) and (c) in Theorem 1 are also satisfied. However, arbitrarily choosing $\bar{q} \geq 0$ and $\bar{r} \geq 0$ will not lead to the satisfaction of the conditions (b) and (c). As a result, the IOVC control problem (1.2) may be viewed as *a search for a positive semi-definite diagonal matrix diag [q] in (2.2b), and a positive semi-definite diagonal matrix diag [r] in (2.2c), to be used in the Lagrangian problem (2.1), so that the conditions (b) and (c) in Theorem 1 are satisfied.* A similar observation is also made in [2]. The following theorem shows that even if the solution of the Lagrangian problem with arbitrarily chosen $\bar{q} \geq 0$ and $\bar{r} \geq 0$ does not satisfy all the conditions in Theorem 1, it provides the solution to a closely related problem;

Theorem 2 [13]

Let $\bar{u} \in \Omega$ be the solution of the Lagrangian problem with arbitrarily chosen $\bar{q} \geq 0$ and $\bar{r} \geq 0$, and define

$$\bar{\sigma}_i^2 \equiv E_\infty \bar{y}_i^2, \; i = 1, \cdots, n_y, \text{ and}$$
$$\bar{\mu}_j^2 \equiv E_\infty \bar{u}_j^2, \; j = 1, \cdots, n_u,$$

where $\bar{y}$ is the output vector achieved by $\bar{u}$. Then $\bar{u}$ solves the following modified IOVC control problem;

$$\left. \begin{array}{l} \min\limits_{u \in \Omega} \; J(u) \\ \text{subject to } \begin{cases} E_\infty y_i^2 \leq \bar{\sigma}_i^2, & i = 1, \cdots, n_y \\ E_\infty u_j^2 \leq \bar{\mu}_j^2, & j = 1, \cdots, n_u \end{cases} \end{array} \right| \quad (2.4)$$

■

Suppose there is no solution to the IOVC control problem (1.2). Even in that case, by obtaining a set of $\bar{\sigma}_i^2$'s and $\bar{\mu}_j^2$'s for different $\bar{q}$'s and $\bar{u}$'s, Theorem 2 may still tell us which input and output variance constraints can not be satisfied and how closely we can achieve the required performance, and so forth.

Now we provide a dual version of the IOVC control problem, which is motivated by Theorem 1;

A Dual Version of IOVC Control Problem

$$\max_{(q,r) \in \Psi} \; h(q,r) \qquad (2.5a)$$

where

$$h(q,r) \equiv \min_{u \in \Omega} \; L(u,q,r), \qquad (2.5b)$$

$$\Psi \equiv \left\{ (q,r) \;\middle|\; \begin{array}{l} q \geq 0, \; r \geq 0 \text{ and} \\ \min\limits_{u \in \Omega} L(u,q,r) \text{ exists} \end{array} \right\}. \qquad (2.5c)$$

■

Notice that the Lagrangian problem is an LQG control problem, and that by Corollary 1 the unique solution of the LQG problem always exists under the assumption of stabilizability and detectability. Hence the domain of definition of h(q,r) becomes

$$\Psi = \{ (q,r) \mid q \geq 0 \text{ and } r \geq 0 \}. \qquad (2.6)$$

In general, the set of all stabilizing control laws, Ω, is open and unbounded. However, under a mild restriction, we can get a compact set. The following theorem is very important for developing the relationship between the IOVC control problem (1.2) and its dual version (2.5);

Theorem 3 [14]

For given $q \geq 0$, $r \geq 0$ and a stabilizing control law $\hat{u} \in \Omega$, define a level set

$$\hat{\Omega} \equiv \{ u \mid L(u,q,r) \leq L(\hat{u},q,r) \}. \qquad (2.7)$$

Then the set $\hat{\Omega}$ is closed and bounded, or compact. ■

Notice that by choosing $\hat{u}$ such that $L(\hat{u},q,r)$ is very large, we can make $\hat{\Omega}$ almost the same as Ω. From now on, for the set of stabilizing control laws, we shall use $\hat{\Omega}$ in lieu of Ω.

3. IOVC Algorithm

Before we give a solution algorithm to the dual version to the IOVC problem, some properties of the dual function h(q,r) are given in the following lemmas, whose proofs are given for completeness.

Lemma 1

$h(q,r)$ is a concave function over the convex set Ψ.

Proof :

Let $(\tilde{q}, \tilde{r}) \in \Psi$ and $(\hat{q}, \hat{r}) \in \Psi$. For $0 \le \alpha \le 1$,

$$
\begin{aligned}
& h(\alpha(\tilde{q},\tilde{r}) + (1-\alpha)(\hat{q},\hat{r})) \\
&= \min_{u \in \hat{\Omega}} L(u, \alpha(\tilde{q},\tilde{r}) + (1-\alpha)(\hat{q},\hat{r})) \\
&= \min_{u \in \hat{\Omega}} [\alpha L(u,\tilde{q},\tilde{r}) + (1-\alpha) L(u,\hat{q},\hat{r})] \\
&\ge \alpha \min_{u \in \hat{\Omega}} L(u,\tilde{q},\tilde{r}) + (1-\alpha) \min_{u \in \hat{\Omega}} L(u,\hat{q},\hat{r}) \\
&= \alpha h(\tilde{q},\tilde{r}) + (1-\alpha) h(\hat{q},\hat{r}) .
\end{aligned}
$$

This implies that $h(q,r)$ is a concave function. The convexity of the set Ψ (2.6) is obvious.

■

Lemma 2

The inequality $h(q,r) \le J(u)$ holds for all $(q,r) \in \Psi$, and for all $u \in \hat{\Omega}$ satisfying the constraints of the IOVC problem (1.2).

Proof :

From the definition of $h(q,r)$ in (2.5a),

$$
h(q,r) \le J(u) + \sum_{i=1}^{n_y} q_i (E_\infty y_i^2 - \sigma_i^2) + \sum_{j=1}^{n_u} r_j (E_\infty u_j^2 - \mu_j^2) .
$$

Then for $u \in \hat{\Omega}$ satisfying the constraints of the IOVC problem (1.2), it is clear that

$$
h(q,r) \le J(u) .
$$

■

Lemma 2 may be utilized for a stopping criteria in an iterative algorithm since the optimal value of $J(u)$ is bounded by both $h(q,r)$ and $J(u)$.

Lemma 3

$h(q,r)$ is differentiable at $(\bar{q}, \bar{r}) \in \Psi$, and the partial derivative is given by

$$
\left.\frac{\partial h}{\partial q_i}\right|_{(q,r)=(\bar{q},\bar{r})} = E_\infty \bar{y}_i^2 - \sigma_i^2 , \qquad (2.8a)
$$

$$
\left.\frac{\partial h}{\partial r_i}\right|_{(q,r)=(\bar{q},\bar{r})} = E_\infty \bar{u}_i^2 - \mu_i^2 , \qquad (2.8b)
$$

where $\bar{y}$ is the output vector when the control law is $\bar{u}$, which is the solution to the Lagrangian problem with $(q,r) = (\bar{q},\bar{r})$.

Proof :

For a given $(\bar{q},\bar{r}) \in \Psi$, $L(u,\bar{q},\bar{r})$ is minimized over $\hat{\Omega}$ at a unique point (see Corollary 1). From Theorem 3, the set $\hat{\Omega}$is closed and bounded. Then from the Corollary 1 in [12, p426], the lemma is proved.

■

The relationship between the IOVC control problem and its dual version is now given in the following theorem;

Theorem 4

Let $(\bar{q},\bar{r}) \in \Psi$ solve the problem (2.5). Then the unique solution to the Lagrangian problem (2.1) with $(q,r) = (\bar{q},\bar{r})$ solves the IOVC control problem (1.2).

Proof :

Since $h(q,r)$ is differentiable at $(q,r) = (\bar{q},\bar{r})$ by Lemma 3, the following optimality conditions for the problem (2.5) should hold;

(a) If $\bar{q}_i > 0$ and $\bar{r}_j > 0$,

$$
\left.\frac{\partial h}{\partial q_i}\right|_{(q,r)=(\bar{q},\bar{r})} = E_\infty \bar{y}_i^2 - \sigma_i^2 = 0 ,
$$

$$
\left.\frac{\partial h}{\partial r_i}\right|_{(q,r)=(\bar{q},\bar{r})} = E_\infty \bar{u}_i^2 - \mu_i^2 = 0 ,
$$

(b) If $\bar{q}_i = 0$ and $\bar{r}_j = 0$,

$$
\left.\frac{\partial h}{\partial q_i}\right|_{(q,r)=(\bar{q},\bar{r})} = E_\infty \bar{y}_i^2 - \sigma_i^2 \le 0 ,
$$

$$
\left.\frac{\partial h}{\partial r_i}\right|_{(q,r)=(\bar{q},\bar{r})} = E_\infty \bar{u}_i^2 - \mu_i^2 \le 0 ,
$$

The above conditions together may be written as

$$
\begin{cases}
E_\infty \bar{y}_i^2 \le \sigma_i^2, & i = 1, \cdots, n_y , \\
E_\infty \bar{u}_j^2 \le \mu_j^2, & j = 1, \cdots, n_u ,
\end{cases}
$$

and

$$
\begin{cases}
\bar{q}_i (E_\infty \bar{y}_i^2 - \sigma_i^2) = 0, & i = 1, \cdots, n_y , \\
\bar{r}_j (E_\infty \bar{r}_j^2 - \mu_j^2) = 0, & j = 1, \cdots, n_u .
\end{cases}
$$

These conditions, together with the fact that $\bar{u}$ solves the Lagrangian problem with $(q,r) = (\bar{q},\bar{r})$, imply that $(\bar{u},\bar{q},\bar{r})$ is a saddle point (see Theorem 1). So, by Theorem 1, $\bar{u}$ solves the IOVC control problem (1.2).

■

Based upon the above theorem, we provide a new algorithm to solve the IOVC control problem. Notice that since the dual version (2.5) is a convex programming problem (see Lemma 1), any solution of (2.5) will be a globally maximum point [15].

IOVC Algorithm

Enter A_p, B_p, C_p, D_p, W_p, $Q_o \geq 0$, $R_o > 0$, $q^{(0)} \geq 0$ and $r^{(0)} \geq 0$.

Step 1. Set k = 0.

Step 2. Solve the Lagrangian problem (2.1) with $Q(q) \equiv Q_o + \text{diag}[q^{(k)}]$ and $R(r) \equiv R_o + \text{diag}[r^{(k)}]$.

Step 3. Calculate the value of the dual objective function h(q,r) in (2.5b) and, if necessary, the derivatives $\frac{\partial h}{\partial q}$ and $\frac{\partial h}{\partial r}$ by (2.8).

Step 4. Set k = k + 1 and utilize the information obtained in Step 3 to update $q^{(k)}$ and $r^{(k)}$ as follows;

$$\begin{Bmatrix} q^{(k+1)} \\ r^{(k+1)} \end{Bmatrix} = \begin{Bmatrix} q^{(k)} \\ r^{(k)} \end{Bmatrix} + \alpha s^{(k)}$$

where α is the step size and $s^{(k)}$ is the search direction.

Step 5. Repeat Steps 2 to 4 until convergence. ■

Notice that in Step 4, α and $s^{(k)}$ should be calculated so that $q^{(k+1)} \geq 0$ and $r^{(k+1)} \geq 0$ must be satisfied. They may be obtained by any existing algorithm [15] under the constraints $q^{(k+1)} \geq 0$ and $r^{(k+1)} \geq 0$. For the LQG control law, h(q,r) and its derivatives at the k-th iteration are given by [7],

$$h(q^{(k)},r^{(k)}) = \text{tr}\,[(K + L)D_pW_pD_p^T + LFVF^T] - q^{(k)T}\sigma^2 - r^{(k)T}\mu^2 , \quad (2.9a)$$

$$\begin{cases} \left.\frac{\partial h}{\partial q}\right|_{(q^{(k)},r^{(k)})} = Y - \sigma^2 , \\ \left.\frac{\partial h}{\partial r}\right|_{(q^{(k)},r^{(k)})} = U - \mu^2 \end{cases} \quad (2.9b)$$

where

$$0=KA_p+A_p^TK-KB_pR^{-1}(r^{(k)})B_p^TK+C_p^TQ(q^{(k)})C_p , \quad (2.9c)$$

$$0=PA_p^T+A_pP-PM_p^TV^{-1}M_pP+D_pW_pD_p^T , \quad (2.9d)$$

$$0=(A_p-FM_p)^TL+L(A_p-FM_p)+G^TR(r^{(k)})G , \quad (2.9e)$$

$$0=(A_p+B_pG)X_c+X_c(A_p+B_pG)^T+FVF^T , \quad (2.9f)$$

$$G=-R^{-1}(r^{(k)})B_p^TK , \quad (2.9g)$$

$$F=PM_p^TV^{-1} , \quad (2.9h)$$

$$Y \equiv \begin{Bmatrix} [C_p(P + X_c)C_p^T]_{11} \\ [C_p(P + X_c)C_p^T]_{22} \\ \vdots \\ [C_p(P + X_c)C_p^T]_{n_y n_y} \end{Bmatrix} , \quad (2.9i)$$

$$U \equiv \begin{Bmatrix} [GX_cG^T]_{11} \\ [GX_cG^T]_{22} \\ \vdots \\ [GX_cG^T]_{n_u n_u} \end{Bmatrix} , \quad (2.9j)$$

where $[\,\bullet\,]_{ii}$ denotes the (i,i) element. The following Kalman filter is used above;

$$\left.\begin{aligned} \dot{x}_c &= (A_p + B_pG - FM_p)\, x_c + Fz , \\ z &= M_px + v , \\ u &= G z , \end{aligned}\right\} \quad (2.10)$$

where M_p is the measurement distribution matrix and the measurement noise v is assumed to be a zero-mean Gaussian white noise with intensity V. Notice that (2.9d) is not dependent upon the iteration number k so it is calculated only once during iteration. In other words, the filter gain, F, is fixed. Hence one Riccati equation and one Lyapunov equation are to be solved to evaluate either h(q,r) or its derivatives for Steps 2 to 4. The IOVC algorithm given above may be used extensively for selecting a set of actuators from given set of candidate actuators while output variance constraints are satisfied [16].

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