'94 KACC (1994. 10. 17 ~ 20)

On the admissibility condition in the model matching problem

Kiheon Park

Dept. of Electrical Engineering, Sung-Kyun-Kwan University, 300, Chunchun-dong, Jangan-gu, Suwon, Kyounggi-do 440-746, Korea
TEL. (0331)290-5416 FAX. (0331)292-8789

ABSTRACT

A new approach to deal with the model matching problem for square plants is suggested. Admissibility conditions of the model matching error are derived in terms of state-space parameters and the derived formulas are exploited to obtain the solution to the model matching problem in H_2 norm.

I. INTRODUCTION

Since the so-called Youla's parametrization lemma[8] for stabilizing controllers, the equation $E(s) = T_0(s) - T_a(s)Q(s)T_b(s)$ has been one of the most important equations in linear system theory. The equation is widely used in H_2 or H_∞ optimization problems and various methods were developed to solve these optimization problems[2,3,6]. In general, H_2 problem is easier to be solved and it is also known that the H_∞ problem with T_a and T_b square is relatively easy so that an explicit formula is possible[6]. In most cases, however, the solving procedure for the H_2 and H_∞ problems are differently set. The main purpose of this paper is to present a state-space form for $E(s)$ that is convenient to develop H_2 or H_∞ solutions in a unified framework for square T_a and T_b. The main results are described in Theorem 2.

Throughout the paper, only real rational matrices are considered and the notations G^{-1} and G^T are used for the inverse and transpose of G. The matrix $G_*(s)$ stands for $G^T(-s)$. A real constant matrix A is said to be stable if the real part of each eigenvalue of A is negative. A rational matrix $G(s)$ is said to be stable if $G(s)$ is analytic in $\mathrm{Re}\, s \geq 0$. The linear fractional transformation(LFT) of $K(s)$ with respect to $G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}$ is denoted by $G(s) \circ K(s)$. That is $G(s) \circ K(s) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$. The conventional notation $C(sI-A)^{-1}B+D = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \{A, B, C, D\}$ is used.

II. PROBLEM FORMULATION

In this section, we define the admissibility of the model matching error and explain the relationship with the model matching probelm. We first define the model matching problem in α norm.

Model Matching Problem in α norm : For given $T_0(s) \in RH_\infty^{p\times m}$, $T_a(s) \in RH_\infty^{p\times p}$ and $T_b(s) \in RH_\infty^{m\times m}$, find all $Q(s) \in RH_\infty^{p\times m}$ such that the α norm, where α could be H_2, H_∞ or any other norm, of the model matching error

$$E(s) = T_0(s) - T_a(s)Q(s)T_b(s) \tag{1}$$

is less than the prescribed value β.

Definition 1 : A rational matrix $E(s) \in RH_\infty^{p\times m}$ is said to be admissible for the triple $(T_0(s), T_a(s), T_b(s))$ if there exists $Q(s) \in RH_\infty^{p\times m}$ such that $E(s) = T_0(s) - T_a(s)Q(s)T_b(s)$.

The inner-outer factorization of a stable matrix plays an important role in the MMP. We make the following assumptions on $T_a(s)$ and $T_b(s)$ in relation with the inner-outer factorizability of these matrices.

Assumption 1 : The ranks of $T_a(j\omega)$ and $T_b(j\omega)$ are p and m, respectively, for $0 \leq \omega \leq \infty$.

Under these assumptions, we can always obtain the following inner-outer factorization for $T_a(s)$ and $T_b(s)$; that is

$$T_a(s) = U_a(s)V_a(s),\ T_b(s) = V_b(s)U_b(s) \quad (2)$$

with the properties that

$$\begin{aligned} &U_a(s) \in RH_\infty^{p\times p},\ U_{a*}(s)U_a(s) = I_p \\ &U_b(s) \in RH_\infty^{m\times m},\ U_b(s)U_{b*}(s) = I_m \end{aligned} \quad (3)$$

$$\begin{aligned} &V_a(s),\ V_a^{-1}(s) \in RH_\infty^{p\times p} \\ &V_b(s),\ V_b^{-1}(s) \in RH_\infty^{m\times m} \end{aligned} \quad (4)$$

Let $T_a(s) = \{A_{ta}, B_{ta}, C_{ta}, D_{ta}\}$ and $T_b(s) = \{A_{tb}, B_{tb}, C_{tb}, D_{tb}\}$ where A_{ta} and A_{tb} are stable and the realizations are not necessarily minimal. Then state-space formulas of the above inner and outer matrices are obtained as follows [7]; That is,

$$\begin{aligned} U_a(s) &= \{A_{ta} - B_{ta}K_{ta},\ B_{ta}\tilde{D}_{ta}, \\ &\qquad C_{ta} - D_{ta}K_{ta},\ D_{ta}\tilde{D}_{ta}\} \\ &:= \{\hat{A}_a,\ \hat{B}_a,\ \hat{C}_a,\ D_a\} \end{aligned} \quad (5)$$

$$V_a(s) = \{A_{ta},\ B_{ta},\ \tilde{D}_{ta}^{-1}K_{ta},\ \tilde{D}_{ta}^{-1}\} \quad (6)$$

where

$$\tilde{D}_{ta} = (D_{ta}^T D_{ta})^{-1/2} \quad (7)$$

$$K_{ta} = (D_{ta}^T D_{ta})^{-1}(D_{ta}^T C_{ta} + B_{ta}^T M_{ta}) \quad (8)$$

and M_{ta} is a symmetric solution of the equation

$$\begin{aligned} &(A_{ta} - B_{ta}D_{ta}^{-1}C_{ta})^T M_{ta} + M_{ta}(A_{ta} - B_{ta}D_{ta}^{-1}C_{ta}) \\ &- M_{ta}B_{ta}(D_{ta}^T D_{ta})^{-1}B_{ta}^T M_{ta} = 0 \end{aligned} \quad (9)$$

with the property that $A_{ta} - B_{ta}K_{ta}$ is stable. And also

$$\begin{aligned} U_b(s) &= \{A_{tb} - K_{tb}C_{tb},\ B_{tb} - K_{tb}D_{tb}, \\ &\qquad \tilde{D}_{tb}C_{tb},\ \tilde{D}_{tb}D_{tb}\} \\ &:= \{\hat{A}_b,\ \hat{B}_b,\ \hat{C}_b,\ D_b\} \end{aligned} \quad (10)$$

$$V_b(s) = \{A_{tb},\ K_{tb}\tilde{D}_{tb}^{-1},\ C_{tb},\ \tilde{D}_{tb}^{-1}\} \quad (11)$$

where

$$\begin{aligned} &\tilde{D}_{tb} = (D_{tb}D_{tb}^T)^{-1/2} \\ &K_{tb} = (M_{tb}C_{tb}^T + B_{tb}D_{tb}^T)(D_{tb}D_{tb}^T)^{-1} \end{aligned} \quad (12)$$

and M_{tb} is a symmetric solution of the equation

$$\begin{aligned} &(A_{tb} - B_{tb}D_{tb}^{-1}C_{tb})M_{tb} + M_{tb}(A_{tb} - B_{tb}D_{tb}^{-1}C_{tb})^T \\ &- M_{tb}C_{tb}^T(D_{tb}D_{tb}^T)^{-1}C_{tb}M_{tb} = 0 \end{aligned} \quad (13)$$

with the property that $A_{tb} - K_{tb}C_{tb}$ is stable

III. MAIN RESULT

The following lemma is the starting point to relate the admissibility conditions in the frequency domain and the state-space domain. The proof is trivial and hence is omitted.

Lemma 1 : A rational matrix $E(s) \in RH_\infty^{p\times m}$ is admissible for (T_0, T_a, T_b) if and only if $U_{a*}(E - T_0)U_{b*}$ is contained in $RH_\infty^{p\times m}$.

The result of Lemma 1 shows that $E(s) \in RH_\infty$ is admissible if and only if $\tilde{E}(s) := E(s) - T_0(s)$ eliminates all the poles of $U_{a*}(s)$ and $U_{b*}(s)$. In the next section, we investigate the pole elimination conditions of rational matrices.

3.1 Pole Elimination of Rational Matrices

Definition 2 : 1) A rational matrix $G(s)$ is said to be a right(left) eliminator of $G_1(s)$ ($G_2(s)$) if the product $G_1(s)G(s)$ ($G(s)G_2(s)$) does not have a pole of $G_1(s)$ ($G_2(s)$). 2) A rational matrix $G(s)$ is said to be a central eliminator of $G_1(s)$ and $G_2(s)$ if the product $G_1(s)G(s)G_2(s)$ does not have a pole of $G_1(s)$ nor $G_2(s)$.

The formula in the next lemma is a basic tool to develop many useful theories in this paper and will be used repeatedly in the following section.

Lemma 2 : Suppose that two constant square matrices A_1 and A_2 do not have a common eigenvalue. Then we have the identity

$$\begin{aligned} &H_1(sI - A_1)^{-1}H(sI - A_2)^{-1}H_2 \\ &= -H_1(sI - A_1)^{-1}MH_2 \\ &\quad + H_1M(sI - A_2)^{-1}H_2 \end{aligned} \quad (14)$$

where M is the unique solution of the Sylvester equation

$$A_1M - MA_2 = -H \quad (15)$$

Proof : Adding $-sM + Ms = 0$ to the left side of (15), we obtain $(sI - A_1)M - M(sI - A_2) = H$. Now multiplying $H_1(sI - A_1)^{-1}$ on the left and $(sI - A_2)^{-1}H_2$ on the right yields the identity.

Lemma 3 : Consider the matrices $G_1(s) = \{A_1, B_1, C_1, D_1\}$, $G_2(s) = \{A_2, B_2, C_2, D_2\}$ and $G(s) = \{A, B, C, D\}$ where A does not have a common eigenvalue with A_1 nor A_2. Then, 1) $G(s)$ is a central eliminator of $G_1(s)$ and $G_2(s)$ if and only if

$$\begin{aligned} &G_0(s) \\ &:= C_1(sI-A_1)^{-1}(B_1D - M_1B)C_2(sI-A_2)^{-1}B_2 \\ &+ C_1(sI-A_1)^{-1}(B_1D - M_1B)D_2 \\ &+ \{D_1(DC_2 + CM_2) + C_1M_1M_2\}(sI-A_2)^{-1}B_2 \\ &= 0 \end{aligned} \quad (16)$$

where M_1 and M_2 are the unique solutions of

$$A_1M_1 - M_1A = -B_1C \tag{17}$$

and

$$AM_2 - M_2A_2 = -BC_2 \tag{18}$$

2) $G(s)$ is a right eliminator of $G_1(s)$ if

$$B_1D - M_1B = 0 \tag{19}$$

When (A_1, C_1) is observable, the converse statement is also true.

3) $G(s)$ is a left eliminator of $G_2(s)$ if

$$DC_2 + CM_2 = 0 \tag{20}$$

When (A_2, B_2) is controllable. the converse statement is also true.

Proof : 1) It is not difficult to show that state-space paramters of $G_1(s)G(s)G_2(s)$ are given by

$$G_1GG_2 = \left\{\begin{bmatrix} A_f & B_fC_2 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_fD_2 \\ B_2 \end{bmatrix}, [\, C_f \;\; D_1DC_2],\, D_1DD_2\right\} \tag{21}$$

where

$$A_f = \begin{bmatrix} A_1 & B_1C \\ 0 & A \end{bmatrix},\; B_f = \begin{bmatrix} B_1D \\ B \end{bmatrix} \text{ and } C_f = [\, C_1 \;\; D_1C] \tag{22}$$

Converting the state-space form in (21) to a transfer matrix form, we obtain

$$\begin{aligned} &G_1GG_2 \\ &= C_f(sI-A_f)^{-1}B_f(D_2 + C_2(sI-A_2)^{-1}B_2) \\ &\quad + D_1DC_2(sI-A_2)^{-1}B_2 + D_1DD_2 \end{aligned} \tag{23}$$

Since, $C_f(sI-A_f)^{-1}B_f = C_1(sI-A_1)^{-1}B_1D + C_1(sI-A_1)^{-1}B_1C(sI-A)^{-1}B + D_1C(sI-A)^{-1}B$, applying Lemma 2 to the second term of the above equation yields the identity $C_f(sI-A_f)^{-1}B_f = C_1(sI-A_1)^{-1}(B_1D - M_1B) + (D_1C + C_1M_1)(sI - A)^{-1}B$.
Substituting this into (23), we obtain

$$\begin{aligned} &G_1(s)G(s)G_2(s) = G_0(s) \\ &\quad + (D_1C+C_1M_1)(sI-A)^{-1}(BD_2-M_2B_2) \\ &\quad + D_1DD_2 \end{aligned} \tag{24}$$

Notice that the poles of $G_0(s)$ come from those of $G_1(s)$ or $G_2(s)$. Since G_1GG_2 is devoided of the poles of G_1 or G_2, $G_0(s)$ showed vanish and this complete the proof. 2) This is a special case that $G_2(s) = I$. Hence, take $B_2 = 0$ and $D_2 = I$. Then we obtain from (24) and (16) that

$$\begin{aligned} G_1(s)G(s) &= C_1(sI-A_1)^{-1}(B_1D-M_1B) \\ &\quad + (D_1C+C_1M_1)(sI-A)^{-1}B + D_1D \end{aligned} \tag{25}$$

Hence, if $B_1D - M_1B = 0$, then G is a right eliminator of G_1. Suppose that (A_1, C_1) is observable and G is a right eliminator of G_1. Then, the first term in (25) should vanish for G_1G to be devoided of the poles of $G_1(s)$. This is possible only when $B_1D - M_1B = 0$ because (A_1, C_1) is observable.

3) The proof is dual to that of 2) and is omitted.

Rational matrices $G(s) \in RH_\infty^{p\times m}$ and $F(s) \in RH_\infty^{q\times n}$ are called inner and co-inner if $G_*(s)G(s) = I_m$ and $F(s)F_*(s) = I_q$, respectively. The following well-known results can be easily derived from the results of Lemma 2 or Lemma 3.

Lemma 4 [1] : 1) A $p\times m$ rational matrix $G(s) = \{A, B, C, D\}$ with A stable is inner if

$$B^TM_1 + D^TC = 0 \text{ and } D^TD = I_m \tag{26}$$

where M_1 is the unique symmetric solution of

$$A^TM_1 + M_1A = -C^TC \tag{27}$$

When (A, B) is controllable, the converse statement is also true. The solution matrix M_1 becomes observability Gramian, hence positive definite, if (A, C) is observable.

2) A $q\times n$ rational matrix $F(s) = \{A, B, C, D\}$ with A stable is co-inner if

$$BD^T + M_2C^T = 0 \text{ and } DD^T = I_q \tag{28}$$

where M_2 is the unique symmetric solution of

$$AM_2 + M_2A^T = -BB^T \tag{29}$$

When (A, C) is observable, the converse statement is also true. The solution matrix M_2 becomes the controllability Gramian, hence positive definite, if (A, B) is controllable.

3.2 Admissibility conditions of the model matching error

In section II, state-space parameters of $U_a(s)$ and $U_b(s)$ were expressed in terms of the state-space parameters of $T_a(s)$ and $T_b(s)$ and they were denoted as $U_a(s) = \{\hat{A}_a, \hat{B}_a, \hat{C}_a, D_a\}$ and $U_b(s) = \{\hat{A}_b, \hat{B}_b, \hat{C}_b, D_b\}$. In general, these parameters are not minimal and difficulties arise when we apply the eliminator theories developed in the previous section. Hence, we will denote minimal realizations of $\{\hat{A}_a, \hat{B}_a, \hat{C}_a\}$ and $\{\hat{A}_b, \hat{B}_b, \hat{C}_b\}$ as $\{A_a, B_a, C_a\}$ and $\{A_b, B_b, C_b\}$, respectively. This minimization, however, does not cause any trouble because all derived formulas can be reexpressed in terms of the old parameters. Since U_a and U_b are square, they are inner and co-inner at the same time and hence it follows from Lemma 4 that the following equalities hold. For $U_a(s) = \{A_a, B_a, C_a, D_a\}$

$$B_a^T M_a + D_a^T C_a = 0,\ D_a^T D_a = D_a D_a^T = I \qquad (30)$$

and

$$B_a D_a^T + N_a C_a^T = 0 \qquad (31)$$

where M_a and N_a are the unique positive definite to solutions of

$$A_a^T M_a + M_a A_a = -C_a^T C_a \qquad (32)$$

and

$$A_a N_a + N_a A_a^T = -B_a B_a^T \qquad (33)$$

It is not difficult to show that $N_a^{-1} = M_a$ (solve (30) for C_a and substitute this into (32)) where M_a is the controllability Gramian of (A_a, B_a) and N_a the observability Gramian of (A_a, C_a). For $U_b(s) = \{A_b, B_b, C_b, D_b\}$,

$$B_b D_b^T + M_b C_b^T = 0,\ D_b D_b^T = D_b^T D_b = I \qquad (34)$$

and

$$B_b^T N_b + D_b^T C_b = 0 \qquad (35)$$

where M_b and N_b are the unique positive definite solutions of

$$A_b M_b + M_b A_b^T = -B_b B_b^T \qquad (36)$$

and

$$A_b^T N_b + N_b A_b = -C_b^T C_b \qquad (37)$$

It is also true that $N_b^{-1} = M_b$. Now we are ready to state the main theorems of the paper.

Theroem 1 : A rational matrix $E(s) = \{A_e, B_e, C_e, D_e\} \in RH_\infty^{p\times m}$, where A_e is stable and the realization is not necessarily minimal, is admissible for (T_0, T_a, T_b) if and only if the parameters A_e, B_e, C_e and D_e satisfy the following five equations

$$A_a^T M_{11} + M_{11} A_e = -C_a^T C_e \qquad (38)$$

$$M_{11} B_e + C_a^T D_e = X_a \qquad (39)$$

$$A_e M_{21} + M_{21} A_b^T = -B_e B_b^T \qquad (40)$$

$$C_e M_{21} + D_e B_b^T = X_b \qquad (41)$$

$$B_a^T (M_{11} M_{21} + M_{12} M_{22}) = 0 \qquad (42)$$

where X_a and X_b are pre-determined values calculated from the given data(T_0, T_a, T_b) as

$$X_a := C_a^T D_0 + M_{12} B_0 \qquad (43)$$

$$X_b := D_0 B_b^T - C_0 M_{22} \qquad (44)$$

and M_{12} and M_{22} are the unique solutions of the equations.

$$\begin{aligned} A_a^T M_{12} + M_{12} A_0 &= -C_a^T C_0 \\ A_0 M_{22} + M_{22} A_b^T &= B_0 B_b^T \end{aligned} \qquad (45)$$

Proof : By Lemma 1, $E(s)$ is admissible if and only if $\tilde{E}(s) := E(s) - T_0(s) := \{A, B, C, D\}$ is a central eliminator of $U_{a*}(s) = \{-A_a^T, -C_a^T, B_a^T, D_a^T\}$ and $U_{b*}(s) = \{-A_b^T, -C_b^T, B_b^T, D_b^T\}$ whose realizations are minimal. It should be noticed that if $\tilde{E}(s)$ is a central eliminator of U_{a*} and U_{b*}, it is also a right eliminator of U_{a*} and a left eliminator of U_{b*} because $U_a U_{a*} = I$ and $U_{b*} U_b = I$. Now, applying the results of Lemma 3 to $\tilde{E}(s)$ which is a central, a left and a right eliminator, we can conclude after a little thought that $\tilde{E}(s) = \{A,B,C,D\}$ is a central eliminator of U_{a*} and U_{b*} if and only if

$$\begin{aligned} &C_a^T D + M_1 B = 0,\ D B_b^T - C M_2 = 0, \\ &B_a^T M_1 M_2 = 0 \end{aligned} \qquad (46)$$

where M_1 and M_2 are the solutions of the equations

$$\begin{aligned} A_a^T M_1 + M_1 A &= -C_a^T C \\ A M_2 + M_2 A_b^T &= -B B_b^T \end{aligned} \qquad (47)$$

Since $\tilde{E} = E - T_0$, it follows that

$$\begin{aligned} &A = \begin{bmatrix} A_e & 0 \\ 0 & A_0 \end{bmatrix},\quad B = \begin{bmatrix} B_e \\ -B_0 \end{bmatrix}, \\ &C = [\,C_e\ \ C_0\,],\quad D = D_e - D_0 \end{aligned} \qquad (48)$$

Now, substituting the equations in (48) and the partitioned expressions $M_1 = [M_{11}\ M_{12}]$ and $M_2 = [M_{21}^T\ M_{22}^T]^T$ into (46) and (47) yields the desired results.

While the formulas in Theorem 1 are convenient to check the admissibility of a given rational matrix, the formulas given in the next theorem are very useful to synthesize a model matching error that satisfies a given norm bound.

Theorem 2 : 1) A rational matrix $E(s)$ is admissible for (T_0, T_a, T_b) if and only if $E(s)$ is of the form

$$E(s) = \Phi(s) \circ W(s) \qquad (49)$$

where

$$\Phi(s) = \left[\begin{array}{cc|cc} A_a & B_a D_a^T X_{bm} N_b & N_a X_a & B_a D_a^T \\ 0 & A_b & B_b & 0 \\ \hline C_a & X_{bm} N_b & 0 & I \\ 0 & D_b^T C_b & I & 0 \end{array}\right] \qquad (50)$$

$$X_{bm} = X_b - D_a B_a^T M_{12} M_{22} \qquad (51)$$

and $W(s)$ an arbitrary stable matrix. 2) A rational matrix $E(s)$ is admissible for (T_0, T_a, T_b) if and only if $E(s)$ satisfies the equality

$$\begin{aligned} E_{ab}(s) &:= D_a U_{a*}(s) E(s) U_{b*}(s) D_b \\ &= \Lambda_*(s) + W(s) \end{aligned} \qquad (52)$$

where

$$\Lambda(s) := D_b^T U_b(s) X_a^T (sI - A_a)^{-1} B_a D_a^T + D_a^T C_b (sI - A_b)^{-1} X_{bm}^T \tag{53}$$

$$= \left[\begin{array}{cc|c} A_a & 0 & B_a D_a^T \\ B_b X_a^T & A_b & X_{bm}^T \\ \hline X_a^T & D_b^T C_b & 0 \end{array}\right] \tag{54}$$

and $W(s)$ the same arbitrary stable matrix as in (49).

Proof : For proof, see [5].

It should be noticed that the matrix $E_{ab}(s)$ is decomposed of a fixed anti-stable matrix $\Lambda_\bullet(s)$ and a free stable matrix $W(s)$. This decomposition property is a very significant one to develop H_2 and H_∞ optimization theories. In fact, the minimal H_2 norm of the H_2 MMP is equal to $\| \Lambda(s) \|_2$ and the minimal H_∞ norm of the H_∞ MMP is equal to $\| \Lambda(s) \|_h$. The H_2 and H_∞ optimization problems are treated in section 4.1

When an admissible $E(s)$ is given by the LFT form in (49), the corresponding $Q(s)$ should be computed from (1).

Lemma 5 : When an admissible $E(s)$ is given by the form in (49), the corresponding $Q(s)$ in (1) is given by

$$-Q(s) = \Psi(s) \circ W(s) \tag{55}$$

where

$$\Psi(s) = \left[\begin{array}{ccc|cc} A_{ta} - B_{ta} K_{ta} & B_{ta} \tilde{D}_{ta} Y_a & B_{ta} D_{ab} C_{tb} & -B_{ta} D_{ab} & B_{ta} D_{ta}^{-1} \\ 0 & A_0 & -Y_b \tilde{D}_{tb} C_{tb} & Y_b \tilde{D}_{tb} & 0 \\ 0 & 0 & A_{tb} - K_{tb} C_{tb} & K_{tb} & 0 \\ \hline -K_{ta} & \tilde{D}_{ta} Y_a & D_{ab} C_{tb} & -D_{ab} & D_{ta}^{-1} \\ 0 & 0 & -D_{tb}^{-1} C_{tb} & D_{tb}^{-1} & 0 \end{array}\right] \tag{56}$$

$$Y_a = B_a^T M_{12} + D_a^T C_0, \quad Y_b = M_{22} C_b^T - B_0 D_b^T \tag{57}$$

and

$$D_{ab} = D_{ta}^{-1} D_0 D_{tb}^{-1} \tag{58}$$

Proof : For the proof, see [5].

Up to this point, all formulas are described in terms of the minimal realization parameters (A_a, B_a, C_a) and (A_b, B_b, C_b). In computational aspect, however, the formulas expressed by the parameters $(\hat{A}_a, \hat{B}_a, \hat{C}_a)$ and $(\hat{A}_b, \hat{B}_b, \hat{C}_b)$ in (5) and (10) are more convenient to use.

Let $\hat{M}_{12}$ and $\hat{M}_{22}$ be the unique solutions of the equations

$$\hat{A}_a^T \hat{M}_{12} + \hat{M}_{12} A_0 = -\hat{C}_a^T C_0, \quad A_0 \hat{M}_{22} + \hat{M}_{22} \hat{A}_b^T = B_0 B_b^T \tag{59}$$

and define

$$\hat{X}_a = \hat{C}_a^T D_0 + \hat{M}_{12} B_0, \quad \hat{X}_b = D_0 \hat{B}_b^T - C_0 \hat{M}_{22} \tag{60}$$

$$\hat{Y}_a = \hat{B}_a^T \hat{M}_{12} + D_a^T C_0, \quad \hat{Y}_b = \hat{M}_{22} \hat{C}_b^T - B_0 D_b^T \tag{61}$$

and

$$\hat{X}_{bm} = \hat{X}_b - D_a \hat{B}_a^T \hat{M}_{12} \hat{M}_{22} \tag{62}$$

Let $\hat{N}_a$, $\hat{N}_b$, $\hat{M}_a$ and $\hat{M}_b$ be the unique solutions of the equations

$$\hat{A}_a \hat{N}_a + \hat{N}_a \hat{A}_a^T = -\hat{B}_a \hat{B}_a^T, \quad \hat{A}_b^T \hat{N}_b + \hat{N}_b \hat{A}_b = -\hat{C}_b^T \hat{C}_b \tag{63}$$

$$\hat{A}_a^T \hat{M}_a + \hat{M}_a \hat{A}_a = -\hat{C}_a^T \hat{C}_a, \quad \hat{A}_b \hat{M}_b + \hat{M}_b \hat{A}_b^T = -\hat{B}_b \hat{B}_b^T \tag{64}$$

Lemma 6 : 1) The following equalities hold;

$$\hat{M}_a = M_{ta}, \quad \hat{M}_b = M_{tb} \tag{65}$$

$$\hat{Y}_a = Y_a, \quad \hat{Y}_b = Y_b \tag{66}$$

where M_{ta} and M_{tb} are defined in (8) and (13).

2) Let $\hat{\Phi}(s)$ denote the transfer matrix $\Phi(s)$ in (50) with A_a, B_a, C_a, N_a, X_a, A_b, B_b, C_b, N_b, and X_{bm} substituted by the above corresponding head-signed variables. Then

$$\hat{\Phi}(s) = \Phi(s) \tag{67}$$

3) Let $\hat{\Lambda}(s)$ denote the transfer matrix $\Lambda(s)$ in (54) with A_a, B_a, X_a, A_b, B_b, C_b, and X_{bm} substituted by the above corresponding head-signed variables. Then

$$\hat{\Lambda}(s) = \Lambda(s) \tag{68}$$

Proof : For the proof, see [5].

Notice that the equalities in (66) make the matrix $\Psi(s)$ in (56) irrelevant to the minimal realization parameters (A_a, B_a, C_a) and (A_b, B_b, C_b).

IV. APPLICATIONS TO H_2 AND H_∞ OPTIMIZATION

In this section, the admissibility formulas developed in the previous section are applied to the MMPs in H_2 and H_∞ norm.

4.1 Model matching problem in H_2 norm

It is well known that the space RL_2 can be decomposed as $RL_2 = RH_2 \oplus RH_2^\perp$ where $RH_2^\perp$ is the orthogonal complement of RH_2 and the sign $\oplus$ stands for the direct sum. An important property of the above decomposition is that when $G(s) = G_1(s) + G_2(s)$ with $G_1(s) \in RH_2$ and $G_2(s) \in RH_2^\perp$, then $\| G \|_2^2 = \| G_1 + G_2 \|_2^2 = \| G_1 \|_2^2 + \| G_2 \|_2^2 = \| G_1 \|_2^2 + \| G_{2*} \|_2^2$, which is often described as the Pythagorean theorem.

For an admissible $E(s)$ to have finite H_2 norm, it is necessary that $E(s)$ be strictly proper. Hence, D_e will be set zero for the MMP in H_2 norm. Since $D_aU_a(s)$ and $U_b(s)D_b$ are inner, $\| E_{ab}(s) \|_2 = \| E(s) \|_2$ and hence it follows from (52) that

$$\begin{aligned} \| E \|_2^2 &= \| \Lambda_*(s) \|_2^2 + \| W(s) \|_2^2 \\ &= \| \Lambda(s) \|_2^2 + \| W(s) \|_2^2 \end{aligned} \tag{69}$$

Noticing that the first term of (69) has a fixed value and the second term is the free parameter $W(s)$, we can conclude that the admissible $E(s)$ which has the minimal H_2 norm, say $\hat{E}(s)$, is given by $E(s)$ in (49) with $W(s)$ set zero. That is

$$\hat{E}(s) = \left[\begin{array}{cc|c} A_a & B_aD_a^TX_{bm}N_b & N_aX_a \\ 0 & A_b & B_b \\ \hline C_a & X_{bm}N_b & 0 \end{array}\right] \tag{70}$$

It should be remarked that $\hat{E}(s)$ and $\Lambda(s)$ have the same H_2 norm but $\hat{E}(s)$ is admissible while $\Lambda(s)$ is not. Note also that all sub-optimal $E(s)$ can be easily determined by choosing appropriate $W(s)$. Now we have the following theorem.

Theorem 3 : The MMP in H_2 norm described in section II is solved as follows; 1) The optimal solution $\hat{E}(s)$ is given as in (70). Therefore the minimal H_2 norm is equal to $\| \hat{E}(s) \|_2 = \| \Lambda(s) \|_2 = \| \hat{\Lambda}(s) \|_2$. 2) All sub-optimal solution $E(s)$ such that $\| E(s) \|_2^2 \le \| \hat{E}(s) \|_2^2 + \beta$ are given by

$$E(s) = \Phi(s) \circ W(s) = \hat{\Phi}(s) \circ W(s) \tag{71}$$

where $\Phi(s)$ is given in (50) and $W(s)$ an arbitrary stable matrix with $\| W(s) \|_2^2 \le \beta$. In this case, the corresponding $Q(s)$ in (1) is given by

$$-Q(s) = \Psi(s) \circ W(s) = \hat{\Psi}(s) \circ W(s) \tag{72}$$

where $\Psi(s)$ is defined in (56). When $W(s) = 0$, the $Q(s)$ in (72) becomes the optimal one which yields the optimal solution $\hat{E}(s)$.

4.2 Model matching problem in H_∞ norm

In this section, we apply the results of Theorem 2 to obtain a formula that gives the minimal norm value of the MMP in H_∞ norm. Consider the matrix $F(s) = \{ A, B, C, D \}$ where A is stable and the realization is not necessarily minimal.

When $G(s) \in RL_2$ is decomposed as $G(s) = G_1(s) + G_2(s)$, $G_1(s) \in RH_2$, $G_2(s) \in RH_2^\perp$, the H_2 optimization problem is easily solved by exploiting the Pythagorean theorem. The following Nehari extension lemma plays an analogous role in H_∞ optimization. The formulas in the next lemma can be easily obtained from Lemma B.1 of [4]

Lemma 7 : 1) Given a rational matrix $F(s) = \{ A, B, C, D \} \in RH_\infty$ with A stable, the minimal value $\hat{\beta} := \min_{w \in RH_\infty} \| F_*(s) + W(s) \|_\infty$ is given by

$$\hat{\beta} = \| F(s) \|_h \tag{73}$$

2) All $W(s) \in RH_\infty$ such that $\| F_*(s) + W(s) \|_\infty \le \beta$, $\beta > \hat{\beta}$, are given by

$$\begin{aligned} W(s) = &\ (L_{11}(s)\,\phi(s) + L_{12}(s)) \\ &\times (L_{21}(s)\,\phi(s) + L_{22}(s))^{-1} \end{aligned} \tag{74}$$

where

$$L(s) = \begin{bmatrix} L_{11}(s) & L_{12}(s) \\ L_{21}(s) & L_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & Z^{-1}B & Z^{-1}MC^T \\ \hline B^TN + D^TC & I & D^T \\ C & 0 & I \end{array}\right] \tag{75}$$

$$Z = (\beta^2 I - MN) \tag{76}$$

and $\phi(s)$ is an arbitrary stable matrix with $\| \phi(s) \|_\infty \le \beta$.

Now, let us go back to the equation (52) in Theorem 2. Since $\| E_{ab}(s) \|_\infty = \| E(s) \|_\infty$ and $E_{ab}(s)$ is decomposed of Λ_* and $W(s)$, the results of Lemma 8 can be directly exploited. Notice also that $\| \Lambda(s) \|_h = \| \hat{\Lambda}(s) \|_h$ by the definition of hankel norm.

Theorem 4 : The minimal H_∞ norm of the H_∞ MMP, say $\hat{\beta}$, described in section II is given by

$$\hat{\beta} = \| \Lambda(s) \|_h = \| \hat{\Lambda}(s) \|_h \tag{77}$$

The significance of the formula (77) lies in the fact that it enables us to determine the minimal H_∞ norm of the given H_∞ optimization problem without complicated iteration procedures.

V. Conclusions

The admissibility concept of the model matching error with T_a and T_b square is introduced and the sufficient and necessary conditions for the admissibility are derived in state-space forms. A parametrization formula for the admissible $E(s)$ is obtained by using these conditions. It is shown that the formula is very convenient to treat H_2 and H_∞ optimization problems in a unified framework.

REFERENCES

[1]. J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, " State-space solutions to standard H_2 and H_∞ control problems," IEEE Trans. on Automat. Contr. Vol. AC-34, pp. 831-847. 1989.

[2]. K. Glover, D. J. N. Limebeer, J. C. Doyle, E. M. Kasenally and M. G. Safonov, " A Characterization of all solutions to the four block general distance problem," SIAM J. Control and Optimization, Vol. 29, No. 2, pp. 283-324, 1991.

[3]. H. Kimura, Y. Lu and R. Kawatani, " On the structure of H_∞ control systems and related extensions," IEEE Trans. on Automat. Contr. Vol. AC-36, pp. 653-667, 1991.

[4]. D. C. McFralane and K. Glover, Robust controller design using normalized coprime facotr plant descriptions, New York: Springer-Verlag, 1990.

[5] K. Park, " Admissibility conditions of the model matching error and applications to H_2 and H_∞ optimization," Report SKK-IEAC-94-01. Sung-Kyun-Kwan University, 1994.

[6] M. S. Verma and J. C. Romig, " Reduced-order controllers in H_∞ optimal synthesis methods of the first kind," Math. Systems Theory, Vol. 22, pp. 109-148, 1989.

[7] M. Weiss, " Spectral and inner-outer factorizations through the constrained Riccati equation," IEEE Tr. on Automat. Contr. Vol. AC-39, pp. 677-681, 1994.

[8] D. C. Youla and J. J. Bongiorno, Jr., "Modern Wiener-Hopf design of optimal controllers, Patr II: The multivariable case," IEEE Trans. on Automat. Contr. Vol. 21, pp. 319-338, 1976