Adaptive Sliding-Mode Tracking Control in the Presence of Unmodeled Dynamics

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Abstract

To increase the robustness of the feedforward tracking control system, a new discrete time sliding function has been defined and utilized for the formulation of control law. In adaptive case the robustness is achieved by using both a normalized gradient algorithm with deadzone and a sliding function-based nonlinear feedback, while in nonadaptive case by using only a sliding function -based nonlinear feedback.

1 Introduction

When the desired position is time varying,the tracking performance can be significantly improved by the feedforward tracking controller. For minimum phase systems the feedforward tracking controller can be designed to achieve perfect tracking based on stable pole-zero cancellation.

In fact the model/plant mismatches eventually lead to an imperfect pole-zero cancellation, which in turn may significantly degrade the tracking performance and even result in the instabilities of the overall system. Mismatch in pole-zero cancellation causes the motivation of using Diophantine equation for constructing sliding function.

In analyzing these uncertainties we adopt the following two cases,i.e.,nonadaptive and adaptive. In nonadaptive case we construct the nonlinear feedback control law which is based on discrete time sliding function. In adaptive case these uncertainties may cause the divergence of the adaptive process, which has been resulted in the introduction of deadzone in the estimator to bring the robust results in the adaptive control system(*Egardt 1978,Samson 1983,Ioannou and Kokotovic 1983,Kreisselmeier and Anderson 1986*).

Deadzone and sliding boundary layer is intended to provide robustness in control system against modeling errors. In this paper we try that their design concept be set up on the same basis, and try to match some relationship between them.

2 Robust Discrete Time Tracking Control

The controlled plant is assumed to be represented by the following discrete time model:

$$y(k) = \frac{z^{-d}\, B(z^{-1})}{A(z^{-1})}\, u(k) \quad + \eta(k) \tag{1}$$

where

- $u(k)$ and $y(k)$ are the measurable input and output respectively,
- $\eta(k)$ represents the modeling error,
- $A(z^{-1})$ and $B(z^{-1})$ are polynomials of order n and m respectively in the backward shift operator z^{-1}.

The order n and m as well as the delay step d are assumed to be known. It is further assumed that

A1 : $A(z^{-1})$ and $B(z^{-1})$ are coprime.

A2 : $\mid \eta(k) \mid \leq \mu\ m(k)$, where μ is a positive scalar and $m(k)$ is defined by

$$m(k) = \sigma\ m(k-1) + \mid u(k-1) \mid + \mid y(k-1) \mid, \tag{2}$$

with $0 < \sigma < 1$.

A3 : $\|\theta_1\| \leq \rho_1$, where $\theta_1^T = [a_1, \cdots, a_n, b_0, \cdots, b_m]$ and ρ_1 is a known positive scalar.

The plant (1) can also be expressed as

$$y(k) = \theta_1^T\ \phi_1(k) + \eta_1(k) \tag{3}$$

where

$$\begin{aligned} \phi_1^T(k) &= [-y(k-1), \cdots, -y(k-n), u(k-d), \\ &\quad \cdots, u(k-d-m)], \\ \eta_1(k) &= A(z^{-1})\eta(k). \end{aligned}$$

Then it follows from $A2 - A3$ and (2) that

$$\mid \eta_1(k) \mid = \mid A(z^{-1})\eta(k) \mid \leq \nu_1\ \mu m(k) \tag{4}$$

$$\nu_1 = 1 + \rho_1\sigma^{-n}n \tag{5}$$

In this paper we propose a robust discrete time tracking control by combining pole/zero cancellation and discrete time version of sliding control. For assignment of closed loop poles and setting the structure of the control system, we utilize the following Diophantine equation.

$$D_1(z^{-1}) = A(z^{-1})S_1(z^{-1}) + z^{-d}R_1(z^{-1}) \tag{6}$$

where

From equations (6) and (1), the following equation is derived.

$$D_1(z^{-1})y(k) = \theta_2^T\ \phi_2(k-d) + \eta_2(k) \tag{7}$$

where $R_1(z^{-1})$, $S_1(z^{-1})$ and $D_1(z^{-1})$ are polynomials of order n-1,d-1 and n respectively in the backward shift operator z^{-1}.

$$\theta_2^T = [b_0, \cdots, b_m s_{d-1}; r_0, r_1, \cdots, r_{n-1}], \tag{8}$$
$$\phi_2^T(k) = [u(k), u(k-1), \cdots, u(k-m-d+1), y(k), y(k-1), \cdots, y(k-n+1)]. \tag{9}$$

and $\eta_2(k)$ is expressed as follows

$$\eta_2(k) = \theta_\eta^T\ \phi_\eta(k) \tag{10}$$

To derive the bounds of $|\ \theta_2^T\phi_2(k-d)\ |$ and $|\ \eta_2(k)\ |$ it is necessary to have the following assumption.

A4: $\|\theta_d\| \le \rho_2$, where $\theta_d^T = [s_1, \cdots, s_{d-1}, r_0, \cdots, r_{n-1}]$ and ρ_2 is a known positive scalar.

Proposition 1 $\quad |\ \theta_2^T\ \phi_2(k-d)\ | \le K_{m1}\ \sigma^{1-d-r} m(k).$

$$where \quad r = max\ deg(n, m+d),$$

$$K_{m1} = (m+1)\rho_1 + (m+1)(d-1)\rho_1\rho_2 + n\rho_2.$$

It follows with assumption A4 that

$$|\ \eta_2(k)\ | = |\ A(z^{-1})\ S_1(z^{-1})\ \eta(k)\ | \le \nu_2\mu m(k) \tag{11}$$

with

$$\nu_2 = 1 + 3\rho_1\rho_2 n(d-1)\ \sigma^{-n-(d-1)} \tag{12}$$

In designing a robust discrete time tracking controller we define $s(k)$ by

$$s(k) = D_1(z^{-1})[y(k) - y_m(k)] \tag{13}$$

and add a control loop with $s(k)$ to compensate the modeling error $\eta_2(k)$. Outside the boundary layer ($|\ s(k)\ | \ge \Phi$) as well as inside the boundary layer ($|\ s(k)\ | < \Phi$) the control law becomes

$$u(k) = \frac{1}{b_0}[s(k) + D_1(z^{-1})y_m(k+d) - \bar{\theta}_2^T\bar{\phi}_2(k) - K\,sat\left\{\frac{s(k)}{\Phi}\right\}] \tag{14}$$

where

$$sat\left\{\frac{s(k)}{\Phi}\right\} = \begin{cases} +1, & \text{for } \Phi \le s(k), \\ \frac{s(k)}{\Phi}, & \text{for } -\Phi < s(k) < \Phi, \\ -1, & \text{for } s(k) \le -\Phi. \end{cases} \tag{15}$$

and $\bar{\theta}_2$ and $\bar{\phi}_2(k)$ are defined by

$$\theta_2^T = [b_0,\ \bar{\theta}_2^T] \tag{16}$$
$$\phi_2^T(k) = [u(k),\ \bar{\phi}_2^T(k)] \tag{17}$$

Proposition 2 *For arbitrary $d_0 > 0$ there exists a $\mu_0 > 0$ (which depends on d_0) such that for all $0 \le \mu \le \mu_0$ and arbitrary initial conditions, it follows that*

$$|\ s(k)\ | \le d_0\ m(k)$$

$$where \quad d_0 = \nu_2\mu_0.$$

Proposition 3 *$m(k)$ is bounded, in turn $u(k)$ and $y(k)$ are bounded.*

The condition that Φ and K satisfy will be explained in theorem 1.

Notice that from Eqs (13) and (14), $s(k)$ satisfies

$$s(k+d) = s(k) + \eta_2(k+d) - K\,sat\left\{\frac{s(k)}{\Phi}\right\} \tag{18}$$

The robust stability of the robust discrete time tracking control system is proved in the following theorem.

Theorem 1 *Robust Stability of the Robust Discrete Time Tracking Control System :*

The robust discrete time tracking control system, consisting of the plant (1) and the control law (14), is stable in the sense that $|\ s(k)\ |$ decreases when $|\ s(k)\ | > \Phi$ and that the steady state value of s is bounded by Φ.

Proof: Introduce the following discrete Lyapunov function candidate

$$V(k) = |\ s(k)\ | \tag{19}$$

which can be interpreted as the distance to the surface $s(k) = 0$.

Then we can formulate the followig difference equation.

$$\begin{aligned} \Delta V(k+d) &= V(k+d) - V(k) \\ &= |\ s(k+d)\ | - |\ s(k)\ | \\ &= |\ s(k) + \eta_2(k+d) - K\ sgn\{s(k)\}\ | \\ &\quad - |\ s(k)\ | \end{aligned} \tag{20}$$

Based on the Proposition 3 let us assume that the upperbound of modeling error is constant, i.e.

$$|\ \eta_2(k)\ | \le \nu_2\mu m(k) \le F_2 \tag{21}$$

where F_2 is a bounded scalar. Then the following a couple of condition on K make (20) negative.

$$K > F_2,\ K < 2\ |\ s_i(k)\ | - F_2 \tag{22}$$

If K is selected as following

$$K = F_2 + \eta_0 \tag{23}$$

Then (19) will be a Lyapunov function in the following region.

$$| s_i(k) |> F_2 + \frac{\eta_0}{2} \quad (24)$$

Inside the boundary layer, the s-dynamics become :

$$s(k+d) = (1 - \frac{K}{\Phi})s(k) + \eta_2(k+d) \quad (25)$$

The boundary layer thickness Φ can be selected such that (25) shows characteristics of a first-order filter with input $\eta_2(k+d)$ and eigenvalue.

$$1 - \frac{K}{\Phi} = \lambda \quad (26)$$

From (23) and (26):

$$\Phi = \frac{F_2 + \eta_0}{1 - \lambda} \quad (27)$$

From the stability viewpoint the following relation should be satisfied.

$$| 1 - \frac{K}{\Phi} |< 1 \quad (28)$$

For a stable eigenvalue λ the steady state solution becomes:

$$\lim_{k\to\infty} s_i(k) = \lim_{k\to\infty} \sum_{j=0}^{\infty} \lambda^j \eta_2(k - j \cdot d) \quad (29)$$

From (29) the following relations are derived.

$$| \lim_{k\to\infty} s(k) | \leq \left\{1 + | \lambda | + | \lambda |^2 + \cdots\right\} F_2 \quad (30)$$

$$| \lim_{k\to\infty} s(k) | \leq \frac{1}{1 - | \lambda |} F_2 < \frac{K}{1 - | \lambda |} \quad (31)$$

$$| \lim_{k\to\infty} s(k) | < \Phi \quad (32)$$

3 Robust Adaptive Discrete Time Tracking Control

To maintain the control objective for a plant with unknown dynamics due to modeling error in real time, parameter adaptation schemes coupled with the control law needs to be investigated. A standard approach to incorporate the adaptation scheme with the control law in stability analysis is to guarantee that a norm of the difference between estimated parameters and actual parameters decreases at each step. Deadzone concept has been utilized in most robust adaptive schemes. In this paper we apply normalized gradient parameter estimator as our adaptation scheme.

The algorithm is given by:

$$\hat{\theta}_2(k) = \hat{\theta}_2(k-1) + \frac{a(k)\phi_2(k-d)D(e(k), d(k))}{\phi_2^T(k-d)\phi_2(k-d) + c} \quad (33)$$

where $k \geq d$, $0 < a(k) < 2$, $c > 0$,and $D(e(k), d(k))$ is the continuous function defined by

$$D(e(k), d(k)) = \begin{cases} e(k) - d(k)sgn\{e(k)\}, & \text{if } | e(k) |\geq d(k), \\ 0, & \text{otherwise.} \end{cases} \quad (34)$$

and :

$$\begin{aligned} e(k) &= \theta_2^T\phi_2(k-d) + \eta_2(k) - \hat{\theta}_2^T(k-1)\phi_2(k-d) \\ &= \tilde{\theta}_2^T(k-1)\phi_2(k-d) + \eta_2(k) \end{aligned} \quad (35)$$

Proposition 4 *The estimation algorithm defined by (33) – (35),when applied to plant (7),has the following properties:*

$$\lim_{k\to\infty} \|\hat{\theta}_2(k) - \hat{\theta}_2(k-1)\| = 0 \quad (36)$$

$$\lim_{k\to\infty} D(e(k), d(k)) = 0 \quad (37)$$

In the notation of this section the control law (14) becomes:

$$u(k) = \frac{1}{\hat{b}_0}[s(k) + D_1(z^{-1})y_m(k+d) - \hat{\bar{\theta}}_2^T\bar{\phi}_2(k) + K sat\{\frac{s(k)}{\phi}\}] \quad (38)$$

A reasonable criteria to use in shifting from the non-adaptive to adaptive controller is whether s is inside or outside the boundary layer. That is, when inside the boundary layer, the control law of (38) is used and adaptation does not occur. The control laws should be continuous at the boundary layer edge. For this purpose, note that the non-adaptive control law of (38) takes on the following value at the boundary layer edge.

$$u(k) = \frac{1}{\hat{b}_0}[D_1(z^{-1})y_m(k+d) - \hat{\bar{\theta}}_2^T\bar{\phi}_2(k) + \lambda\Phi sgn\{s(k)\}] \quad (39)$$

The control law of (39) reduces to :

$$\begin{aligned} D_1(z^{-1})y_m(k+d) &= -\lambda\Phi\, sgn\{s(k)\} + \hat{b}_0 u(k) + \hat{\bar{\theta}}_2\bar{\phi}_2(k) \\ &= -\lambda\Phi\, sgn\{s(k)\} + \hat{\theta}_2\phi_2(k) \end{aligned} \quad (40)$$

Recalling (7) :

$$D_1(z^{-1})y(k+d) = \theta_2\phi_2(k) + \eta_2(k+d) \quad (41)$$

From (40) and (41) the s-dynamics becomes :

$$s(k+d) = \lambda\Phi\, sgn\{s(k)\} + \tilde{\theta}_2\phi_2(k) + \eta_2(k+d) \quad (42)$$

Combining (42) and estimation algorithm yields :

$$\lim_{k\to\infty} \frac{[s(k) - \lambda\Phi sgn\{s(k-d)\} - d(k)sgn\{e(k)\}]^2}{\phi_2^T(k-d)\phi_2(k-d) + c} = 0 \quad (43)$$

$$e(k) = s(k) - \lambda\Phi\, sgn\{s(k-d)\} \quad (44)$$

Now select $d(k)$ as :

$$d(k) = (1 - \lambda)\Phi \quad (45)$$

From (27) , this guarantees that $d(k) \geq F_2$ as required.

The condition on adaptation remains as $| e(k) |\geq d(k)$, which from (44) and (45) is met if $| s(k) |\geq \Phi$ (being outside the boundary layer). With this condition satisfied, from (44) it can be seen that $sgn\{e(k)\} = sgn\{s(k)\}$, and so using (45), (43) becomes :

$$\lim_{k\to\infty} \frac{[s(k) - \Phi\, sgn\{s(k-d)\}]^2}{\phi_2^T(k-d)\phi_2(k-d) + c} = 0 \quad (46)$$

Note that condition (1) of Key Technical Lemma{*Goodwin and Sin* 1984) is satisfied, with $p(k) = s(k) - \Phi\, sgn\{s(k)\}$, $b_1(k) = c$, $b_2(k) = 1$, and $\sigma(k) = \phi_2(k-d)$. This choice of $b_1(k)$ and $b_2(k)$ also satisfies condition (2). To show that condition (3) of the lemma is satisfied, it is necessary that $[s(k) - d(k) sgn\{s(k-d)\}]$ linearly bounds $\phi(k-d)$. This can be done by first showing that $y(k)$ linearly bounds $\phi(k-d)$. Notice that $\phi(k-d)$ is made up of functions of the output $y(i)$, $k-n+1 \leq i \leq k$ and of the control input $u(i)$.$k-m-d+1 \leq i \leq k$. Since the parameter estimates remain bounded due to the boundedness properties of the parameter estimator, $y(i)$ and $u(i)$ are also bounded as it was demonstrated by Proposition 3. To demonstrate that $[s(k) - \Phi\, sgn\{s(k-d)\}]$ linearly bounds $y(k)$, recall that the selection of function $s(k)$ in Section 2 established the ability to set the dynamics on the sliding surface as desired. Since $s(k)$ is bounded and $y_m(k)$ is assumed to be bounded, $s(k)$ can linearly bound $y(k)$. So $[s(k) - \Phi\, sgn\{s(k-d)\}]$ linearly bounds $y(k)$. With all conditions of the Key Technical Lemma satisfied, this allows for the results that $\phi(k-d)$ is bounded for all k and that

$$\lim_{k \to \infty} [s(k) - \Phi\, sgn\, \{s(k-d)\}] = 0.$$

The results thus proved is summarized in the following theorem.

Theorem 2 *Robust Stability of the Robust Adaptive Discrete Time Tracking Control System :*

The robust adaptive discrete time tracking control system, consisting of the plant (1),*the controller* (38),*and the adaptive law* (33) – (35) *satisfying* (45), *is stable in the sense that* $| s(k) |$ *decreases when* $| s(k) | > \Phi$ *and that the steady state value of* s *is bounded by* Φ.

Both in adaptive and in nonadaptive case we have arrived at the robust same results.

4 Conclusion

Based on the Diophantine equation a new discrete time sliding function has been defined and utilized for the robust feedforward tracking control law. The sliding boundary layer is composed of eigenvalue in s-dynamics and of upperbound in modeling error. The modeling error which ultimately affects on the tracking performance is diluted in the sliding boundary layer, which results in increase of robustness. Parameter estimator with a new deadzone is proposed to robustly estimate the unknown parameters of plant. Using eigenvalue as an intermediate variable,the relationship between deadzone and sliding boundary layer is derived to provide compatibility in robust control parameters. In adaptive case the robustness is achieved by using both a normalized gradient algorithm with deadzone and a sliding function-based nonlinear feedback, while in nonadaptive case by using only a sliding function -based nonlinear feedback.

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