'94 KACC (1994. 10. 17 ~ 20)

Extreme Point Results for Robust Schur Stability

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ABSTRACT

In this paper, we develop two sufficient conditions for Schur stability of convex combinations of discrete time polynomials. We give conditions under which Schur stability of the extremes implies Schur stability of the entire convex combination. These results are based on Bhattacharyya's result (1991), the AHMC theory in Barmish and Kang's paper (1993) and the bilinear transformation. Important applications of the results involves robust Schur stability of a feedback system having degenerate interval plants in an extreme point context.

1. INTRODUCTION

This paper deals with the convex direction problem for the space of Schur stable polynomials. We want to find the condition of the monic polynomial g under which Schur stability of two polynomials f and $f+g$ implies Schur stability of the edge polynomial $p(z,\lambda)=f(z)+\lambda g(z)$ for any $\lambda\in[0,1]$. By Schur stability we mean that all the roots of a polynomial lie in the unit disk. If g is a convex direction, then Schur stability of both f and $f+g$ implies Schur stability of $f+\lambda g$ for any $\lambda\in[0,1]$. Throughout this paper, $p(z,\lambda)$ has an invariant degree n with real coefficients for any $\lambda\in[0,1]$. There exist several known results on the convex direction problem. Perez, Abdallah and Docampo (1992) derived the case of the convex direction when $g(z)\in\{1,z,z^2\cdots,z^{\lceil \frac{n}{2}\rceil}\}$. By $\lceil x\rceil$ we mean the smallest integer larger than or equal to x. See also the paper by Kang (1993). Fu (1991) tells us that the symmetric and the antisymmetric polynomial of degree n are convex directions. Mansour and Kraus (1991) tell us that $(z+a)k(z)$ is a convex direction where $a\in R$ and $k(z)$ is the symmetric or the antisymmetric polynomial of degree $n-1$. In addition they show that $(z+a)(z-1)^r k(z)$ is a convex direction where the degree of $k(z)+r$ is equal to $n-1$. In this paper, we will show that the polynomial $(z+a)z^i$ is a convex direction for $i\in\{0,1,2,\cdots,\lfloor \frac{n}{2}\rfloor-1\}$. By $\lfloor x\rfloor$ we mean the largest integer less than or equal to x. In addition it will be shown that $b(z)z^i$ is another convex direction for $i\in\{0,1,2,\cdots,\lfloor \frac{n}{2}\rfloor-1\}$ where $b(z)$ is antiSchur stable or has some restrictions. In Section 2, we present the main results and Section 3 defines the degenerate interval plant and deals with the interpretation of the main results in the degenerate interval plant framework and conclusions are provided in Section 4. Some useful lemmas are presented in the Appendix for proofs of theorems and main results.

2. MAIN RESULTS

We need one definition before we present the main results.

2.1 Definition (Convex Direction)

A monic polynomial g is said to be a *convex direction* for the space of Schur stable n-th order discrete time polynomials if the following condition is satisfied: Given any Schur stable n-th order polynomial f such that $f+g$ is also Schur stable and $\deg(f+\lambda g)=n$ for all $\lambda\in[0,1]$, it follows that $f+\lambda g$ is Schur stable for all $\lambda\in[0,1]$.

Remark: We can define the convex direction for the space of strict left half plane stable n-th order continuous time polynomials by a little modification of the definition.

2.2 Main Results

We are in a position to describe the main results.

Theorem 1: *Suppose* $g(z)=(z+a)z^i$ *for* $i\in\{0,1,2,\cdots,\lfloor \frac{n}{2}\rfloor-1\}$. *Then* $g(z)$ *is a convex direction for the space of the Schur polynomials of degree* $n\geq 3$.

Proof: At first, we want to prove that $(z+a)z^i$

$i = \lfloor \frac{n}{2} \rfloor - 1$ is a convex direction for the space of Schur stable polynomials of degree n. The bilinear transformation gives us the mapping $z = \frac{s+1}{s-1}$. Substituting z into the above direction and multiplying the direction by $(s-1)^n$, we obtain

$$(s-1)^n\left(\frac{s+1}{s-1}+a\right)\left(\frac{s+1}{s-1}\right)^i$$

and rewriting it as follows: If n is even, it becomes

$$(s^2-1)^{\frac{n}{2}-1}(s-1)((s+1)+a(s-1))$$

and if n is odd, it is

$$(s^2-1)^{\frac{n-3}{2}}(s-1)^2(s+1)+a(s-1)).$$

Using Lemma A in the Appendix, the two polynomials are indeed the convex directions for the space of stable polynomials of degree $n \geq 3$. So we proved that $(z+a)z^i$ $i = \lfloor \frac{n}{2} \rfloor - 1$ is a convex direction for the space of Schur stable polynomials of degree n. By similar approaches, we can prove that $g(z)=(z+a)z^i$ for $i \in \{0,1,2,\cdots, \lfloor \frac{n}{2} \rfloor - 2\}$ is a convex direction for the space of Schur stable polynomials of degree $n \geq 3$. Now we complete the proof. []

Theorem 2: *Let $b(z)$ be a monic quadratic polynomial and two roots of $b(z)$ has one of the following three conditions:*

i) $b(z)$ is antiSchur stable;

ii) two roots are nonnegative real, one root σ_1 is Schur stable and the other root σ_2 is not and the condition $\left|\frac{\sigma_1+1}{\sigma_1-1}\right| \leq \frac{\sigma_2+1}{\sigma_2+1}$ is satisfied

iii) two roots are nonpositive real, one root σ_1 is Schur stable and the other root σ_2 is not and the condition $\left|\frac{\sigma_1+1}{\sigma_1-1}\right| \geq \frac{\sigma_2+1}{\sigma_2+1}$ is satisfied

Under these conditions, for any $i \in \{0,1,2,\cdots, \lfloor \frac{n}{2} \rfloor - 1\}$, $b(z)z^i$ is a convex direction for the space of Schur stable polynomials of degree $n \geq 4$.

Before we prove Theorem 2, we need a lemma.

Lemma 1: *If $g(s)$ is a convex direction for the space of the stable n-th order polynomials, then $g(s)A(s)P(s)$ is a convex direction for the space of stable polynomials of the appropriate order where $A(s)$ is antistable and monic and $P(s)$ is a monic polynomial having either all even or all odd powers of s.*

Proof: Let $\overline{A(s)} = (-1)^m A(-s)$ where m is the order of $A(s)$. We obtain $\overline{A(s)}$ is stable. Since g is a convex direction, stability of f and $f+g$ implies stability of the edge polynomial $f+\lambda g$ for $\lambda \in [0,1]$. Now we assume that f and $f+Ag$ are stable. Then $f(s)\overline{A(s)}$ and $(f+Ag)\overline{A(s)}$ are clearly stable. By Lemma B in the Appendix, $Ag\overline{A(s)}$ is a convex direction. So $(f+\lambda Ag)\overline{A(s)}$ is stable for any $\lambda \in [0,1]$. Hence so is $f+\lambda gA$ for any $\lambda \in [0,1]$. Now we obtain that gA is a convex direction. Again by Lemma B, we complete the proof. []

We are in a position to prove Theorem 2.

Proof: At first, we want to prove that $b(z)z^i$ $i = \lfloor \frac{n}{2} \rfloor - 1$ is a convex direction for the space of Schur stable polynomials of degree n. The bilinear transformation gives us the mapping $z = \frac{s+1}{s-1}$. Substituting z into the above direction and multiplying it by $(s-1)^n$, we obtain

$$(s-1)^n b\left(\frac{s+1}{s-1}\right)\left(\frac{s+1}{s-1}\right)^i$$

and rewriting it as follows: If n is even, it becomes

$$(s^2-1)^{\frac{n}{2}-1}(s-1)^2 b\left(\frac{s+1}{s-1}\right)$$

and if n is odd, it is

$$(s^2-1)^{\frac{n-3}{2}}(s-1)(s-1)^2 b\left(\frac{s+1}{s-1}\right).$$

Now the remaining thing is that we will prove that the above two polynomials are convex directions for the space of the stable continuous time polynomials. If the condition (i) is satisfied, then $(s-1)^2 b\left(\frac{s+1}{s-1}\right)$ is antistable and Lemma 1 tells us that the above two directions are convex direction for the space of Schur stable polynomials of degree n. We use that fact that any real number may be antistable. If the condition (ii) or (iii) is satisfied and n is even, then $(s-1)^2 b\left(\frac{s+1}{s-1}\right)$ is unstable and Lemma C in the appendix and Lemma 1 tells us that the above two directions are convex direction for the space of Schur stable polynomials of degree n. If the condition (ii) or (iii) is satisfied and n is odd, then the roots of $(s-1)(s-1)^2 b\left(\frac{s+1}{s-1}\right)$ has some sorts of the interlacing properties satisfying conditions in Lemma C. So we proved that $(s-1)(s-1)^2 b\left(\frac{s+1}{s-1}\right)$ is a convex direction. Lemma 1 tells us that the above two directions are convex direction for the space of Schur stable polynomials of degree n.

By similar approaches, we can prove that for any $i \in \{0,1,2,\cdots, \lfloor \frac{n}{2} \rfloor - 2\}$, $b(z)z^i$ is a convex direction for the space of Schur stable polynomials of degree $n \geq 4$. Now, we complete the proof. []

3. EXTREME POINT RESULTS FOR DEGENERATE INTERVAL PLANTS

3.1 Proportional Compensators and the Degenerate Interval Plant

A degenerate interval plant family P is described in terms of uncertain parameter vector q and r entering into the coefficients of a strictly proper transfer function $P_d(z,q,r)$. We write

$$P_d(z,q,r)=\frac{N_d(z,q)}{D_d(z,r)}$$

where $N_d(z,q)=\sum_{i=0}^{m} q_i z^i$; $D_d(z,r)=s^n+\sum_{i=0}^{n-1} r_i z^i$. In addition, interval bounds $[q_i^-,q_i^+]$ are allowed for $i=0,1,2,\cdots,\min(m,\lfloor\frac{n}{2}\rfloor-1)$ and $q_i^-=q_i^+$ for $i>\min(m,\lfloor\frac{n}{2}\rfloor-1)$; interval bounds $[r_i^- r_i^+]$ are allowed for $i=0,1,2,\cdots,\lfloor\frac{n}{2}\rfloor-1$ and $r_i^-=r_i^+$ for $i\geq\lfloor\frac{n}{2}\rfloor$. The first result involves the degenerate interval plant family P with a proportional compensator $C(s)=k$. The proportional compensator robustly stabilizes $P_d(z,q,r)$ (unit disk stability) if and only if it stabilizes each of the extreme plants. Furthermore, we may change the interval conditions such that interval bounds $[q_i^-,q_i^+]$ are allowed for $i=0,1,2,\cdots,\min(m,\lceil\frac{n}{2}\rceil)$ and $q_i^-=q_i^+$ for $i>\min(m,\lceil\frac{n}{2}\rceil)$; interval bounds $[r_i^- r_i^+]$ are allowed for $i=0,1,2,\cdots,\lceil\frac{n}{2}\rceil$ and $r_i^-=r_i^+$ for $i>\lceil\frac{n}{2}\rceil$. With this new degenerate interval plants, the extreme point result for robust Schur stability still holds.

3.2 First Order Compensator

The second result involves the degenerate interval plant family P with a first order compensator

$$C(s)=k\frac{z-z_i}{z-p_i}.$$

The next theorem comes from the modification of Hollot and Yang's result (1990), Theorem 1 and Lemma D in the appendix.

Theorem 3: *A first order compensator C robustly stabilizes $P_d(z,q,r)$ (unit disk stability) if and only if it stabilizes each of the extreme plants.*

Proof: The necessity is trivial. We concentrate on the proof of the sufficiency. The closed loop polynomial with the first order compensator is

$$D_d(z,r)(z-p_i)+N_d(z,q)(z-z_i).$$

Constructing all the edge polynomials from the closed loop polynomials, the edge polynomials are described by

$$f(z)+\lambda(z+a)z^i$$

for $i=0,1,2,\cdots,\lfloor\frac{n}{2}\rfloor-1$.

By Theorem 1, we obtain the extreme point results for the edge polynomials. By Lemma D (Edge theorem) in the appendix, we complete the proof. []

3.3 Second Order Compensators

Given a degenerate interval plant family P with a compensator

$$C(s)=k\frac{(z-1.1)(z-1.2)}{(z-1.5)(z-1.7)},$$

we obtain the extreme point result by Theorem 2. Using either the condition 2 or the condition 3, we may choose the compensator as

$$C(s)=k\frac{(z-0.1)(z-0.5)}{(z-0.2)(z-1.7)}$$

or

$$C(s)=k\frac{(z+0.1)(z+1.5)}{(z+0.2)(z+1.7)}.$$

Thirdly, we note that we can mix the results from the conditions 2 and 3. For example, we obtain an extreme point result for the compensator

$$C(s)=k\frac{(z-0.1)(z-1.5)}{(z+0.1)(z+1.5)}$$

and for the compensator $\overline{C}=\frac{1}{C}$ as well.

4. CONCLUSIONS

We developed two sufficient conditions for Schur stability of convex combination of the discrete time polynomials in an extreme point context. Important application of the results involved robust Schur stabilty of a feedback system having the degenerate interval plants. In the future, we will want to obtain the extreme point result for robust Schur stability of degenerate interval plants beyond the second order compensator in terms of the root locations.

5. APPENDIX

Lemma A (Bhattacharyya (1991)): The continuous time polynomial $(s+a)A(s)P(s)$ is a convex direction for the space of stable polynomials of the appropriate order where $A(s)$ is antistable and monic and $P(s)$ is a monic polynomial having either all even or all odd powers of s.

Lemma B (Rantzer (1992)): Consider an edge polynomial

$$f(s,\lambda)=f_0(s)+\lambda p(s)\quad \lambda\in[0,1].$$

Assume that $F(s, \lambda)$ has a constant degree n for any $\lambda \in [0,1]$ and the inequality

$$\frac{d}{d\omega} \angle p(j\omega) \leq \left| \frac{\sin(2\angle p(j\omega))}{2\omega} \right|$$

holds for all $\omega > 0$ such that $p(j\omega) \neq 0$. Then $F(s, \lambda)$ is stable for all $\lambda \in [0,1]$ if and only if $F(s,0)$ and $F(s,1)$ are stable.

Lemma C (Barmish and Kang (1992)): Consider a monic third order polynomial $g(s)$ and the three roots $\sigma_1, \sigma_2, \sigma_3$ of of $g(s)$ satisfies one of the following three conditions:

(i) $\sigma_i \geq 0$ for $i = 1,2,3$.

(ii) $\sigma_1 \leq 0, \sigma_2 \leq 0, \sigma_3 \geq 0$ and the "interlacing condition" $|\sigma_2| \leq \sigma_3 \leq |\sigma_1|$ is satisfied.

(iii) $\sigma_1 \leq 0, \sigma_2 \geq 0, \sigma_3 \geq 0$.

Under these conditions, $g(s)$ is a convex direction for the space of stable polynomials of degree $n \geq 4$.

Lemma D (Bartlett, Hollot and Huang (1988)): The zeros of a polytope of polynomials **P** is robustly Schur stable if and only if the edges of **P** are robustly Schur stable. Hence, one need only test for robust Schur stability for all convex combinations of the form

$$\alpha P^i(z) + (1-\alpha) P^j(z) \qquad \alpha \in [0,1]$$

where $P^i(z)$ and $P^j(z)$ are the generators for **P**.

6. REFERENCES

1. B. R. Barmish and H. I. Kang, "Extreme point results for robust stability of interval plants: beyond first order compensators," *Automatica*, vol. 28, no. 6, pp. 1169--1180, 1992.

2.. A. C. Bartlett, C. V. Hollot and L. Huang, ''Root locations of an entire polytope of polynomials: it suffices to check the edges," *Mathematics of Control, Signals and Systems*, vol. 1, no. 1, pp. 61--71, 1988.

3. S. P. Bhattacharyya, "Vertex results in robust stability," Tech. Rep., TCSP Report, Texas A & M Univ. May, 1991.

4. F. Perez, C. Abdallah and D. Docampo, "Extreme--point stability tests for discrete--time polynomials," *Proceedings of the Conference on Decision and Control*, pp. 1552--1553, Tucson, AZ, 1992.

5. H. I. Kang, "Alternative proofs of Perez, Abdallah and Docampo's theorem," *Proceedings of KTTE fall Conference*, pp. 235--236, 1993.

6. M. Fu, "A class of weak Kharitonov regions for robust stability of linear uncertain systems," *IEEE Transactions on Automatic Control*, vol. AC-36, pp. 975--978, 1991.

7. M. Mansour and F. J. Kraus, "Argument conditions for Hurwitz and Schur stable polynomials and robust stability probems, in Control of Uncertain Dynamic Systems, (L. H. Keel and S. P. Bhattacharyya, eds.), pp. 511--524, Littletown, MA: CRC Press, September, 1991.

8. C. V. Hollot and F. Yang, "Robust stabibization of interval plants using lead or lag compensators," *Systems and Control Letters*, vol. 14, pp. 9--12, 1990.

9. A. Rantzer, "Stability conditions for polytopes of polynomials," *IEEE Transactions on Automatic Control*, vol. AC-37, pp. 79--89, 1992.