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# On Iterative Learning Control For Some Distributed Parameter Systems

◦Won Cheol Kim, Kwang Soon Lee, *Arkadii V. Kim

*Institute of Mathematics & Mechanics
Ural Branch Acad. Sci. of Russia
Dept. Chem. Eng. Sogang Univ.

## ABSTRACT

In this paper, we discuss a design method of iterative learning control systems for parabolic linear distributed parameter systems(DPSs). First, we discuss some aspects of boundary control of the DPS, and then propose to employ the Karhunen-Loeve procedure to reduce the infinite dimensional problem to a low-order finite dimensional problem. An iterative learning control(ILC) for non-square transfer function matrix is introduced finally for the reduced order system.

## 1. INTRODUCTION

Some processes such as heat treatment of metallic or ceramic products can be described by - batch operation of - PDE (partial differential equation ) systems with boundary control. Since a PDE which describe a DPS is of infinite-dimension, it is hard to manage in practical use, especially, in the control system design. A PDE with boundary control can be reduced to a set of ODEs (ordinary differential equations) by using some reduction techniques [7]. Once it is reduced to a set of ODEs, we can employ various control techniques developed for MIMO(multi-input multi-output) systems.

In this paper, we consider an ILC for a linear parabolic PDE with boundary control. We will first convert the boundary control problem to a distributed control problem and then show the system can be decomposed into an infinite number of ODEs using eigenfunction expansions. In practical purpose, it can be approximated by a finite dimensional ODE system. This system can be controlled by known control methods.

Usually the approximated system has more outputs than inputs. We propose an ILC method for this problem.

## 2. STATEMENT OF THE PROBLEM

In the following, $\Omega$ denotes a domain in $R^3$ with boundary $\Gamma$. We assume a time over a finite interval, $t \in [\ 0, T]$ , $T < \infty$. and define

$Q = \Omega \times [\ 0, T]$ , and $\Sigma = \Gamma \times [\ 0, T]$

and $\Delta$ as the Laplacian operator.

Consider the heat conduction equation in Fig. 1 in $\Omega$

$$\begin{cases} \dfrac{\partial y}{\partial t} = a\Delta y \quad \text{in } Q \\ y \mid_{\Gamma_i} = f_i\ ,\ i = 1,2,3 \\ \dfrac{\partial y}{\partial n} \mid_{\Gamma_i} = 0, i = 4,5 \\ y(x, t_0) = y^0(x) \quad \text{in } \Omega \end{cases} \tag{1}$$

where $a$ is a constant and $\partial / \partial n$ denotes the normal derivative toward the exterior of $\Omega$. It is assumed that the initial condition $y^0(x)$ is fixed.

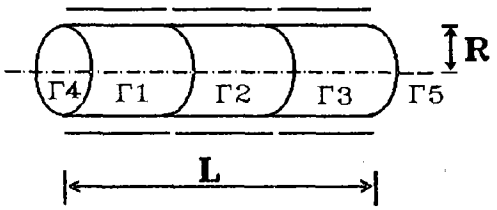


Fig.1 Schematic diagram of three-dimensional heat conduction system

Our aim is to realize an optimal trajectory which minimize the cost function given by

$$J(f)=\int_Q (y-y_d)^2 dxdt+q\sum_{i=1}^{3}\int_{\Sigma_i} f_i^2 d\Gamma dt \tag{2}$$

through iterative learning operation where $y_d$ is a desired trajectory function in $L^2(Q)$ and $q$ is a given positive constant. Then $f_i$ should be taken in $L^2(\Sigma_i)$ in order for (2) to make sense.

## 3. Uniqueness and Existence

To understand the solution of boundary control problem (1), let's consider a test function space given by

$$\begin{aligned}\Phi=\{\phi \mid\ & \phi\in H^{2,1}(Q),\ \phi(x,T)=0,\\ & \phi\mid_{\Gamma_1\cup\Gamma_2\cup\Gamma_3}=0,\ \frac{\partial\phi}{\partial n}\mid_{\Gamma_4\cup\Gamma_5}=0\}\end{aligned} \tag{3}$$

where

$$\begin{aligned}H^{2,1}=\{\ & \phi\mid\phi,\ \partial\phi/\partial t,\ \partial\phi/\partial x_i,\\ & \partial^2\phi/\partial x_i\partial x_j\in L^2(Q)\}\end{aligned} \tag{4}$$

It is known that, if the control space is closed and convex, there exists a unique $y=y(f_i)\in L^2(Q)$ such that the following equality is satisfied [3].

$$\begin{aligned}\int_Q y(-\frac{\partial\phi}{\partial t}-a\triangle\phi)dxdt\ =\ & \int_\Omega y^0\phi(x,t_0)dx\\ & -\sum_{i=1}^{3}\int_{\Sigma_i} f_i\frac{\partial\phi}{\partial n}\,dxdt,\ \forall\phi\in\Phi\end{aligned} \tag{5}$$

where $f_i\in L^2(\Sigma_i)$, $y^0\in L^2(\Omega)$

## 4. Conversion of a boundary control problem to a distributed control problem

The solution of (1) is equal to the sum of two parts $y_s(x,t)+y_t(x,t)$ in the sense of generalized function such that $y_t(x,t)$ is a solution of the system given by

$$\begin{cases}\dfrac{\partial y_t}{\partial t}=a\triangle y_t-\dfrac{\partial y_s}{\partial t}\\ y_t\mid_{\Gamma_i}=0,\ i=1,2,3\\ \dfrac{\partial y_t}{\partial n}\mid_{\Gamma_i}=0,\ i=4,5\\ y_t(x,t_0)=y^0(x)-y_s(x,t_0)\ \text{ in }\Omega\end{cases} \tag{6}$$

and $y_s(x,t)$ satisfies BCs(boundary conditions)

$$y_s(x,t)=\sum_{i=1}^{3} y_{si}(x)f_i(t) \tag{7}$$

where $y_{si(x)}$ is a solution of in the following equations

$$\begin{cases}\triangle y_{si}=0\ \text{ in }\ Q\\ y_{si}\mid_{\Gamma_i}=f_j\delta_{ij},\ j=1,2,3\\ \dfrac{\partial y_{si}}{\partial n}\mid_{\Gamma_i}=0,j=4,5\end{cases} \tag{8}$$

$$y(x,t)=y_s(x,t)+y_t(x,t) \tag{9}$$

By substituting (9) in the original problem (1), we obtain the following equations for $y_t(x,t)$:

$$\begin{cases}\dfrac{\partial y_t}{\partial t}=a\triangle y_t-\sum_{i=1}^{3} y_{si}(x)f_i(t)\\ y_t\mid_{\Gamma_i}=0,\ i=1,2,3\\ \dfrac{\partial y_t}{\partial n}\mid_{\Gamma_i}=0,\ i=4,5\\ y_t(x,t_0)=y^0(x)\ \text{ in }\Omega\end{cases} \tag{10}$$

We now have a new problem with zero BCs. This PDE can be solved by eigenfunction expansions.

## 5. Eigenfunction Expansions

Let $\phi_i(x)$'s be the eigenfunctions of $a\triangle$:

$$\begin{cases}a\triangle\phi_j=\lambda_i\phi_j\ \ in\ \ \Omega\\ \phi_j\mid_{\Gamma_i}=0,\ i=1,2,3\\ \dfrac{\partial\phi_j}{\partial n}\mid_{\Gamma_i}=0,\ i=4,5\\ (\phi_j,\phi_k)=\delta_{jk}\end{cases} \tag{11}$$

where (,) is the inner product defined by

$$(\phi_i, \phi_j) = \int_\Omega \phi_i \phi_j dx \tag{12}$$

The solution of (11) can be written by

$$y_i(x, t) = \sum_{i=1}^{\infty} \phi_i(x)\, a_i(t) \tag{13}$$

where $a_i(t) = (y_i, \phi_i)$,

and the solution of (8) can be written by

$$y_{si} = \sum_{j=1}^{\infty} \phi_j(x)\, \beta_{ji} \tag{14}$$

where $\beta_{ji} = (y_{si}, \phi_j)$.

By substituting (13) and (14) into (9), we obtain an infinite- dimensional ODE system given by

$$\frac{da_k}{dt} = \lambda_k a_k - \sum_{i=1}^{3} \beta_{ki} f_i(t) \ , \quad k = 1, 2, \ldots \tag{15}$$

$$y_i(x, t) = \sum_{i=1}^{\infty} \phi_{i(x)} a_i(t) \tag{16}$$

Therefore, the original system (1) can be written in a vector form as following :

$$\begin{cases} \dot{x} = Ax + B\dot{u} \\ y = Cx + Du \end{cases} \tag{17}$$

where

$$x = [\ a_1\, a_2 \ldots]^{T},\ u = [\ f_1\, f_2\, f_3\ ]^{T},\ y = y(x, t),$$

$$A = \begin{vmatrix} \lambda_1 & 0 & \ldots \\ 0 & \lambda_2 & \ldots \\ . & . & . \\ . & . & . \\ . & . & . \end{vmatrix}, \quad B = \begin{vmatrix} -b_{11} & -b_{12} & -b_{13} \\ -b_{21} & -b_{22} & -b_{23} \\ . & . & . \\ . & . & . \\ . & . & . \end{vmatrix},$$

$$C = [\ \phi_1\ \phi_2\ \phi_3\ \ldots]\ ,\ D = [\ y_{s1}\, y_{s2}\, y_{s3}\ ] \tag{18}$$

Control of an infinite dimensional system is not feasible. Indeed, higher eigen modes have minimal participation in the response. Hence, we can approximate the system by taking a finite number of dominant modes where the number is sufficiently large such that the accuracy is not affected. In this way, the infinite - dimensional problem can be reduced to a finite - dimensional problem.

## 6. Karhunen-Loeve Approximation

In the eigenfunction expansions, we generally have to take a large number of modes to get a sufficient accuracy, which still have the problem with a high dimensionality.

This trouble can be overcome by the Karhunen-Loeve (K-L) procedure which is a rational procedure to determine empirical eigenfunctions from an ensemble of system responses. Details of the K-L procedure can be referred to [8]. We will describe the performance of the K-L procedure by numerical simulations.

Consider the evolution equation

$$\frac{\partial y}{\partial t} = L(y) \tag{19}$$

where L is a linear differential operator. It is assumed the variable $y(x, t)$ taken on homogeneous boundary conditions. Consider the two points spatial correlation $K$ formed as follows:

$$K(x, x') = \lim_{T \to \infty} \frac{1}{T} \int_0^T y(x, t)\, \bar{y}(x', t) dt \tag{20}$$

where the overbar indicate complex conjugation. K and L have the same eigenfunctions. The eigenvalues calculated by the K-L procedure represent the average energy of corresponding eigenfunction in the system. We might take the eigenfunctions which capture most of the energy.

To apply the K-L procedure, we need to take an ensemble from (1). For simplicity, we consider about two dimensional domain bounded by $r = [\ 0, R\ ]$ , $z = [\ 0, 1\ ]$ and we set $a$=0.8 in (1). We simulated the problem by using the finite difference method and took an ensemble consisting of 40 realizations separated 0.1 time units. In fig. 2, the six dominant K-L eigenfunctions and eigenvalues which have 99.8% of energy were shown. When we gave changes on the boundary, the responses of two systems constructed by the empirical eigenfunctions and by the finite difference method were shown in figure 3 where (i,j) denotes the grid point at (z,r).

## 7. ILC for non-square transfer function matrix

The final approximated system using above procedures can be written by

$$y(s) = \{C(sI - A)^{-1}Bs + D\}u(s) = G(s)u(s) \tag{21}$$

in the transformed domain where $y(s) \in R^n$ and $u(s) \in R^m$, $n > m$.

The transfer function matrix generally appears as

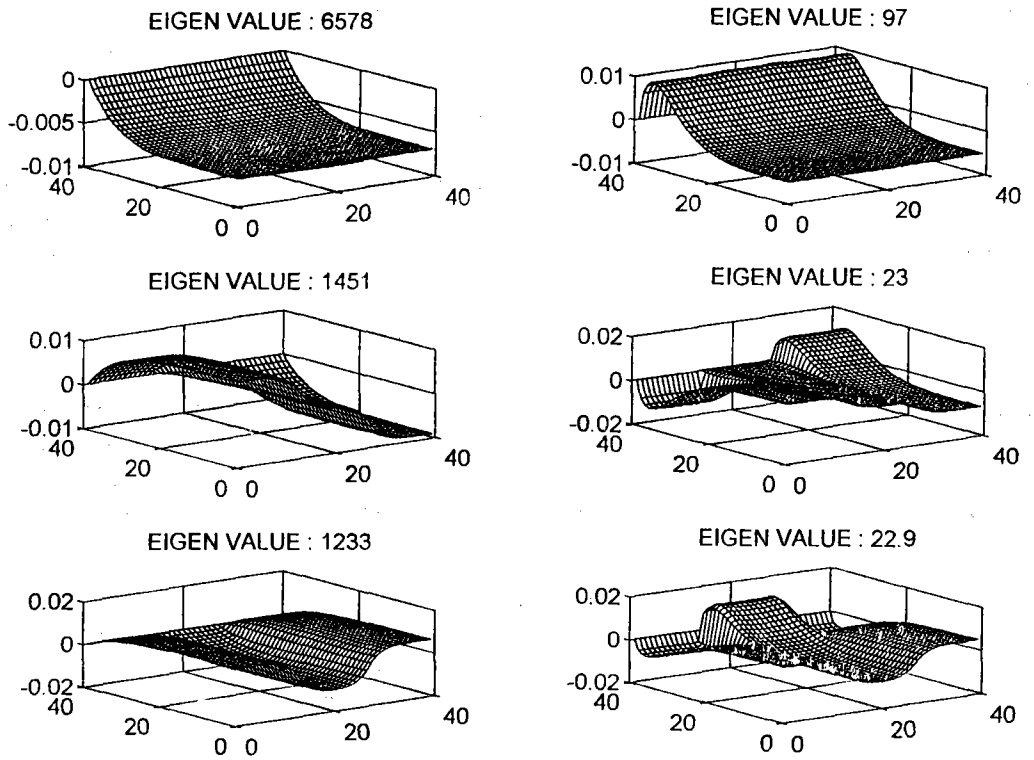


Fig. 2. Six dominant K-L eigenfunctions and eigenvalues.

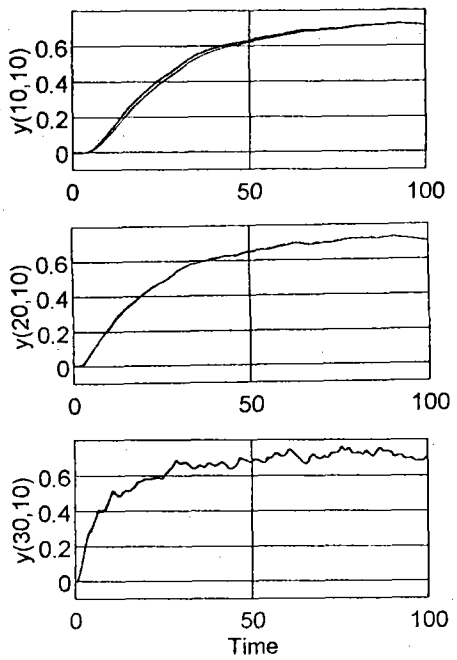


Fig. 3. Responses of two systems constructed by the K-L eigenfunctions and by FDM.

non-square. The necessary input update law to minimize the cost function (2) is written by

$$u_{k+1} = u_k + H_1 e_k - \varepsilon u_k \tag{22}$$

where $\varepsilon$ is a positive constant corresponding to the control weighting and the subscript k denotes the k-th performance cycle[6] . The smaller $\varepsilon$ means the lower control weighing. Substituting (21) into (28), we get the following sufficient condition for convergence.

$$\|(1-\varepsilon)I - H_1 G\|_i < 1 \tag{23}$$

A design method of decentralized MIMO ILC was proposed for a square transfer function matrix in [1]. In that paper, the diagonal dominance of the transfer matrix plays an important role for the design of the learning transfer matrix. In the non-square case, we can not take the inverse of diagonal terms for learning transfer matrix.

The $m$ linear combinations of the $n$ outputs may be used for learning control. Then the input update law can be written by

$$u_{k+1} = (1-\varepsilon)u_k + H_1 L e_k \tag{24}$$

where $L \in R^{m \times n}$ and the compensated matrix $LG$ satisfies diagonal dominance as required. Now we suggest to use the following learning controller

$$H_1 = (LG)_d^{-1} \tag{25}$$

where $(LG)_d$ denotes the diagonal part of $LG$.

In the case of $\varepsilon = 0$, if the input update law satisfies the convergence condition, it minimizes the cost function

$$\begin{aligned} J(u) &= \int_{t_0}^{t_f} e(u)^T L^T L e(u)\, dt \\ &= \int_{t_0}^{t_f} e(u)^T P\, e(u)\, dt \end{aligned} \tag{26}$$

where P is a positive semi-definite matrix. When the learning control with the control weighting is applied to the system, $\varepsilon$ can be used to adjust the convergence rate.

## 8. Conclusions

In this paper, an iterative learning control for linear parabolic DPSs with boundary control was discussed. We converted the PDE to an infinite ODEs by using eigenfunction expansions. We employed K-L procedure to get dominant empirical eigenfunctions and showed its efficiency by numerical simulation. We could get a lower order ODEs which described the original system sufficiently. Finally an iterative learning control for non-square transfer function matrix was discussed.

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