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Robust Pole Assignment of Proportional Integral Control System

Hwan Seong Kim*, Ken'ichi Ogasawara and Shigeyasu Kawaji

Department of Electrical Engineering and Computer Science,
Faculty of Engineering, Kumamoto University
2-39-1 kurokami, Kumamoto 860, JAPAN

Abstract

This paper is concerned with assess the possibility of robust pole assignment of proportional integral(PI) state feedback control system. First, the equivalence relations between a PI control system and an argumented control system proposed by Kawaji and Kim(1994) are extended from the new points of views of invariant closed loop poles. Second, on the relations, a remarkable result that the integral gain of PI control system is directly related to the insensitivity of system is presented. And, it is shown that the design of robust PI pole assignment is possible under the certain conditions.

1. Introduction

The importance of robust state feedback control has been recognized by many researchers and emphasized in the model with uncertainty. In the designs of state feedback law, the pole assignment is central and several algorithms have been proposed. The best known approaches from a numberical point of view are (1) implicit QR methods[10] (2) matrix equation method[4] (3) solution via real Schur form[12] (4)SVD based method[8]. The last one is known a robust pole assignment method and iterative eigenstructure method used in some control design software *e.g.*, MATLAB(by the MathWorks, Inc.,1987).

In the practical control systems, however, there are many disturbances, parameter variations and noises, and these unpredictable disturbances have an effect on the steady-state responses. So, a more robust controller is required.

As is well known, a PI controller can reject constant disturbances, and has attractive regulation of states in the system with parameter variations. From these properties, the PI control have been applied for many processing control fields[1, 2, 11].

On the other hand, a PI observer has been proposed by Wojciechowsk(1978) as an interesting part, and time recovery(steady - state recovery) with PI observer based control system has been shown by pole assignment method[3]. Also, the simultaneous recovery of loop transfer property and disturbance attenuation property by PI observer is studied by Kawaji and Kim(1994) using the conventional LQG/LTR and the duals of equivalence relations between a PI control system and an argumented control system. But, in the design method of PI controller, the robustness of pole assignment is an open problem.

The aim of this paper is to assess the possibility of robust PI pole assignment while keeping the structural advantages of PI controller, such as constant disturbance cancellation, perfect regulation and reduced sensitivity to parameter variations under the condition of full state feedback control. A proposed proof and a remakable facts are shown as follows: First, the equivalence relations proposed by Kawaji and Kim(1994) are extended from the new points of views of invariant poles. Second, under the pole assignment method by Kautsky *et.al.* (1985), the insensitivity of PI control system is compared with that of argumented control system by condition number, and it is shown that the integral gain of PI controller is related to the insensitivity of system. And, it is shown that the design of robust PI pole assignment is possible under the certain conditions.

Notation

$\Sigma(A, B, C)$	Realization of linear system specified by the time domain description $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$
$K_f(X)$	Condition number of matrix X *i.e.*, $K_f(X) = \lVert X \rVert_F \cdot \lVert X^{-1} \rVert_F$
$\lVert \cdot \rVert_F$	Frobenius norm
I_n	n-square matrix with 1's on the diagonal and 0's elsewhere
0_n	n-square matrix with 0's

$I_{n\times m}$	$n\times m$ dimension matrix with 1's on the diagonal of $\min(n,m)$ and 0's elsewhere
$0_{n\times m}$	$n\times m$ dimension matrix with 0's
A^*	Complex conjugate matrix of A
A^+	Pseudoinverse matrix of A
$\|\cdot\|_s$	Spectral norm
σ_{max}	Maximum singular value
σ_{min}	Minimum singular value

2. Equivalent systems and problem statements

We consider the following linear time invariant system:

$$\Sigma_{PI} : \Sigma(A,B,C) \tag{1}$$

where $x(t)\in R^n$ is state vector, $u(t)\in R^m$ control input, and $y(t)\in R^p$ measurement output. It is assumed that (A,B) is controllable.

A proportional integral control input is given as

$$u(t) = -H_P x(t) - \omega(t) + v(t) \tag{2}$$
$$;\ \omega(t) = H_I \int x(t)\,dt$$

where $v(t)$ is the external noise, and H_P and H_I are proportional and integral gains.

First, we assume that

$$\textbf{(A1)}\quad B = \begin{bmatrix} B_m \\ 0_{(n-m)\times m}\end{bmatrix}$$

where B_m is arbitrary matrix. For without loss of generality, If matrix B is not a form of **(A1)** in the system Σ_{PI}, we should be reconstruct the system's matrices of the Σ_{PI} and input's gains of (2) by transformation matrix U^* as

$$\tilde{A} = U^*AU^{*^{-1}},\quad \tilde{B} = U^*B,\quad \tilde{C} = CU^{*^{-1}}$$
$$\tilde{H}_P = H_P U^{*^{-1}},\ \text{ and }\ \tilde{H}_I = H_I U^{*^{-1}} \tag{3}$$

where U^* is obtained by SVD(Singular Value Decomposition) of B as $B \triangleq U\Pi V^*$.

The reconstructed system whose system's matrices are consisted as (3) is denoted as $\tilde{\Sigma}_{PI}$, and is illustrated by Fig. 1.(a). And, its closed loop transfer function is obtained as

$$\tilde{G}_{PI}(s) = \tilde{C}(sI - \tilde{A} + \tilde{B}\tilde{H}_P + \tilde{B}s^{-1}\tilde{H}_I)^{-1} \tag{4}$$

Second, we consider an argumented system Σ_E.

$$\Sigma_E : \Sigma(A_e, B_e, C_e) \tag{5}$$

where,

$$A_e = \begin{bmatrix} \tilde{A} & I_{n\times m} \\ I_{m\times n} & 0_m \end{bmatrix},\ B_e = \begin{bmatrix} \tilde{B} \\ 0_m \end{bmatrix},\ C_e = \begin{bmatrix} \tilde{C} & 0_{p\times m}\end{bmatrix}$$

and, $x_e(t) = [x(t)^T \ \ \xi(t)^T]^T$ is state vector.

Let the input of the argumented system Σ_E be given by

$$u_e(t) = -H_e x_e(t) + v(t) = -[H_1 \ \ H_2]x_e(t) + v(t) \tag{6}$$

Then, the closed loop transfer function of Σ_E is given by

$$G_E(s) = C_e(sI_{(n+m)} - A_e + B_e H_e)^{-1} B_e \tag{7}$$

and, its closed loop system is illustrated by Fig. 1.(b).

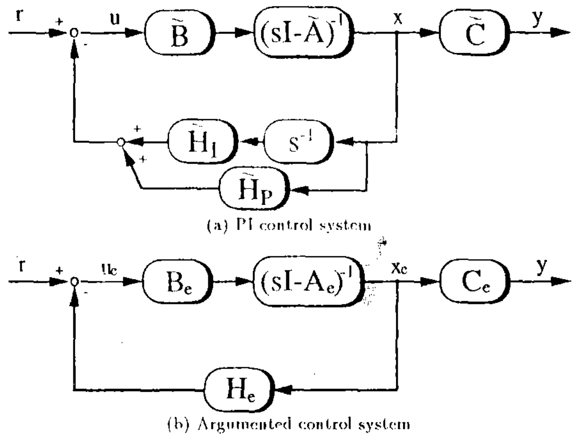


Fig. 1. Configurations of PI control system and argumented control system

In general equivalent systems[6], there are four invariant conditions betweens equivalent systems, *e.g.*, closed loop transfer function, poles of closed loop system, gramians of controllability and observability, and invariant zeros[7].

Kawaji and Kim(1994) have shown that equivalency of two systems $\tilde{\Sigma}_{PI}$ and Σ_E from the invariant closed loop transfer functions. Here, we will extend the result to the equivalency of two systems by the point of views invariant poles.

Following proposition shows the equivalency of systems by using invariant poles.

Proposition 1 : *Consider the two systems $\tilde{\Sigma}_{PI}$ and Σ_E, and suppose $\tilde{B}$ is form of the* **(A1)**. *If*

$$\tilde{B}\tilde{H}_P = \tilde{B}H_1 \tag{8}$$
$$\tilde{B}\tilde{H}_I = (\tilde{B}H_2 - I_{n\times m})I_{m\times n} \tag{9}$$

are satisfied, then the poles of closed loop system of $\tilde{\Sigma}_{PI}$ equal to those of Σ_E.

(The proof is given in Appendix A.)

From the proposition, it is found that the equivalency of system exists, and the argumented control system is the system with state feedback controller. So, if the state feedback gain of argumented system is obtained, then the gains of PI controller are calculated by subsitituting the state feedback gain into equivalence relations (8) and (9). But, it is an open problem that guaranteeing the robustness of PI control system.

The problem to be discussed, in this paper, is defined as follows:

Problem : *When the PI controller is designed by using equivalence relations, is it guaranteed the robustness of PI control system ? And what relation exists betweens the robustness of two systems ?*

3. Assessment of robust PI pole assignment

The robust pole assignment problem is to choose the matrices H_P and H_I so that the eigenvalues of closed loop system are as insensitive to pertubations in the closed loop system as possible.

Let's define $M = A - BH_P - BH_I\xi I_{m\times n}$. If M is diagonalizable and $X \in R^{(n+m)}$ is a matrix whose columns are the eigenvector of M then a measure of the sensitivity of the eigenvalues that applies to both small and large perturbations is the condition number

In order to minimize the sensitivity of the closed-loop eigenvalues, the condition number $K_f(X)$ should be minimized under the constraint of given A, B and eigenvalues $\Lambda_e = \{\lambda_1 \cdots \lambda_n \cdots \lambda_{n+m}\}$ as

$$\min K_f(X) \tag{10}$$

where,

$$MX = X\,\mathrm{diag}\{\Lambda_e\}$$

Thus, the problem of robust pole assignment reduces to design controller gain H_P and H_I such that minimize (10).

On the equivalence relations of systems mentioned in section 2, first we consider pole assignment problem of argumented system, and it can be parameterized using following lemma.

Lemma 1 :[8] Given the eigenvalues Λ_e and non-singular X_e, there exist H_e, a state feedback law of (6), if and only if

$$U_1^T(A_eX_e - X_e \cdot \mathrm{diag}\{\Lambda_e\}) = 0 \tag{11}$$

where

$$B_e = \begin{bmatrix} U_0 & U_1 \end{bmatrix}\begin{bmatrix} Z \\ 0 \end{bmatrix} \tag{12}$$

with $U = [U_0 \ \ U_1]$ orthogonal and Z non-singular. H_e is then given by

$$H_e = Z^{-1}U_0^T(X_e\Lambda_eX_e^{-1} - A_e) \tag{13}$$

In the lemma 1, the problem is to obtain the matrix X_e minimizing condition number of X_e, and it is discussed as a topic by some researcher[5, 8]. Kautsky *et.al.*,(1985) proposed a method and it has been used as a robust pole assignment method in MATLAB software. However, when the PI controller is designed by using equivalence relations in section 2, minimizing of the condition number of PI control system is an open problem.

Thus, under the pole assingment method by Kautsky *et.al.*(1985) in this section the problem is to study that what relation exists between condition numbers, and what element is subjected to insensitivity of system.

The general property of condition number has been known by following lemma.

Lemma 2 :[13] *The condition number $K_f(X_e)$ has invariant property with respect to unitary similarity transformation.*

In order to investigate the transformation matrix between the two systems, a following matrices are considered. Let the closed loop transfer function matrices, G_E and G_{PI}, be considered in the $G_E(s)$ and $G_{PI}(s)$ which is closed loop transfer function of Σ_{PI}, as follows

$$G_E = \begin{bmatrix} \tilde{A} - \tilde{B}H_1 & -(\tilde{B}H_2 - I_{n\times m}) \\ I_{m\times n} & 0_m \end{bmatrix} \tag{14}$$

$$G_{PI} = \begin{bmatrix} A - BH_P & -B \\ H_I & 0_m \end{bmatrix} \tag{15}$$

Proposition 2 : *If $\tilde{B}$ is a form of* **(A1)**, *then a non-singular transformation matrix exists between closed loop transfer relations matrices G_E and G_{PI} as*

$$TG_ET^{-1} = G_{PI} \tag{16}$$

(The proof is given in Appendix B.)

In the two closed loop system Σ_E and Σ_{PI}, in order to the condition numbers to be equal each other, the following condition is proposed.

Proportion 3 : *If the integral gain matrix of $G_{PI}(s)$ is satisfied as*

$$\sigma_{max}[H_I] = \sigma_{min}[H_I] = 1 \tag{17}$$

Then, the closed loop system $G_{PI}(s)$ has equal insensitivity as $G_E(s)$.

(The proof is given in Appendix C.)

According to the proposition 3, to make equals the condition numbers of two closed loop system each other, it should be satisfied that $\sigma_{max}[H_I] = \sigma_{min}[H_I] = 1$. However it is difficult to design the intergral gain of PI controller satisfied as above. And, it is a future problem to design method satisfying (17).

Although integral gain not satisfied above conditions, the PI control system can be keeping the structure advantages that constant disturbances cancellation, perfect regulation and reduced sensitivity to parameter variations.

4. Numberical example

To illustrate the robustness of PI control system, following example is given, and the responses of PI control system and the argumented control system are compared with each other.

We consider a nominal system matrix A_0, B and C which was used as an example in Beale and Shafai(1989).

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1.89 & 0.39 & -5.555 \\ 0 & -0.034 & -2.98 & 2.43 \\ 0.034 & -0.0011 & -0.99 & -0.21 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0.36 & -1.6 \\ -0.95 & -0.032 \\ 0.03 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The system included the uncertainties δA as

$$\delta A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.222 \\ 0 & 0 & 0.387 & 0.729 \\ 0 & 0 & 0 & 0.18 \end{bmatrix}$$

First, because B matrix is not a form of **(A1)**, we reconstruct the system's matrices by using transformation matrix as mentioned in Appendix A as follows,

$$\tilde{A}_0 = U^* A_0 U^{*^{-1}}, \quad \tilde{B} = U^* B, \quad \tilde{C} = C U^{*^{-1}}, \text{ and}$$
$$\delta\tilde{A} = U^* \delta A U^{*^{-1}} \tag{18}$$

where U^* is obtained by SVD of B as

$$U^* = \begin{bmatrix} -0.0000 & 0.9876 & -0.1568 & 0.0055 \\ 0.0000 & -0.1569 & -0.9871 & 0.0308 \\ 0.9995 & -0.00001 & 0.0009 & 0.0314 \\ -0.0314 & -0.00062 & 0.0313 & 0.9990 \end{bmatrix}$$

Let the desired closed loop poles be given as

$$\Lambda_e = \{-2.036 \pm 2.034j \quad -3.404 \pm 2.96j \quad -10 \quad -15\}$$

We construct the argumented system Σ_E and design the gains of PI controller by robust pole assignment method(Kautsky *et.al*, 1986). The gains of PI controller are calculated by (8) and (9), and it should be transform by (29) and (30) mentioned in Appendix A as

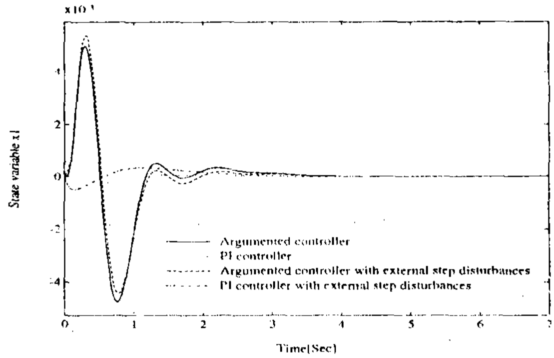
$$H_P = \begin{bmatrix} -689283 & 1.3603 & 52.965 & 2070.01 \\ 1273740 & -8.5479 & -151.57 & -4772.59 \end{bmatrix}$$

$$H_I = \begin{bmatrix} -0.000 & 689310 & 2193.9 & 363.31 \\ 0.000 & -1273812 & -5177.8 & -636.18 \end{bmatrix}$$

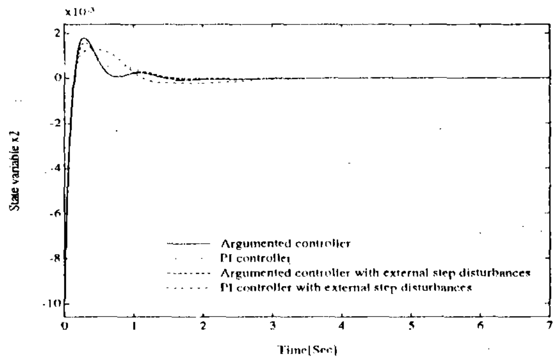
Let's the initial value of state variable are given as

$$x(0) = [0.000 \quad -0.01 \quad 0.000 \quad 0.000]^T$$

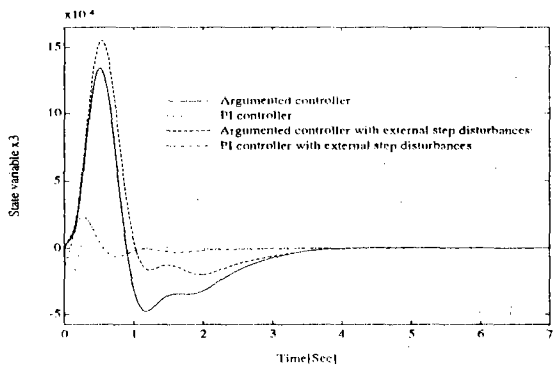
And, implementations of argumented controller and PI controller are presented under the external step disturbances is given, where the external step disturbances are given as



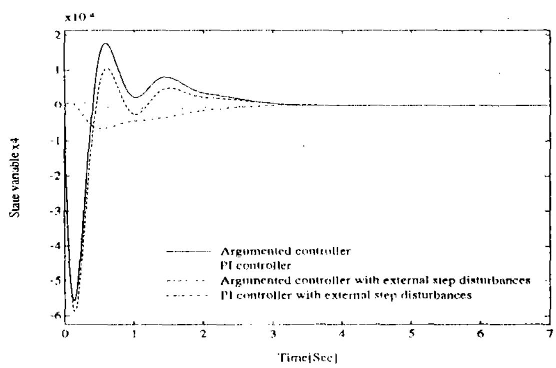
(a) Results of state variable x_1



(b) Results of state variable x_2



(c) Results of state variable x_3



(d) Results of state variable x_4

Fig. 2. Results of argumented control system and PI control system with uncertainties

$v(t) = [-0.005 \quad -0.005]^T$

The state responses of simulation are shown by Fig. 2.(a) to (d) in PI and argumented control systems including uncertainties with or without external step disturbances, respectively.

From the results, we can know the facts that the argumented control system keep a step disturbances cancellation, regulation and reduced sensitivity to parameter variations as PI control system, but the speed of PI control system's regulation is better than that of argumneted control system. So, the effectivness of the robustness of PI control system was verified by numerical example.

5. CONCLUSION

In this paper we have derived a new proof of closed loop equivalency between PI control system and argumented control system under the conditions of invariant poles. By based on the equivalency, the fact is considered : in the robust PI pole assignment the insensitivity of PI control system is scaled by condition number, and it is shown that the condition number is related to the integral gain of PI control system.

Thus, it is asserted that the robust PI controller can be designed by robust pole assignment method under the certain conditions. And, the robustness of PI controller is shown by the numberical example in the system with uncertainties and external step disturbances.

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Appendix

In this appendix, we show the facts affiliated with section 2 that the stable condition of PI control system and the necessary condition of assumption **(A1)** in proposition A.1 and A.2, respectively.

First, a basic stable condition of PI control system is represented as following.

Proposition A.1 : *The PI control system is stable if and only if all the eigenvalues of the matrix*

$$R_c = \begin{bmatrix} A - BH_P & -B \\ H_I & 0 \end{bmatrix} \tag{19}$$

have negative real parts.

Proof: From the Σ_{PI} and (2), it is rewritten as

$$\begin{aligned} \dot{x}(t) &= Ax(t) - BH_P x(t) - B\omega(t) + Bv(t) \\ \dot{\omega}(t) &= H_I x(t) \end{aligned} \tag{20}$$

Suppose the state variable is $[x(t)^T \;\; \omega(t)^T]^T$. From (20), it is reconstructed as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\omega}(t) \end{bmatrix} = \begin{bmatrix} A-BH_P & -B \\ H_I & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \omega(t) \end{bmatrix} + \begin{bmatrix} B \\ 0_m \end{bmatrix} v(t) \tag{21}$$

Thus, the proof is proved easily and it is shown as the dual property of PI observer[3]. ■

Second, the necessary condition of **(A1)** is discussed as following proposition.

Proposition A.2 : *Suppose an equivalence relations between two closed loop systems of* $\tilde{\Sigma}_{PI}$ *and* Σ_E *as follows*

$$\tilde{B}\widetilde{H}_P = \tilde{B}H_1 \tag{22}$$

$$\tilde{B}\tilde{H}_I = (\tilde{B}H_2 - I_{n\times m})I_{m\times n} \quad (23)$$

If $\tilde{B}$ is a form of **(A1)**, *then there exists mutual equivalence relationship.*

Proof: From (23), it is rewritten as

$$\tilde{B}H_2 I_{m\times n} = \tilde{B}\tilde{H}_I + I_{n\times m}I_{m\times n} \quad (24)$$

By premultiplying the (24) by $\tilde{B}^+$, it follows

$$H_2 I_{m\times n} = I_m \tilde{H}_I + \tilde{B}^+ I_{n\times m} I_{m\times n} \quad (25)$$

Let's premultifying the (25) by $\tilde{B}$, then we get

$$\tilde{B}H_2 I_{m\times n} = \tilde{B}I_m\tilde{H}_I + \tilde{B}\tilde{B}^+ I_{n\times m} I_{m\times n} \quad (26)$$

For equal (24) to (26), the right term of (24) should be satisfy that of (26) as follows

$$\tilde{B}\tilde{H}_I = \tilde{B}I_m\tilde{H}_I \quad (27)$$

$$\tilde{B}\tilde{B}^+ = I_{n\times m}I_{m\times n} \quad (28)$$

Thus, for satisfying mutual equal in (27) and (28), the necessary condition is that $\tilde{B}$ is a form of **(A1)**. ■

If matrix B is not a form of **(A1)** and the PI controller is designed by (8) and (9) in the system $\tilde{\Sigma}_{PI}$, then the gains of PI controller should be transformated as

$$H_P = \tilde{H}_P U^* \quad (29)$$

$$H_I = \tilde{H}_I U^* \quad (30)$$

Appendix A

Proof of proposition 1 : Let's obtain the $\det(G_E(s))$

$$\det(G_E(s)) = \det\begin{bmatrix} sI_n - \tilde{A} + \tilde{B}H_1 & -(\tilde{B}H_2 - I_{n\times m}) \\ I_{m\times n} & sI_m \end{bmatrix}$$
$$= \det(sI_n - \tilde{A} + \tilde{B}H_1)\cdot\det\{I_m + s^{-1}I_{m\times n}\cdot (sI_n - \tilde{A} + \tilde{B}H_1)^{-1}(\tilde{B}H_2 - I_{n\times m})\} \quad (31)$$

and obtain the $\det(\tilde{G}_{PI}(s))$

$$\det(\tilde{G}_{PI}(s)) = \det(sI_n - \tilde{A} + \tilde{B}\tilde{H}_P)\cdot\det\{I_m + s^{-1}I_m\tilde{H}_I(sI_n - \tilde{A} + \tilde{B}\tilde{H}_P)^{-1}\tilde{B}\} \quad (32)$$

Substituting (8) and (9) into (32), then we have

$$\det(\tilde{G}_{PI}(s)) = \det(sI_n - \tilde{A} + \tilde{B}H_1)\cdot\det\{I_m + \tilde{B}^+(\tilde{B}H_2 - I_{n\times m})s^{-1}I_{m\times n}(sI_n - \tilde{A} + \tilde{B}H_1)^{-1}\tilde{B}\}$$
$$= \det(sI_n - \tilde{A} + \tilde{B}H_1)\cdot\det\{I_m + s^{-1}I_{m\times n}\cdot (sI_n - \tilde{A} + \tilde{B}H_1)^{-1}\tilde{B}\tilde{B}^+(\tilde{B}H_2 - I_{n\times m})\}$$
$$= \det(sI_n - \tilde{A} + \tilde{B}H_1)\cdot\det\{I_m + s^{-1}I_{m\times n}\cdot (sI_n - \tilde{A} + \tilde{B}H_1)^{-1}(\tilde{B}H_2 - I_{n\times m})\}$$
$$= \det(\tilde{G}_E(s)) \quad (33)$$

The proof is completed. ■

Appendix B

Proof of Propositon 2 : Let the transformation matrix T be

$$T = \begin{bmatrix} T_1 & 0_{n\times m} \\ 0_{m\times n} & T_2 \end{bmatrix} \quad (34)$$

where, $T_1 = (U^*)^{-1}$ and $T_2 = \tilde{H}_I I_{n\times m}$.

Then

$$TG_ET^{-1} = \begin{bmatrix} T_1 & 0_{n\times m} \\ 0_{m\times n} & T_2 \end{bmatrix}\cdot \begin{bmatrix} \tilde{A} - \tilde{B}H_1 & -(\tilde{B}H_2 - I_{n\times m}) \\ I_{m\times n} & 0_m \end{bmatrix}\begin{bmatrix} T_1^{-1} & 0_{n\times m} \\ 0_{m\times n} & T_2^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} T_1(\tilde{A} - \tilde{B}H_1)T_1^{-1} & -T_1(\tilde{B}H_2 - I_{n\times m})T_2^{-1} \\ T_2 I_{m\times n}T_1^{-1} & 0_m \end{bmatrix}$$
$$= \begin{bmatrix} T_1\tilde{A}T_1^{-1} - T_1\tilde{B}H_1T_1^{-1} & -T_1(\tilde{B}H_2 - I_{n\times m})I_{n\times m}\tilde{H}_I^+ \\ \tilde{H}_I I_{n\times m}I_{m\times n}T_1^{-1} & 0_m \end{bmatrix}$$
$$= \begin{bmatrix} A - BH_P & -B \\ H_I & 0_m \end{bmatrix} = G_{PI} \quad (35)$$

Because of $(\tilde{B}H_2 - I_{n\times m})I_{m\times n}\tilde{H}_I^+ = \tilde{B}$ from (9). ■

Appendix C

Proof of proposition 3 :

$$K_f(X) = \| TX_e \|_F \cdot \| (TX_e)^{-1} \|_F$$
$$\le \| T \|_s \cdot \| X_e \|_F \cdot \| X_e^{-1} \|_F \cdot \| T^{-1} \|_s$$
$$\le K_f(X_e) \| T \|_s \cdot \| T^{-1} \|_s$$
$$\le K_f(X_e)\sigma_{max}[T]\cdot(\sigma_{min}[T])^{-1} \quad (36)$$

In order to $K_f(X) = K_f(X_e)$, it should be $\sigma_{max}[T]\cdot(\sigma_{min}[T])^{-1} = 1$. Thus $\sigma_{max}[H_I] = \sigma_{min}[H_I] = 1$, because of $\sigma_{max}[T_1] = \sigma_{min}[T_1] = 1$. ■