Asymptotic Behavior of Adaptive Systems: Convergence Analysis without the Barbalat's Lemma

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ABSTRACT

Convergence of the state error e to zero in adaptive systems is shown using the uniqueness of solutions and the existence of a Lyapunov function in which the adaptation laws are constructed. Results in the paper are general, and therefore applicable to any adaptive control of a linear/nonlinear, time-varying or distributed-parameter system. Since the approach taken in the paper does not require the boundedness of the derivative of the state error e for all $t \geq 0$, it is particularly useful in the adaptive control of infinite dimensional systems.

I. INTRODUCTION

When a new control algorithm or a mathematical model for a physical system is proposed, it is natural to investigate whether the proposed algorithm or the model provides existence and uniqueness of forward-time solutions for all possible initial data, otherwise control action can not be continued forward in time forever or the mathematical equation does not accurately describe the physical system. Once the existence and uniqueness is assured then the stability of the algorithm or the model is investigated. However in the area of adaptive control the order is interestingly reversed: An adaptative control algorithm is first derived considering the stability and then the existence of solutions for all $t \geq 0$ is assured. In this paper the asymptotic convergence of the state error to zero in an adaptive system is shown using the existence of solution and a Lyapunov function. This reveals a fundamental fact in an adaptive control of a general system that if the adaptation law is derived in the way that $\dot{V}(x,y,z) \leq -\alpha(\|x\|)$, where V is a Lyapunov function, x denotes the error dynamics between plant and model, and $\alpha(\cdot)$ is a monotone function, then the trajectory of the plant follows that of the model. Although the approach taken in the paper provides a different proof for the convergence analysis in the finite dimensional adaptive control, it is particularly useful in the adaptive control of distributed parameter systems since it does not require the boundedness of the state error derivative for all $t \geq 0$.

In the adaptive systems utilizing the Lyapunov direct method in constructing control law, the adaptation laws are derived in such a way that the time derivative of the Lyapunov function V is negative semi-definite, which implies that the origin is (uniformly) stable (in the large). Therefore even if it is necessary to assure the existence and uniqueness of solutions for all $t \geq 0$ before the application of the Lyapunov method, the existence and uniqueness question of the closed loop adaptive system comes naturally after the assurance of stability since the feedback adaptative control law is designed in the fashion that stability is guaranteed. However the obtained overall adaptive system does not admit the global Lipschitz condition which suffices the global uniqueness.

The analysis of adaptive systems consists of investigating the stability and the asymptotic behavior of the solutions. The fundamental idea of the model reference adaptive control for the finite dimensional system is well documented in (Narendra and Annaswamy, 1989, p.99; Sastry and Bodson, 1989, p.99) using a scalar differential equation. Outlining briefly, the adaptative control law consists of some adjustable parameters which permit the closed loop equation to coincide exactly the reference model equation when the tuning parameters converge to their nominal values. The stability of the whole adaptive system is obtained by considering a Lyapunov function and making it to be negative semi-definite. The Lyapunov function involves the state error defined as the difference between the plant and the reference model, and the parameter errors defined as the differences between the current parameter values and their nominal values. Since the adaptation laws are derived in the way that all terms involving the controller parameters in the derivative of the Lyapunov function cancel out each other, the global uniform stability of the origin is at most obtained. Finally to assert $\lim_{t\to\infty} e(t) = 0$, two facts are used in the literature (Narendra and Annaswamy, 1989, p.85; Sastry and Bodson, 1989, p.19; Slotine and Li, 1991, p.123). One is $e(t) \in L^2(0,\infty)$ and the other is that $\dot{e}(t)$ is bounded for all $t \geq 0$, which enable the application of the Barbalat's theorem. Fortunately in the finite dimensional adaptive system the second fact follows from the Lyapunov function and a nature of finite dimensionality. Also further analysis reveals that the persistency of excitation of the reference input makes the whole adaptive system to be exponentially stable.

Compared to the finite dimensional case the adaptive control of infinite dimensional systems is not well understood and has only recently been studied. Wen (1985) proposed adaptive control laws and analyzed the Lagrange stability of direct model reference adaptive control in infinite dimensional Hilbert space by using command generator tracker approach. Hong and Bentsman (1992b,c; 1993; 1994) have investigated a direct adaptive control of parabolic systems and analyzed the stability using the averaging method. One of the main difficulties in synthesizing control algorithms for a distributed parameter system is obtaining the stability of whole closed loop system (Hong and Bentsman, 1992a; Hong and Wu, 1992; Wu and Hong, 1994).

Now we consider the following example of infinite dimensional adaptive control of parabolic partial differential equation (Hong and Lee, 1993), and obtain the asymptotic convergence of the state error to zero without relying on the Barbalat's theorem. The example below is taken for the purpose of illustrating where and how the Barbalat's theorem is not easily applicable. Parabolic partial differential equations arise in many physical, biological and engineering problems, for instance in the area of heat transfer, nuclear reactor dynamics, chemical reactions, crystal growth, population genetics, flow of electrons and holes in a semiconductor, nerve axon equations, hydrology, petroleum recovery area, and fluid mechanics. For more examples (Friedman, 1969; Henry, 1981) and their references are referred. For vibrational control of parabolic systems (Bentsman and Hong 1991, 1993; Bentsman et al., 1991, 1992) are mentioned.

Example : Consider a class of distributed parameter systems described by a linear parabolic partial differential equation with spatially-varying coefficients as

$$\frac{\partial \xi(p,t)}{\partial t}=\frac{\partial}{\partial p}\left(a(p)\frac{\partial \xi(p,t)}{\partial p}\right)+b(p)\xi(p,t)+u(p,t),\ t>0 \tag{1.1}$$

where t is the time, $p\in\Omega\subset R$ denotes the spatial variable, and $u(p,t)$ is a control input function. $a(p)$ and $b(p)$ are unknown but $a(p)>0$ is assumed to be parabolic. Boundary and initial conditions are given as

$$\xi(p,t)=\beta(t),\ p\in\partial\Omega$$
$$\xi(p,0)=\xi_o(p).$$

It is assumed that $a(p)$, $b(p)$ and the boundary data $\beta(t)$ are analytic in their appropriate domains. It is also assumed that $\beta(t)$ is a priori known, and distributed sensing and actuation are available. A reference model is defined as

$$\frac{\partial \xi_m(p,t)}{\partial t}=\frac{\partial}{\partial p}\left(a_m(p)\frac{\partial \xi_m(p,t)}{\partial p}\right)+b_m(p)\xi_m(p,t)+r(p,t),\ t>0$$
$$\xi_m(p,t)=\beta(t),\ p\in\partial\Omega$$
$$\xi_m(p,0)=\xi_{mo}(p) \tag{1.2}$$

where $r(p,t)$ is a bounded reference input. It is assumed that $a_m(p)\ge a_o>0, b_m(p)<0,\quad |b_m(p)|\ge b_o>0$, and that $a_m(p), b_m(p)$ are analytic in Ω. Now consider the following control law $u(p,t)$ with adjustable parameters $\phi_a(p,t)$ and $\phi_b(p,t)$ such that

$$u(p,t)=\frac{\partial}{\partial p}\left(\phi_a(p,t)\frac{\partial \xi(p,t)}{\partial p}\right)+\phi_b(p,t)\xi(p,t)+r(p,t). \tag{1.3}$$

The closed loop plant equation becomes identical to the equation of the reference model when $\lim_{t\to\infty}\phi_a(p,t)=\phi_a^*$ and $\lim_{t\to\infty}\phi_b(p,t)=\phi_b^*$, where $\phi_a^*(p)$ and $\phi_b^*(p)$ are nominal functions defined as $\phi_a^*(p)=a_m(p)-a(p)$ and $\phi_b^*(p)=b_m(p)-b(p)$. Define the state error e as $e(p,t)=\xi(p,t)-\xi_m(p,t)$, and the controller parameter errors ψ_a and ψ_b as $\psi_a(p,t)=\phi_a(p,t)-\phi_a^*(p)$ and $\psi_b(p,t)=\phi_b(p,t)-\phi_b^*(p)$, respectively. Subtracting (1.2) from (1.1) yields the state error equation with homogeneous boundary conditions as

$$\frac{\partial e(p,t)}{\partial t}=\frac{\partial}{\partial p}\left(a_m(p)\frac{\partial e(p,t)}{\partial p}\right)+b_m(p)e(p,t)$$
$$+\frac{\partial}{\partial p}\left(\psi_a(p,t)\frac{\partial \xi(p,t)}{\partial p}\right)+\psi_b(p,t)\xi(p,t)$$
$$e(p,t)=0,\ p\in\partial\Omega$$
$$e(p,0)=\xi_o(p)-\xi_{mo}(p). \tag{1.4}$$

Consider the adaptation laws given by

$$\frac{\partial \phi_a(p,t)}{\partial t}=\varepsilon\frac{\partial e(p,t)}{\partial p}\frac{\partial \xi(p,t)}{\partial p},\ \phi_a(p,0)=\phi_{ao} \tag{1.5}$$

$$\frac{\partial \phi_b(p,t)}{\partial t}=-\varepsilon\, e(p,t)\xi(p,t),\ \phi_b(p,0)=\phi_{bo} \tag{1.6}$$

where $\varepsilon>0$ is the adaptation gain. Then by considering a functional $V:(L_2(\Omega))^3\to R^+$ as

$$V(e,\psi_a,\psi_b)=\frac{1}{2}\int_\Omega\left(e^2(p,t)+\frac{1}{\varepsilon}\left(\psi_a^2(p,t)+\psi_b^2(p,t)\right)\right)dp$$

and differentiating V with respect to t along the trajectories of (1.4)-(1.6) employing integration by parts and boundary conditions yields

$$\frac{dV}{dt}=\int_\Omega\left(-a_m(p)\left(\frac{\partial e(p,t)}{\partial p}\right)^2+b_m(p)e^2(p,t)\right)dp$$
$$\le -b_o\int_\Omega e^2(p,t)dp \tag{1.8}$$
$$\le 0.$$

Therefore the global uniform stability of the origin (i.e. $(e,\psi_a,\psi_b)=(0,0,0)$) in $L_2(\Omega)^3$ is concluded. Furthermore (1.8) implies that $e(p,t)\in L_2(\Omega\mathrm{x}[0,\infty))$. However the assertion that $\lim_{t\to\infty}\|e(p,t)\|_{L_2}=0$ is not obvious through a similar analysis as the case of finite dimensional system which requires the boundedness of $\dot{e}(p,t)$ in (1.4).

This paper develops a new approach in asserting the convergence of the state error to zero which does not rely on the Barbalat's theorem. This approach is applicable to any adaptive systems which is constructed in the way that (i) the existence and uniqueness of solutions is assured, (ii) there exists a Lyapunov function which determines the stability of the overall adaptive system, and finally (iii) $\alpha(\|e\|_X)\in L_1(0,\infty)$, where e is the state error of an adaptive system, $\alpha(\bullet)$ is a monotone function with $\alpha(0)=0$, and $\|\bullet\|_X$ denotes a norm in a Banach space X. This approach is particularly crucial in the adaptive control of infinite dimensional system since it does not require the boundedness of the derivative of the state error for all $t\ge 0$.

II. FINITE DIMENSIONAL ADAPTIVE SYSTEMS

The adaptive control of finite dimensional systems is now well developed. In this section we revisit the finite dimensional case and show that the asymptotic convergence of the state error is well guaranteed once the adaptation laws are designed using the Lyapunov redesign method. Let a general finite dimensional adaptive system of a linear/nonlinear, time-varying plant with bounded external disturbances be given in the following form as in the work of Polycarpou and Ioannou (1993).

$$\dot{x}=f(t,x,y),\ x(0)=x_o \tag{2.1}$$

$$\dot{y}=g(t,x,y,\eta),\ y(0)=y_o \tag{2.2}$$

$$\dot{\eta}=-\delta_o\eta+h(t,x,y),\ \eta(0)=\eta_o \tag{2.3}$$

where $x\in R^n$, $y\in R^m$, $\eta\in R^1$ and $\delta_0>0$ is a constant. f, g and h are in general nonlinear time-varying functions. The state x represents the error dynamics between the closed loop plant with filters and the model. The state y denotes the estimated parameter vector which is referred to as the adaptation law. η is a design variable known as the normalizing signal. The explicit dependence of the functions f, g, and h on t could be due to time variation in the plant parameters and/or exogenous signals such as plant disturbances and reference input.

Assumptions

(A1) $f(t,0,0)=0$. $g(t,0,0,\eta)=0$. f, g and h are piecewise continuous in t, and are continuous in other variables. Furthermore f and h are locally Lipschitz in x and y. g is locally Lipschitz in x, y and η.

(A2) (a) $\|f(t,x,y)\| \le \alpha_0(y)\|x\| + c_0, \forall t \ge 0$ (2.4)

(b) $|h(t,x,y)| \le \alpha_1(y)\|x\|^2 + \alpha_2(y)\|x\| + c_1, \forall t \ge 0$ (2.5)

where c_0, c_1 are constants and $\alpha_0, \alpha_1, \alpha_2 : R^m \to R^+$ are bounded for finite values of y.

(A3) there exists a function $V: R^{m+l} \to R^+$ such that

$$k_1\|Cx\|^2 + k_2\|y\|^2 \le V(Cx,y) \le k_3\|Cx\|^2 + k_4\|y\|^2 \qquad (2.6)$$

where k_1, k_2, k_3, k_4 are positive constants, and $C \in R^{l \times n}$ is a constant matrix.

The overall adaptive system does not admit the global Lipschitz condition in general. However if there exists a Lyapunov function for the whole adaptive system, the existence and uniqueness for all $t \ge 0$ can be asserted from the Lyapunov function together with the local existence and uniqueness resulting from the condition (A1) (Narendra and Annaswamy, 1989, p.117, Comment 3.2). Furthermore if a considered Lyapunov function involves only part of the state of the whole adaptive system like (2.6), the global existence and uniqueness can still be obtained with the conditions like (A2). For completeness this is summarized in the following Lemma (Polycarpou and Ioannou, 1993).

Lemma : Consider an adaptive system (2.1)-(2.3) with the assumptions above. Assume that

$$\dot{V}(Cx,y)\Big|_{(2.1)-(2.2)} \le 0. \qquad (2.7)$$

Then there exists a unique solution of (2.1)-(2.3) defined for all $t \in [0,\infty)$.

Proof: The proof follows (polycarpou and Ioannou, 1993). Defining $z = [x^T, y^T, \eta]^T$ with $z(0) = [x^T(0), y^T(0), \eta(0)]^T$, (2.1)-(2.3) can be rewritten as

$$\dot{z} = p(t,z) = \begin{bmatrix} f(t,x,y) \\ g(t,x,y,\eta) \\ -\delta_o\eta + h(t,x,y) \end{bmatrix}, \quad z(0) = \begin{bmatrix} x_o \\ y_o \\ \eta_o \end{bmatrix}.$$

Since p is locally Lipschitz in z, by the standard local existence theorem (see Hale, 1969) there exists a unique solution defined on an interval $J_T = [0,T)$ for some T > 0. Also the existence of a Lyapunov function V satisfying (2.7) implies that a set $E_\beta = \{(Cx,y): V(Cx,y) \le \beta, \beta \in R^+\}$ is positive invariant. (The possibility of existence of a finite escape time is removed by the function V with (2.7)). Hence $y(t) \le \beta, \forall t \ge 0$, where β is a constant not depending on T. Now in the rest of proof it will be shown that neither any component of the state x nor η does "explode" in finite time. The solutions of (2.1) and (2.3) on the interval J_T are

$$x(t) = x(0) + \int_0^t f(\tau, x(\tau), y(\tau))d\tau, \qquad (2.8)$$

$$\eta(t) = e^{-\delta_o t}\eta(0) + \int_0^t e^{-\delta_o(t-\tau)}h(\tau, x(\tau), y(\tau))d\tau, \quad (2.9)$$

respectively. Taking norms on both sides of (2.8) using the condition (A2-a)

$$\|x(t)\| \le \|x(0)\| + \int_0^t (\alpha_0(y)\|x(\tau)\| + c_0)d\tau$$

$$\le \|x(0)\| + \overline{\alpha}_0 \int_0^t \left(\|x(\tau)\| + \frac{c_0}{\overline{\alpha}_0} \right) d\tau$$

where $\overline{\alpha}_0 = \sup_{|y(t)|<\beta} \alpha_0(y(t))$. Applying the Bellman-Gronwall's inequality yields

$$\|x(t)\| \le \left(\|x(0)\| + \frac{c_0}{\overline{\alpha}_0} \right) e^{\overline{\alpha}_o t} \qquad (2.10)$$

for all $t \in J_T$. Similarly using (2.10) and the assumption (A2-b) in (2.9) obtains

$$|\eta(t)| \le c_1 + c_2 e^{\overline{\alpha}t}, \forall t \in J_T$$

for some constant $c_1, c_2, \overline{\alpha} \ge 0$. Therefore the solutions can be continued past $t = T$ and since the solutions cannot grow faster than an exponential function, they can not have finite escape times and thus the solutions exist and are unique for all $t \in [0,\infty)$. Q.E.D.

Remark 1: In the special case that $C = I$, the condition (A1) and the Lyapunov function (2.6) satisfying (2.7) are sufficient for the global existence and uniqueness of $x(t)$ and $y(t)$. Taking $C = I$ will not lose any generality in analysis since those components of the vector x corresponding to the filters can be specifically included in the Lyapunov function. In this case both vectors $x(t)$ and $y(t)$ are both bounded by some constant β for all $t \ge 0$. Observing the boundedness of $x(t)$ and $y(t)$, the boundedness of the state error derivative can be obtained relying on some conditions like (2.4) or directly (2.1) in finite dimensional case.

Theorem 1. Consider an adaptive system (2.1)-(2.3) with the assumptions above. Assume that

$$\dot{V}(x,y)\Big|_{(2.1)-(2.2)} \le -\alpha(\|x\|). \qquad (2.11)$$

where $\alpha(\bullet)$ is a monotone function with $\alpha(0)=0$. Then $x(t) \to 0$ as $t \to \infty$.

Proof: Let the unique solution of (2.1) at time t starting with initial state $x(s)$ at initial time s be of the form

$$x(t) = x(s) + \int_s^t f(\tau, x(\tau), y(\tau))d\tau \qquad (2.12)$$

and denote it as $x(t) = x(t, x(s), s)$. Define a two parameter family of map $S(t,s)$ on R^n as

$$S(t,s)x(s) = x(t,x(s),s), \quad 0 \le s \le t < \infty. \qquad (2.13)$$

Then by the uniqueness and continuous dependence of the solutions $x(t) = x(t,x(s),s)$ on the triple $(t, x(s), s)$, the mapping $S(t,s)$ on R^n becomes an evolution process such that (Walker, 1980, p.12)

(i) $S(\bullet,s)x(s): R^+ \to R^n$ is continuous (right continuous at $t = s$)

(ii) $S(t,\bullet)(\bullet): R \times R^n \to R^n$ is continuous

(iii) $S(s,s)x(s) = x(s)$

(iv) $S(t,s)x(s) = S(t,r)S(r,s)x(s)$, for all $x(s) \in R^n$ and $0 \le s \le r \le t < \infty$.

Note also that the condition (2.11) implies that

$$\int_0^\infty \alpha(\|S(t,0)x_o\|)\, dt < \infty. \qquad (2.14)$$

Indeed, the conclusion of the theorem can be proven by contradiction. Suppose $S(t,0)x_o \not\to 0$ as $t \to \infty$, then there exist an ε > 0 and an infinite sequence $t_j \to \infty$ such that

$$\|S(t_j,0)x_o\| \ge \varepsilon.$$

Now however small the ε is, there exist constants $M > 0$ and $\varepsilon_o > 0$ such that

$$M \geq \overline{\alpha}_0 = \sup_{\|y(t)\|<\beta} \alpha_0(y(t)), \text{ and}$$

$$\frac{\varepsilon}{e} - \frac{c_o}{M} \geq \varepsilon_o > 0. \tag{2.15}$$

Note that if $c_o = 0$, (2.15) is always satisfied. Therefore taking norms on both sides of (2.12)

$$\|x(t)\| \leq \|x(s)\| + \int_s^t M\left(\|x(\tau)\| + \frac{c_o}{M}\right) d\tau.$$

Applying the Bellman-Gronwall's inequality yields

$$\|x(t)\| \leq \left(\|x(s)\| + \frac{c_o}{M}\right) e^{M(t-s)} \tag{2.16}$$

for all $t \geq s \geq 0$.

Now without loss of generality we can assume that $t_{j+1} - t_j > M^{-1}$. If we set $\Delta_j = [t_j - M^{-1}, t_j]$, then $m(\Delta_j) = M^{-1} > 0$ (m = Lebesgue measure) and the intervals Δ_j do not overlap. For $t \in \Delta_j$

$$\begin{aligned} \varepsilon &\leq \|S(t_j, 0)x_o\| \\ &= \|S(t_j, t)S(t, 0)x_o\| = \|S(t_j, t)x(t)\| \\ &\leq \left(x(t) + \frac{c_o}{M}\right) e^{M(t_j - t)} \\ &\leq \left(x(t) + \frac{c_o}{M}\right) e \end{aligned}$$

where the second inequality above is obtained from (2.16). Therefore we have

$$\|x(t)\| \geq \varepsilon_o$$

for all $t \in \Delta_j = [t_j - M^{-1}, t_j]$. Hence

$$\begin{aligned} \int_0^\infty \alpha(\|S(t,0)x_o\|)dt &\geq \sum_{j=1}^\infty \int_{\Delta_j} \alpha(\|S(t,0)x_o\|)dt \\ &\geq \sum_{j=1}^\infty \int_{\Delta_j} \alpha(\varepsilon_o)dt \\ &= \alpha(\varepsilon_o) \sum_{j=1}^\infty m(\Delta_j) \\ &= \infty \end{aligned}$$

contradicting (2.14). Thus we must have $x(t) \to 0$ as $t \to \infty$. Q.E.D.

Remark 2: The above theorem suggests the following general design procedure. Designing a model following adaptive system consisting of a plant, a model, filters, tuners and some normalizing signals, i) derive an adaptalive control law which permits exact equation matching between the plant and the model when the adjustable parameters in the controller converge to some values, ii) assure the existence and uniqueness of solutions, iii) there exists a Lyapunov function for the whole adaptive system and the derivative of the Lyapunov function is of the form

$$\dot{V} \leq -\alpha(\|x\|)$$

where x is the state error between plant and model, and α is monotone. Then $\lim_{t\to\infty} x(t) = 0$.

Remark 3: Note that (2.14) must hold for all initial conditions $x_o \in B_\beta = \{x: \|x\| \leq \beta\}$ due to the positive invariance of B_β. Therefore (2.14) excludes the typical situation that f in (2.1) is a function of only t and y. Indeed if f is of the form (this will never happen in an adaptive control since x denotes the plant with filters)

$$\dot{x} = f(t, y),\ x(0) = x_o \tag{2.17}$$

the solution is of the form

$$x(t) = x_o + \int_0^t f(\tau, y(\tau))d\tau. \tag{2.18}$$

Then (2.14) is never achieved for an arbitrary $x_o \neq 0$ because (2.14) can be satisfied for one particular non zero x_o by offsetting the second term in (2.18) but not for all initial conditions.

Remark 4: The above theorem also concludes the following. In general $x(t) \in L_p(0,\infty)$ does not imply $\lim_{t\to\infty} x(t) = 0$. The uniform continuity of $x(t)$ is needed as is shown in the Barbalat's theorem. However besides the fact that $x(t) \in L_p$, if the signal comes through a dynamical system as $\dot{x} = f(t, x, y)$, where a unique solution exist for all $t \geq 0$ and y is a bounded parameter, then $\lim_{t\to\infty} x(t) = 0$. Let us take a pathological signal $x(t)$ which belongs to L_p but does not tend to zero (this signal will violate the uniform continuity condition for all $t \geq 0$). And let the derivative of $x(t)$ be $\zeta(t)$. Then $x(t)$ can be considered as a signal generated through a dynamical system of the form

$$\dot{x}(t) = \zeta(t),\ x(0) = 0$$

which is exactly the form of (2.17) and the only 0 initial condition will provide $x(t) \in L_p$.

The above observation is summarized in the following corollary.

Corollary: Let $x(t) \in L_p(0,\infty), p \geq 1$, and be a unique solution of $\dot{x} = f(t, x, y)$, $x \in R^n$, $y \in R^m$ where y is a bounded parameter. Let f satisfy $\|f(t,x,y)\| \leq \alpha(y)\|x\| + c_o$, where c_o is a constant and $\alpha(\bullet)$ is bounded for a finite value of y. Then $\lim_{t\to\infty} x(t) = 0$.

III. INFINITE DIMENSIONAL ADAPTIVE SYSTEMS

The overall adaptive system of the Example in Section I can be represented as

$$\begin{aligned} \dot{e} &= \left((a_m + \psi_a)e'\right)' + (b_m + \psi_b)e + \left(\psi_a \xi_m{}'\right)' + \psi_b \xi_m; \\ & e(p,t) = 0, p \in \partial\Omega;\ e(p,0) = e_o \end{aligned} \tag{3.1}$$

$$\dot{\psi}_a = \varepsilon e'\left(e' + \xi_m{}'\right),\ \psi_a(p,0) = \psi_{ao} \tag{3.2}$$

$$\dot{\psi}_b = -\varepsilon e\left(e + \xi_m\right),\ \psi_b(p,0) = \psi_{bo} \tag{3.3}$$

where $\bullet$ and $'$ denote the derivatives with respect to t and p, respectively, and $\xi_m(p,t)$ is an exogenous signal.

Substituting (3.2) and (3.3) into (3.1), (3.1) has the form

$$\dot{e} = B(t,e)e + g(t,e) \tag{3.4}$$

where

$$\begin{aligned} B(t,e)\bullet &= \left(\left(a_m + y_{ao} + e\int_0^t \left((e')^2 + e' x_m{}'\right)dt\right)(\bullet)'\right) \\ &\quad + \left(b_m + y_{bo} - e\int_0^t \left(e^2 + e x_m\right)dt\right)\bullet \end{aligned}$$

$$g(t,e) = \left(\left(\Psi_{ao} + \varepsilon\int_0^t \left((e')^2 + e' \xi_m{}'\right)dt\right)\xi_m{}'\right)$$

$$+\left(\psi_{bo}-\varepsilon\int_0^t\left(e^2+e\xi_m\right)dt\right)\xi_m$$

Since $\xi_m(p,t)$ is smooth, there exists a $t_o>0$ such that the principal term of (3.4) is strongly elliptic for all $t\in[0,t_o]$, i.e.

$$<-B(t,e)e,e>\ \geq c<e,e>,\ \ \forall t\in[0,t_o],\quad c>0.$$

Therefore (3.4) is parabolic (Friedman, 1969, p.134), and there exists a unique solution for $t\in[0,t_o]$. Typical values of those α, σ, ρ on page 170 of (Friedman, 1969) for (3.4) can be chosen as $\alpha=1/2$, and $\sigma=\rho=1$. Finally the Lyapunov function defined as in (1.16) ensures that all solutions belong to a closed bounded set, and hence their existence for all $t\geq 0$ is guaranteed as well.

Theorem 2: Consider an evolution equation as

$$\dot{x}(t)+A(y(t))x(t)=f(t,x,y),\ x(0)=x_o \tag{3.5}$$

$$\dot{y}(t)=g(t,x,\eta),\ y(0)=y_o \tag{3.6}$$

$$\dot{\eta}(t)=-\delta_o\eta(t)+h(t,x,y),\ \eta(0)=\eta_o \tag{3.7}$$

where $x,y,\eta\in X$, X is a Banach space and $\delta_0>0$ is a constant. Let the state x denote the error dynamics between plant and model, the state y represent the parameter vector to be tuned, and η refer some normalizing signal. Assume that

(i) there exist unique solutions to (3.5)-(3.7), and the unique solution of (3.5) has the form

$$x(t)=\Phi(t,0)x_o+\int_0^t\Phi(t,\tau)f(t,x(\tau),y(\tau))d\tau \tag{3.8}$$

where $\Phi(t,s)$ is an evolution system corresponding to $-A(y(t))$.

(ii) (a) $\|f(t,x,y)\|\leq\alpha_0(y)\|x\|+c_0,\ \forall t\geq 0$

(b) $|h(t,x,y)|\leq\alpha_1(y)\|x\|^2+\alpha_2(y)\|x\|+c_1,\ \forall t\geq 0$

where c_0, c_1 are constants and $\alpha_0,\alpha_1,\alpha_2:X\rightarrow R^+$ are bounded for finite values of y.

(iii) there exists a functional $V: R\times X\times X\rightarrow R^+$ such that

$$k_1\|x\|^2+k_2\|y\|^2\leq V(t,x,y)\leq k_3\|x\|^2+k_4\|y\|^2$$

where k_1, k_2, k_3, k_4 are positive constants.

(iv) there exists a continuous non-decreasing function $\alpha(\bullet)$ with $\alpha(0)=0$ such that

$$\dot{V}(t,x,y)\Big|_{(3.5)-(3.6)}\leq-\alpha(\|x\|).$$

Then $\|x(t)\|\rightarrow 0$ as $t\rightarrow\infty$.

Proof: Using contradiction, a similar strategy as in Theorem 1 is applied. Suppose that $\|x(t)\|\not\rightarrow 0$ as $t\rightarrow\infty$, then there exist an $\varepsilon>0$ and an infinite sequence $t_j\rightarrow\infty$ such that

$$\|x(t_j)\|=\|S(t_j,0)x_o\|\geq\varepsilon$$

where $S(t,0)x_o$ is the unique solution of (3.5) starting at initial condition x_o at time 0. Taking norms on the equation (3.8) with the initial state $x(s)$ and time s using the condition on f yields

$$\|x(t)\|\leq M_1\|x(s)\|+\int_s^t M_1\left(M_2\|x(\tau)\|+c_o\right)d\tau$$

$$\leq M_1\|x(s)\|+M_1M_2\int_s^t\left(\|x(\tau)\|+\frac{c_o}{M_2}\right)d\tau$$

where $M_1=\sup_{s,t\in[0,\infty)}\|\Phi(t,s)\|$, and M_2 is chosen to be sufficiently large so that $\left(\varepsilon/e-c_o/M_2\right)\geq\varepsilon_o>0$. Applying the Bellman-Gronwall's inequality

$$\|x(t)\|\leq\left(M_1\|x(s)\|+\frac{c_o}{M_2}\right)e^{M_1M_2(t-s)}$$

for all $t\geq s\geq 0$. Now we take the sequence t_j such that $t_{j+1}-t_j>(M_1M_2)^{-1}$. Then the intervals Δ_j defined as $\Delta_j=[t_j-(M_1M_2)^{-1},t_j]$ do not overlap and $m(\Delta_j)>0$. For any $t\in\Delta_j$

$$\varepsilon\leq\|S(t_j,0)x_o\|=\|S(t_j,t)S(t,0)x_o\|=\|S(t_j,t)x(t)\|$$

$$\leq\left(M_1x(t)+\frac{c_o}{M_2}\right)e^{M_1M_2(t_j-t)}$$

$$\leq\left(M_1x(t)+\frac{c_o}{M_2}\right)e$$

Therefore we have

$$\|x(t)\|\geq\varepsilon_o$$

for all $t\in\Delta_j$, which leads to contradiction to the condition (iv) in the theorem. Therefore we must have $\|x(t)\|\rightarrow 0$ as $t\rightarrow\infty$.

IV. CONCLUSIONS

Asymptotic convergence of the state error to zero for a general adaptive control system which includes both finite and infinite dimensional adaptive systems is investigated. The method developed in the paper is general and therefore is applicable to any adaptive system in assuring convergence of the state error to zero if the adaptive system is constructed in the way that (i) the existence and uniqueness of solutions is assured, (ii) there exists a Lyapunov function which determines the stability of the overall adaptive system, and finally (iii) $\alpha(\|e\|_X)\in L_1(0,\infty)$, where e is the state error of an adaptive system, $\alpha(\bullet)$ is a monotone function with $\alpha(0)=0$, and $\|\bullet\|_X$ denotes a norm in a Banach space X. This approach is particularly crucial in the adaptive control of infinite dimensional system since it does not require the boundedness of the state error for all $t\geq 0$.

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