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An Empirical Comparison of Static Fuzzy Relational Model Identification Algorithms

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ABSTRACT

An empirical comparison of static fuzzy relational models which are identified with different fuzzy implication operators and inferred by different composition operators is made in case that all the information is represented by the fuzzy discretization. Four performance measures(integral of mean squared error, maximal error, fuzzy equality index and mean lack of sharpness) are adopted to evaluate and compare the quality of the fuzzy relational models both at the numerical level and logical level. As the results, the fuzzy implication operators useful in various fuzzy modeling problems are discussed and it is empirically shown that the selection of data pairs is another important factor for identifying the fuzzy model with high quality.

1. Introduction

Fuzzy relational calculus makes it realizable to handle the ambiguity of the relationships between concepts. Fuzzy relational equations(f.r.e for short) can be viewed as the computational structure for the calculus and it's effectiveness for the approximate reasoning was well illustrated in [1]. The overall presentation concerned with the resolutions and some applicational issues of them was made by Pedrycz[2]. The special statement on the fuzzy modelling was made in [3]. The great importance of the f.r.e lies in the fact that provided with the resolution for the practical usage, the systematic and creative analysis of the existing topics in fuzzy set theory could be made further progress.

Since Sanchez[4] presented a greatest resolution of sup-min type f.r.e, many resolutions were proposed and applied to system identification by several authors[5-8]. In general, there are two ways in constructing fuzzy relational model for a process. The first way is to formalize the linguistic description of operator for the process. The second way is to directly compute the fuzzy relation from the I/O(input/output) data of the process. In both ways the definition of fuzzy implication operator(FIO for short) which is used to quantize and interpret the linguistic rule " IF ~ THEN ~ " is the most important factor which influences the quality of the fuzzy model identified. Many authors investigated the influence of FIO on the quality of fuzzy model identified in the first way. Kiszka[9] compared seventy-two FIO's by the performance measures of integral of mean squared error. Cao and Kandel[10] compared nine FIO's by various performance measures which have practical meaning. But is not made yet the practical and general investigation of the influence of FIO on the quality of fuzzy models identified in the second ways.

The purpose of this paper is twofold. First, an empirical comparison of some fuzzy relational models which are identified with the different FIO's and inferred by different composition operators are made under the criteria of GMP(generalized modus ponens) and GMT(generalized modus tollens). Second, it is elucidated that when the fuzzy sets are represented by the use of fuzzy discretization the selection of data set is also an important factor which seriously affects the quality of the identified fuzzy relational models. To these purposes, four numerical and logical performance measures(integral of mean squared error, maximal error, fuzzy equality index and mean lack of sharpness) are adopted and an extensive simulation is performed for three representative I/O functions with two types of data sets. The inputs of data sets of the first type are equally spaced and the inputs of data sets of the second type are random data uniformly distributed.

In section 2, the forms of fuzzy reasoning system and the structures of fuzzy relational model for them are illustrated. In section 3, the procedures for computing the fuzzy relation are described. In section 4, the performance measures adopted to evaluate and compare the quality of the fuzzy relational models at the logical and numerical level are given. In section 5, the comparison of the fuzzy relational models with different composition operators and FIO's under the criteria of GMP and GMT is given by examining the extensive simulation results for the three I/O functions. Finally, the conclusions obtained in this paper are discussed in section 6.

2. Structures of fuzzy relational model

In any system identification procedure, we can distinguish three stages.

a) determination of the structure of the systems

b) calculation of the parameters of the structure imposed in the first stage

c) validation of the model of the system

Dealing with the fuzzy systems which are described by the f.r.e, the structure of the system is mainly determined by the composition operator used to perform the approximate reasoning. The parameters of the structure is the fuzzy relation. The validity of the fuzzy model is examined in two point of view: numerical and logical sense. Further discussions for the calculation of fuzzy relation and validation of fuzzy model will be given in section III and IV, respectively. In this section, the structures of the fuzzy model are discussed.

In order to perform the approximate reasoning, Zadeh[1] suggested a rule called ' compositional rule of inference (CRI for short) '. There are two forms of CRI, one is generalized modus ponens (GMP) and the other is generalized modus tollens (GMT) which can be represented as follows.

■ generalized modus ponens(GMP);

antecedent: A is U
antecedent: if A is U_i then B is Y_i, i=1,2,...,N (2.1)

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consequence: Y = ?

■ generalized modus tollens(GMT);

antecedent: B is Y
antecedent: if A is U_i then B is Y_i, i=1,2,...,N (2.2)

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consequence: U = ?

These two CRI's are frequently applied to the approximate reasoning in decision making, control, prediction problems, etc. In the approximate reasoning by use of fuzzy relation eq. (2.1) and (2.2) are represented by the following f.r.e's (2.3) and (2.4), respectively.

$$Y = U \circ R \tag{2.3}$$

$$U = Y \circ R^{-1} \tag{2.4}$$

where $U \in F(\mathbf{U})$, $Y \in F(\mathbf{Y})$ are fuzzy subsets defined on the respective universe of discourse and $R \in F(\mathbf{U} \times \mathbf{Y})$ ($\times$: cartesian product) is the fuzzy relation representing the relationship between U and Y. $F(\cdot)$ denotes a family of fuzzy sets (or relations) defined on a relevant universe of discourse (or Cartesian product space). In system identification, U and Y represent the physical variables of a process to be identified and R represents the static behavior of the process. The notation $\circ$ represents the composition operator which mainly determine the structure of a fuzzy model. In this paper, three types of composition operator which are discussed in [11] are considered : max-min, max-prod and max-$\wedge$ composition.

The other factors which determine the structure of a fuzzy model and relate with the practical use of it are fuzzification, defuzzification procedures and representation procedure of fuzzy sets in eq. (2.3) and (2.4). In this paper, fuzzification and defuzzification are done by the singleton method and center of gravity, respectively. As the fuzzy set representation procedure, the fuzzy discretization[8] is used.

By the fuzzy discretization procedure, every fuzzy set in eq. (2.3) and (2.4) is represented with respect to the reference fuzzy sets defined on the underlying universe of discourse. For example, let the reference fuzzy sets defined on $\mathbf{Y}$ be $Y^1, Y^2, \cdots, Y^{c_Y}$. Then any fuzzy set $\tilde{y}$ can be represented by its possibility measure as follows.

$$Y = [\ \mathrm{Poss}(\tilde{y} \mid Y^1), \mathrm{Poss}(\tilde{y} \mid Y^2), \cdots, \mathrm{Poss}(\tilde{y} \mid Y^{c_Y})] \tag{2.5}$$

Using this approach to the approximation of any fuzzy sets can greatly reduce the burden of memory capacity and make the unified treatment of fuzzy and nonfuzzy information possible in the fuzzy modelling problem and other applications of f.r.e. In addition to the above advantages, the relation between the fuzzy reasoning by the f.r.e and that by the implication statements can be easily elucidated. That is, each reference fuzzy set can be thought as a fuzzy set corresponding to a linguistic label. Therefore, a set of implication statements can be induced from the fuzzy relation R in eq. (2.3) as follows.

IF u *is* U^1 THEN y *is* Y^1
IF u *is* U^1 THEN y *is* Y^2
....
IF u *is* U^{c_u} THEN y *is* Y^{c_y}

where c_u and c_y are the numbers of reference fuzzy sets for the input and output of the process. Then each element of R can be thought as representing the truth degree of corresponding statement.

3. Fuzzy model identification procedures.

In the fuzzy modelling problem based on the f.r.e approach, the fuzzy model identification procedure is concerned with the calculation of fuzzy relation R in eq. (2.3) and eq. (2.4). In off-line fuzzy modelling problem, the procedure is performed for the selected I/O data and in on-line fuzzy modelling problem, for the I/O data observed in real-time operation. The FIO plays central role in the fuzzy model identification procedures and has most serious influence on the quality of the fuzzy model identified.

The implication operators selected for the comparison purpose are summarized in Table 1. I_2(α-operator) and I_8(β-operator) are the FIO's which are theoretically verified to provide the maximal and least solution of f.r.e (2.3), respectively. But, they have meaning only when the assumption of solvability is satisfied for all the selected I/O data. If the assumption is violated, a useless model may often be obtained. I_9 presented by Baboshin and Naryshikin[7] is the FIO to ensure the least solution of the f.r.e (2.3). I_9 is selected because the solution by it is asserted to provide the inferred output with low fuzziness if the I/O data are properly selected. The other FIO's are those developed in logical and empirical background.

Table 1 Fuzzy implication operators

	Fuzzy Implication operators
I_1	$a \rightarrow b = \begin{cases} 1 & \text{, if } a \le b \\ 0 & \text{, otherwise} \end{cases}$
I_2	$a \rightarrow b = \begin{cases} 1 & \text{, if } a \le b \\ b & \text{, otherwise} \end{cases}$
I_3	$a \rightarrow b = a \wedge b$
I_4	$a \rightarrow b = a \cdot b$
I_5	$a \rightarrow b = 1 \wedge (1-a+b)$
I_6	$a \rightarrow b = 1-a+a \cdot b$
I_7	$a \rightarrow b = 0 \vee (a+b-1)$
I_8	$a \rightarrow b = \begin{cases} b & \text{, if } a \le b \\ 0 & \text{, otherwise} \end{cases}$
I_9	$a \rightarrow b = \begin{cases} b & \text{, if } a = 1 \\ 0 & \text{, otherwise} \end{cases}$

In general, the procedure for constructing the fuzzy relational models consists of three steps. First step is the selection of data pairs to be used for constructing the fuzzy model from the I/O data pairs of the process. Second step is the computation of subrelation R_k for each data pair (U_k, Y_k) with an FIO. Final step is the aggregation of subrelations to produce R which is performed by one of the following two aggregation methods.

$$R = \bigcup_{k=1}^{M} R_k = \bigcup_{k=1}^{M} U_k \rightarrow Y_k \qquad (3.1)$$

$$R = \bigcap_{k=1}^{M} R_k = \bigcap_{k=1}^{M} U_k \rightarrow Y_k \qquad (3.2)$$

where $\rightarrow$ represents an implication operator and M is the number of selected I/O data pairs. For the aggregation purpose, eq. (3.1) is used for the implication operators I_3, I_4, and $I_7 \sim I_9$, eq. (3.2) for the implication operators I_1, I_2, I_5 and I_6.

4. Performance measures

The model evaluation is an essential process in the model identification procedures. In the conventional modelling exercises, some numerical performance measures between the model and data are usually employed. In fuzzy modelling, using only the numerical measure may be inadequate and some more logical measures are required because the fuzzy model is the model which are constructed at the logical level. At the same point of view, the 'mean lack of sharpness' is introduced in [12] and the fuzzy equality index and fuzzy confidence interval are recommended in [13]. The mean lack of sharpness measures the fuzziness of the output fuzzy set estimated from the fuzzy model. The fuzzy equality index measures the set-theoretic difference between the estimated output fuzzy set and the real output fuzzy set. These performance measures can be thought as reasonable ones for evaluating the identified fuzzy model in fuzzy sense. But the ultimate purpose of constructing the fuzzy model is usually to apply it in the crisp world, therefore the evaluation by the numerical performance measure is inevitable. Another useful performance measure for evaluating the identified fuzzy model in numerical sense is the maximal error of the fuzzy model adopted by Cao and Kandel[10] to compare the applicability of FIO's. The maximal error is very important because it may often cause serious problem when the identified fuzzy model is implemented for the practical use.

In this paper, four performance measures are adopted to evaluate and compare the representing capability of the fuzzy relational models identified with different implication operators: integral of mean squared(IMS) error, maximal error, mean lack of sharpness and the fuzzy equality index. The detailed descriptions of them are as follows.

IMS error is computed by eq. (4.1).

$$p_1 = \sum_{k=1}^{M_e} (\hat{y}_k - y_k)^2 / M_e \qquad (4.1)$$

where $\hat{y}_k$ is crisp output estimated from fuzzy model and y_k is the real output. M_e is the number of I/O data pairs used in the fuzzy model evaluation process.

The maximal error is computed by eq. (4.2).

$$p_2 = \max_k (\hat{y}_k - y_k), \qquad k = 1, 2, \ldots, M_e \qquad (4.2)$$

The mean lack of sharpness of the estimated output fuzzy set is computed by eq. (4.3)

$$p_3 = \sum_{k=1}^{M_e} (1 - \max_j \hat{Y}_k(j)) / M_e \qquad (4.3)$$

where $j = 1, 2, \cdots, c_y$.

The fuzzy equality index between the estimated output fuzzy set and real output fuzzy set is computed by eq. (4.4).

$$p_4 = \sum_{k=1}^{M} \delta_k / M_e \qquad (4.4)$$

where

$$\delta_k = [\hat{Y}_k \equiv Y_k] = \min_j [\hat{Y}_k(j) \equiv Y_k(j)] \qquad (4.5)$$

where $\equiv$ denotes the fuzzy equality index between two fuzzy quantities and pointwise computation of it is performed by eq. (4.6)

$$[\hat{Y}_k(j) \equiv Y_k(j)]' = \begin{cases} 1 + \hat{Y}_k(j) - Y_k(j) & \text{, if } \hat{Y}_k(j) < Y_k(j) \\ 1 & \text{, if } \hat{Y}_k(j) = Y_k(j) \\ 1 + Y_k(j) - \hat{Y}_k(j) & \text{, if } \hat{Y}_k(j) > Y_k(j) \end{cases} \qquad (4.6)$$

The fuzzy equality index of eq. (4.5) is called as a pessimistic one. If we use max-operator instead of min-operator, an optimistic fuzzy equality index is obtained. If we take the average of pointwise fuzzy equality indices, the average type fuzzy equality index is obtained. The other definitions of fuzzy equality index are obtained by using different definitions of φ-operator(pseudocomplement). The characteristics of other definitions of fuzzy equality index are well discussed in [14].

5. Comparison results

The fuzzy model identification procedures may often show different performance according to the process to be identified. In order to make the comparison more practical and general, in this paper, following three I/O functions which can be thought as the representative characteristics of the process are identified.

$$F_1:\ y = u \tag{5.1}$$

$$F_2:\ y = u^2 - 2u + 1 \tag{5.2}$$

$$F_3:\ y = e^{-2u}\sin(4\pi u) \tag{5.3}$$

If a fuzzy relational model identified with one of the FIO's suffices to represent above three I/O functions, the FIO would be an useful one for the fuzzy modelling of the real processes. Simulation conditions are as follows.

In all the cases, the universes of discourse are defined as [0,1] for u and y. The number of reference fuzzy sets is 9 for u and y. The shape of membership functions are symmetrical triangular and the centers of reference fuzzy sets are equally spaced by the amount of 0.125 as in Fig.1. Two types of data sets are used for the identification of the I/O functions. The data sets of the first type are the ones of which u's are equally spaced. The data sets of the second type are the ones of which u's are uniformly distributed.

D1 and D2 are the data sets pertaining to the first type. D1 consists of nine data pairs(M=9) and u of each data pair is equal to the center of each reference fuzzy set. D2 consists of seventeen data pairs(M=17) and u's of D2 are equally spaced by 0.0625. These data sets are for emphasizing the importance of selection of I/O data set.

Sixty data sets pertaining to the second type are used for the comparison purpose in more practical and general situation of modelling problem. The average and standard deviation of u's of each data set are about 0.5 and 0.24, respectively. They are grouped into three(D3, D4, D5) according to the numbers of data pairs of data sets. Each group consists of twenty data sets. The numbers of data pairs of each data set pertaining to D4, D5 and D6 are 100, 500 and 1000, respectively. The evaluation of the identified fuzzy models are performed for the data set which consists of 101 I/O data pairs(M_e=101) and u's of it is equally spaced by the amount of 0.01. Especially for the D3, D4 and D5, each performance measure is averaged by the number of data sets, that is, twenty.

The simulation results for the fuzzy models identified in the structure of max-product composition operator are summarized in Table 2 and 3 where R_i represents the fuzzy relational model identified with FIO I_i. The comparisons between the fuzzy relational models identified with different FIO's and between the composition operators are as follows.

In order to simplify the discussions on the comparison, at first, we describe the performance of fuzzy models R_1, R_2 and R_9 identified with all the data sets and fuzzy models $R_3 \sim R_8$ identified with data set D1.

For the I/O function F_1, all the fuzzy models identified with data set D1 and inferred by all the three composition operators perfectly satisfy the f.r.e's (2.3) and (2.4). All the fuzzy models identified accord with the intuitional linguistic rules for F_1: only the diagonal elements of fuzzy relation are unity. For the I/O function F_2 and F_3, all the fuzzy models except for R_1 shows equal performance in the same composition operator. For the F_2, the fuzzy models with max-product composition operator shows best performance followed by those with max-min composition operator. For the F_3, the fuzzy models with max-min composition operator shows best performance followed by those with max-product composition operator. The fuzzy models with max-$\wedge$ composition operator shows worst performance for both I/O functions. From these facts it can be said that once the composition operator is decided and data set like D1 is selected the definition of FIO does not influence on the quality of fuzzy models identified.

For the I/O function F_1, R_1 and R_2 show best performance for all the data sets. But, for the I/O function F_2 and F_3, R_1 with all the data sets, R_2 with all the data sets except for D1 and R_9 with all the data sets except for D1 and D2 show worst performance so as not to be applicable in practical usage. As can be seen from the definition of FIO I_9, the reason for bad performance of R_9 with random data sets is that a feasible fuzzy model cannot be obtained until the data pairs of which u's are equal to the centers of r.f.s does not occur. From the above discussions for the fuzzy models R_1, R_2 and R_9 and the fact that the data set like D1 and D2 hardly can be obtained in practical modelling problem, it can be concluded that the FIO's I_1, I_2 and I_9 are inadequate in the fuzzy relational modelling which directly compute the fuzzy relation from the I/O data.

Now we compare the performance of $R_3 \sim R_8$ identified with the data sets D2~D5.

5.1 Comparison between fuzzy relational models with max-min composition operator

For the I/O function F_1, R_5 and R_7 identified with the data sets D2 perfectly and R_6 nearly perfectly satisfy the f.r.e's (2.3) and (2.4). R_3, R_4 and R_8 shows good performance in crisp sense, but not in fuzzy sense. When the data sets of D3, D4 and D5 are used, R_7 nearly perfectly satisfy the f.r.e's (2.3) and (2.4). The performances of R_5 and R_6 getting better as the number of data pairs increases. In this linear relationship case, the increment of the data pairs does not make serious influence on the performance of the identified fuzzy model except for R_5 and R_6. For the I/O function F_2 which is a nonlinear function, R_4 and R_7 show best performance followed by R_3 and R_8. For the I/O function F_3, R_5 and R_6 show best performance followed by R_4 and R_7.

5.2 Comparison between fuzzy relational models with max-product composition operator

The relative comparison results between fuzzy models with max-product composition operator is similar to the previous ones in subsection 5.1 except that the performance p_3 of R_5 is better than that of R_7. The comparison between max-min and max-product composition operators is as follows.

For the I/O function F_1, the fuzzy models except R_7 show better numerical performance than those with max-min composition operator. In logical sense, the fuzzy models except R_6 and R_7 show better performance than those with max-min composition operator. For the I/O function F_2 and F_3, the numerical performance p_1 and p_2 of the fuzzy models with max-product composition operator are better than and the logical performance p_3 and p_4 are worse than those of fuzzy models with max-min composition operator.

A notable thing is that the fuzzy models R_5 and R_6 identified with data set D5 and max-product composition operator nearly perfectly represent the I/O function F_3.

5.3 Comparison between fuzzy relational models with max-∧ composition operator

In this model structure, the relative comparison results between fuzzy models shows somewhat different pattern from the above two comparison results. R_4 and R_7 show best performance in this model structure. The overall performance of fuzzy models in this structure except for some cases are worse than those of fuzzy models in the other two structures. R_3 and R_8 shows better performance being compared with those with max-min composition operator.

5.4 Summarization of comparison results

From the above discussions, it can be seen that it is very difficult to decide which FIO is most adequate for all the environment in system identification. But we think the recommendation of FIO adequate for some restricted environment is available. In this paper, we recommend FIO's for some cases in modelling problem.

At first, I_5 is most adequate FIO for the case that I/O relationship of a process is very nonlinear and a lot of data pairs are available. But, I_5 and I_7 are expected to show similar performance when a lot of data can not be obtained. At second, I_7 is most adequate FIO for the case that the I/O relationship of a process is somewhat nonlinear and monotone and a few I/O data pairs are available. In this case I_4 is also a probable FIO, but it shows worse logical performance than I_7.

6. Conclusions

An empirical comparison of fuzzy relational models which are identified with different FIO's and inferred by different composition operators is made in case that all the information is represented by the fuzzy discretization. The FIO's which are adequate for the identification of fuzzy relational model are recommended for some cases in system identification. Four performance measures are adopted to evaluate and compare the fuzzy relational models not only at the numerical level but also at the logical level. IMS and maximal estimation error is employed for the comparison at the numerical level, and fuzzy equality index and mean lack of sharpness for the comparison at the logical level. It is also shown that the selection of data pairs is another important factor for the identification of fuzzy relational model of high quality.

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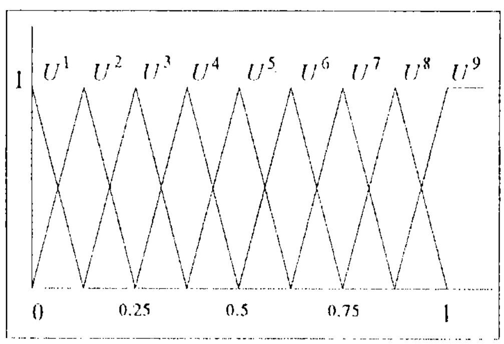

Fig.1 Reference fuzzy sets for input.

Table 2 Performance of static fuzzy relational models (GMP, Max-product composition)

Data Set	FM	Function 1				Function 2				Function 3			
		p_1	p_2	p_3	p_4	p_1	p_2	p_3	p_4	p_1	p_2	p_3	p_4
D1	R_1	0.00000	0.0000	1.0000	0.2471	0.08778	0.7569	0.3785	0.7421	0.21268	0.8851	0.3378	0.4912
	R_2-R_9					0.00002	0.0184	0.7585	0.3791	0.01020	0.2242	0.5512	0.3500
D2	R_1	0.00000	0.0000	1.0000	0.2471	0.20300	1.0000	0.2953	0.9330	0.21268	0.8851	0.2610	1.0000
	R_2	0.00000	0.0000	1.0000	0.2471	0.05712	0.7569	0.4920	0.7214	0.05401	0.6513	0.3546	0.8330
	R_3	0.00033	0.0416	0.6390	0.2471	0.00035	0.0416	0.6670	0.3791	0.01387	0.4082	0.5304	0.3501
	R_4	0.00015	0.0250	0.8220	0.2471	0.00021	0.0407	0.7246	0.3791	0.03276	0.3273	0.5510	0.3501
	R_5	0.00000	0.0000	1.0000	0.2471	0.00004	0.0184	0.7273	0.4581	0.00221	0.1222	0.5276	0.5768
	R_6	0.00000	0.0000	0.8117	0.4353	0.00006	0.0189	0.6877	0.5081	0.00238	0.1222	0.5137	0.5978
	R_7	0.00000	0.0000	1.0000	0.2471	0.00020	0.0398	0.7573	0.3791	0.02124	0.2966	0.5560	0.3501
	R_8	0.00033	0.0416	0.6390	0.2471	0.00041	0.0141	0.6983	0.3791	0.02657	0.3308	0.5310	0.3501
	R_9	0.00000	0.0000	1.0000	0.2471	0.00002	0.0184	0.7585	0.3791	0.01020	0.2242	0.5512	0.3501
D3	R_1	0.00000	0.0000	1.0000	0.2471	0.20300	1.0000	0.2994	0.9284	0.21268	0.8852	0.2611	1.0000
	R_2	0.00000	0.0152	0.9975	0.2471	0.09062	0.7691	0.3084	0.8151	0.19530	0.8852	0.2799	0.9492
	R_3	0.00037	0.0452	0.6625	0.2790	0.00076	0.0832	0.6601	0.3516	0.06622	0.5000	0.5322	0.3865
	R_4	0.00021	0.0326	0.8157	0.3074	0.00039	0.0621	0.7231	0.3914	0.05196	0.4499	0.5714	0.4218
	R_5	0.00830	0.2792	0.9536	0.2471	0.01490	0.3097	0.7101	0.4681	0.02525	0.3394	0.5281	0.6444
	R_6	0.01223	0.3150	0.8116	0.4348	0.01619	0.3155	0.6709	0.5207	0.02679	0.3467	0.5187	0.6601
	R_7	0.00001	0.0098	0.9280	0.3101	0.00032	0.0172	0.7435	0.3958	0.01253	0.4176	0.5780	0.4302
	R_8	0.00036	0.0440	0.6886	0.2790	0.00097	0.1250	0.6388	0.4031	0.05243	0.4626	0.5583	0.3981
	R_9	0.24029	0.9375	0.3063	0.8868	0.17972	0.9692	0.2062	0.9149	0.20458	0.8852	0.2954	0.9018
D4	R_1	0.00000	0.0000	1.0000	0.2471	0.20300	1.0000	0.2954	0.9331	0.21268	0.8852	0.2611	1.0000
	R_2	0.00000	0.0027	0.9997	0.2471	0.15258	0.8006	0.3857	0.8355	0.19530	0.8852	0.2775	0.9587
	R_3	0.00033	0.0119	0.6456	0.2540	0.00076	0.0809	0.6533	0.3258	0.07010	0.5118	0.4985	0.3439
	R_4	0.00016	0.0250	0.8221	0.2607	0.00031	0.0557	0.7300	0.3639	0.05534	0.4630	0.5386	0.3614
	R_5	0.00076	0.0841	0.9000	0.2471	0.00173	0.1023	0.6973	0.4824	0.00409	0.1518	0.4922	0.6788
	R_6	0.00122	0.1049	0.8118	0.4353	0.00187	0.1043	0.6642	0.5242	0.00414	0.1550	0.4901	0.6821
	R_7	0.00000	0.0017	0.9842	0.2608	0.00027	0.0407	0.7524	0.3653	0.01858	0.4456	0.5423	0.3630
	R_8	0.00033	0.0117	0.6511	0.2540	0.00096	0.1250	0.6221	0.3672	0.06739	0.4984	0.5108	0.3468
	R_9	0.13697	0.6925	0.4665	0.6617	0.06417	0.6777	0.4267	0.7145	0.13875	0.8148	0.3855	0.7363
D5	R_1	0.00000	0.0000	1.0000	0.2471	0.20300	1.0000	0.2954	0.9331	0.21268	0.8852	0.2611	1.0000
	R_2	0.00000	0.0016	0.9998	0.2471	0.17274	0.9392	0.3849	0.8369	0.19530	0.8852	0.2775	0.9592
	R_3	0.00033	0.0117	0.6431	0.2501	0.00077	0.0811	0.6515	0.3221	0.07104	0.5139	0.4905	0.3330
	R_4	0.00016	0.0255	0.8221	0.2531	0.00030	0.0552	0.7307	0.3601	0.05572	0.4638	0.5309	0.3479
	R_5	0.00025	0.0540	0.9955	0.2471	0.00062	0.0688	0.6950	0.4842	0.00287	0.1530	0.4877	0.6810
	R_6	0.00043	0.0691	0.8118	0.4353	0.00066	0.0702	0.6640	0.5244	0.00288	0.1529	0.4863	0.6835
	R_7	0.00000	0.0009	0.9928	0.2531	0.00026	0.0406	0.7534	0.3613	0.01921	0.4490	0.5336	0.3485
	R_8	0.00033	0.0417	0.6454	0.2501	0.00099	0.1250	0.6177	0.3616	0.06965	0.5100	0.4978	0.3352
	R_9	0.06320	0.4917	0.6456	0.4857	0.02982	0.5401	0.5576	0.5748	0.10126	0.7692	0.4385	0.5682

Table 3 Performance of static fuzzy relational models (GMT, Max-product composition)

Data set	FM	Function 1				Function 2			
		p1	p2	p3	p4	p1	p2	p3	p4
D1	R_1	0.00000	0.0000	1.0000	0.2471	0.03509	0.3800	0.3746	0.5988
	R_2-R_9					0.00157	0.0086	0.6267	0.3536
D2	R_1	0.00000	0.0000	1.0000	0.2471	0.10331	0.6400	0.2764	0.7616
	R_2	0.00000	0.0000	1.0000	0.2471	0.01095	0.2900	0.3781	0.6544
	R_3	0.00033	0.0116	0.6390	0.2471	0.00110	0.1219	0.5863	0.3132
	R_4	0.00015	0.0250	0.8220	0.2471	0.00105	0.1108	0.6288	0.3138
	R_5	0.00000	0.0000	1.0000	0.2471	0.00162	0.0086	0.5710	0.4009
	R_6	0.00000	0.0000	0.8117	0.4353	0.00195	0.0091	0.5525	0.4626
	R_7	0.00000	0.0000	1.0000	0.2471	0.00143	0.0086	0.6409	0.3143
	R_8	0.00033	0.0116	0.6390	0.2471	0.00133	0.1219	0.5872	0.3536
	R_9	0.00000	0.0000	1.0000	0.2471	0.00157	0.0086	0.6267	0.3536
D3	R_1	0.00000	0.0000	1.0000	0.2471	0.10300	0.6400	0.2750	0.7617
	R_2	0.00002	0.0153	0.9872	0.2471	0.04386	0.4760	0.2873	0.7313
	R_3	0.00037	0.0452	0.6625	0.2790	0.00150	0.1402	0.5939	0.3232
	R_4	0.00021	0.0326	0.8157	0.3074	0.00106	0.1162	0.6197	0.3585
	R_5	0.00705	0.1911	0.8823	0.2471	0.01877	0.3552	0.5550	0.4260
	R_6	0.01071	0.2321	0.8093	0.4348	0.01970	0.3588	0.5145	0.4774
	R_7	0.00001	0.0098	0.9280	0.3101	0.00133	0.1021	0.6607	0.3637
	R_8	0.00039	0.0441	0.6809	0.2790	0.00408	0.2093	0.6287	0.3542
	R_9	0.24029	0.9375	0.3063	0.8868	0.18090	0.7187	0.2029	0.8617
D4	R_1	0.00000	0.0000	1.0000	0.2471	0.10332	0.6400	0.2765	0.7617
	R_2	0.00000	0.0027	0.9983	0.2471	0.01953	0.5000	0.2779	0.7448
	R_3	0.00033	0.0419	0.6456	0.2540	0.00150	0.1387	0.5763	0.2931
	R_4	0.00016	0.0259	0.8221	0.2607	0.00106	0.1121	0.6451	0.3266
	R_5	0.00048	0.0497	0.9780	0.2471	0.00531	0.2742	0.5531	0.4406
	R_6	0.00081	0.0644	0.8118	0.4353	0.00561	0.2750	0.5415	0.4851
	R_7	0.00000	0.0017	0.9842	0.2608	0.00141	0.0990	0.6620	0.3280
	R_8	0.00033	0.0417	0.6510	0.2540	0.00402	0.2110	0.6225	0.3175
	R_9	0.13697	0.6925	0.4665	0.6617	0.09520	0.5205	0.4157	0.6743
D5	R_1	0.00000	0.0000	1.0000	0.2471	0.10332	0.6400	0.2765	0.7617
	R_2	0.00000	0.0016	0.9990	0.2471	0.04961	0.5000	0.2769	0.7457
	R_3	0.00033	0.0417	0.6431	0.2501	0.00150	0.1385	0.5734	0.2887
	R_4	0.00016	0.0255	0.8221	0.2531	0.00105	0.1118	0.6443	0.3224
	R_5	0.00011	0.0246	0.9883	0.2471	0.00313	0.2157	0.5528	0.4417
	R_6	0.00019	0.0323	0.8118	0.4353	0.00342	0.2163	0.5414	0.4853
	R_7	0.00000	0.0009	0.9928	0.2531	0.00142	0.0990	0.6614	0.3233
	R_8	0.00033	0.0417	0.6453	0.2501	0.00101	0.2115	0.6191	0.3110
	R_9	0.06320	0.4917	0.6456	0.4857	0.02301	0.2836	0.5352	0.5216