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Design of Nonlinear Optimal Regulators Using Lower Dimensional Riemannian Geometric Models

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Abstract: A new Riemannian geometric model for the controlled plant is proposed by imbedding the control vector space in the state space, so as to reduce the dimension of the model. This geometric model is derived by replacing the orthogonal straight coordinate axes on the state space of a linear system with the curvilinear coordinate axes. Therefore the integral manifold of the geometric model becomes homeomorphic to that of a fictitious linear system. For the lower dimensional Riemannian geometric model, a nonlinear optimal regulator with a quadratic form performance index which contains the Riemannian metric tensor is designed. Since the integral manifold of the nonlinear regulator is determined to be homeomorphic to that of the linear regulator, it is expected that the basic properties of the linear regulator such as feedback structure, stability and robustness are to be reflected in those of the nonlinear regulator. To apply the above regulator theory to a real nonlinear plant, it is discussed how to distort the curvilinear coordinate axes on which a nonlinear plant behaves as a linear system. Consequently, a partial differential equation with respect to the homeomorphism is derived. Finally, the computational algorithm for the nonlinear optimal regulator is discussed and a numerical example is shown.

1. Introduction

Differential geometric approach is extremely useful for solving a special class of nonlinear control problems [1],[2]. The linearization problem is a typical case where the geometric approach acts effectively. Based on Krener's study [1] for the equivalence of control systems, Isidori [2] has shown a linearization method by using the vector fields and their higher order Lie derivatives as the coordinate bases. The method gives a exact solution to the linearization problem, but it requires the high accuracy for the system parameters because of the nature of higher order Lie derivatives [3]. A different method of linearization using Riemannian geometric approach was proposed by the authors [5], which does not require the higher order differentiation. The fundamental idea is that in the state space a nonlinear system referred to appropriate curvilinear coordinates will behave as a linear system. The equations which determine the curvilinear coordinate system were derived for the general case. The Riemannian geometric approach was also applied successfully to design the controller for bilinear plants [6].

In the Riemannian geometric approach the dimension of the state space becomes rather high because it is described as a direct sum of the state vector space in itself and the control vector space. In this paper it is proposed to decrease the dimension of the Riemannian space by a proper choice of the construction of the space, which leads to decrease the computation time remarkably.

First, a lower dimensional nonlinear model is derived by expressing the control vector space with the state space structure. The model is constructed like the derivation of the geodesic curve on the gravitational gauge field in Einstein's principle of general relativity [4] and the integral manifold of this model becomes homeomorphic to that of a linear system.

In the second step, a new quadratic-form performance index is introduced using Riemannian metric tensors, and the corresponding nonlinear optimal regulator is constructed which is homeomorphic to the linear optimal regulator. The optimal control is represented in the state feedback form and constructed in terms of the homeomorphism and the solution of a pseudo Riccati equation. It is expected that the basic properties of the linear regulator such as feedback structure, stability and robustness are to be reflected in those of the nonlinear regulator.

In the third step, it is discussed how to distort the curvilinear coordinates axes fitted to the nonlinear system. Consequently, a partial differential equation with respect to the homeomorphism is derived.

Under the preparation of the above procedures, the partial differential equation is solved and the optimal control law is determined by using the solution and the solution of the above pseudo Riccati equation. Then the boundary condition of the gauge field is determined. Since the new model is homeomorphic to the linear system, the boundary condition is determined by the condition of the linear approximation at a suitable point.

Numerical examples are shown to prove the effectiveness of the proposed design method.

2. Riemannian Space and Geometric model

If the curvilinear coordinate axes is used as the state space coordinate axes instead of the orthogonal straight coordinate axes, the orbits of a linear system are observed as those of a nonlinear system. Using this fundamental idea, a geometric model is derived. To descrive the model, in this paper, the state space is regarded as a Riemannian space which has the curvilinear coordinate axes. The state vector $\tilde{X}$ and the extended control vector $\tilde{U}$ (which has the same dimension as the state vector) are imbeded in this Riemannian space. Accordingly, the space dimension of this Riemannian model is lower than that of the previous model by the authors which is described in a direct sum space of the state vector space and the control vector space [5],[6].

Consider the nonlinear system

$$\dot{x} = \alpha\,(x)x + Bu, \tag{1}$$

where x is an n dimensional vector, u is an r dimensional vector,

$a(x)$ is an $n \times n$ matrix, B is an $n \times r$ matrix.

Let $\tilde{\mathcal{X}}^\mu, \tilde{\mathcal{U}}^\mu$ $(\mu = 1, \ldots, n)$ be two (1,0)-tensors on a curvilinear coordinate system $(\tilde{x}^i)$, $i = 1, \ldots, n$, and $\tilde{\mathcal{A}}^\mu_\nu, \tilde{\mathcal{B}}^\mu_\nu$ $(\mu, \nu = 1, \ldots, n)$ be two (1,1)-tensors on $(\tilde{x}^i)$. Corresponding to these tensors, let $\bar{\mathcal{X}}^\mu, \bar{\mathcal{U}}^\mu, \bar{\mathcal{A}}^\mu_\nu, \bar{\mathcal{B}}^\mu_\nu$ be the tensors on a orthogonal straight coordinate system $(\bar{x}^i)$. From the transformation formula of a tensor component (see Appendix A1), we have

$$\bar{\mathcal{X}}^\mu = \frac{\partial \bar{x}^\mu}{\partial \tilde{x}^\gamma} \tilde{\mathcal{X}}^\gamma \tag{2}$$
$$\bar{\mathcal{U}}^\mu = \frac{\partial \bar{x}^\mu}{\partial \tilde{x}^\gamma} \tilde{\mathcal{U}}^\gamma \tag{3}$$
$$\bar{\mathcal{A}}^\mu_\nu = \frac{\partial \bar{x}^\mu}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{x}^\beta}{\partial \bar{x}^\nu} \tilde{\mathcal{A}}^\alpha_\beta \tag{4}$$
$$\bar{\mathcal{B}}^\mu_\nu = \frac{\partial \bar{x}^\mu}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{x}^\beta}{\partial \bar{x}^\nu} \tilde{\mathcal{B}}^\alpha_\beta \tag{5}$$

From the definition of the matrix representation of a tensor (see Appendix A1), a (1,0)-tensor is expressed as a column vector, and a (1,1)-tensor $\mathcal{T}^\mu_\nu$ is expressed as a matrix with (μ, ν) element $\mathcal{T}^\mu_\nu$. Therefore a linear system

$$\frac{d}{dt}\bar{\mathcal{X}} = \bar{\mathcal{A}}\,\bar{\mathcal{X}} + \bar{\mathcal{B}}\,\bar{\mathcal{U}} \tag{6}$$

is represented as a tensor equation

$$\frac{d}{dt}\bar{\mathcal{X}}^\mu = \bar{\mathcal{A}}^\mu_\nu\,\bar{\mathcal{X}}^\nu + \bar{\mathcal{B}}^\mu_\nu\,\bar{\mathcal{U}}^\nu. \tag{7}$$

Substituting (2),(3),(4) and (5) into (7), we have

$$\begin{aligned}\frac{\partial \bar{x}^\mu}{\partial \tilde{x}^\gamma}\frac{d\tilde{\mathcal{X}}^\gamma}{dt} &+ \frac{\partial^2 \bar{x}^\mu}{\partial \tilde{x}^\beta \partial \tilde{x}^\lambda}\frac{d\tilde{x}^\lambda}{dt}\tilde{\mathcal{X}}^\beta \\ &= \frac{\partial \bar{x}^\mu}{\partial \tilde{x}^\alpha}\frac{\partial \tilde{x}^\beta}{\partial \bar{x}^\nu}\tilde{\mathcal{A}}^\alpha_\beta\frac{\partial \bar{x}^\nu}{\partial \tilde{x}^\rho}\tilde{\mathcal{X}}^\rho + \frac{\partial \bar{x}^\mu}{\partial \tilde{x}^\alpha}\frac{\partial \tilde{x}^\beta}{\partial \bar{x}^\nu}\tilde{\mathcal{B}}^\alpha_\beta\frac{\partial \bar{x}^\nu}{\partial \tilde{x}^\rho}\tilde{\mathcal{U}}^\rho.\end{aligned} \tag{8}$$

Multiplying $\frac{\partial \tilde{x}^\gamma}{\partial \bar{x}^\mu}$ into (8), we have

$$\frac{d\tilde{\mathcal{X}}^\gamma}{dt} + \frac{\partial \tilde{x}^\gamma}{\partial \bar{x}^\mu}\frac{\partial^2 \bar{x}^\mu}{\partial \tilde{x}^\beta \partial \tilde{x}^\lambda}\frac{d\tilde{x}^\lambda}{dt}\tilde{\mathcal{X}}^\beta = \tilde{\mathcal{A}}^\gamma_\rho\,\tilde{\mathcal{X}}^\rho + \tilde{\mathcal{B}}^\gamma_\rho\,\tilde{\mathcal{U}}^\rho \tag{9}$$

From the definition of the Christoffel symbols $\widetilde{\{{}_\lambda{}^\mu{}_\nu\}}$ (see Appendix A2) the equation (9) becomes

$$\frac{d\tilde{\mathcal{X}}^\gamma}{dt} + \widetilde{\{{}_\beta{}^\gamma{}_\lambda\}}\frac{d\tilde{x}^\lambda}{dt}\tilde{\mathcal{X}}^\beta = \tilde{\mathcal{A}}^\gamma_\rho\,\tilde{\mathcal{X}}^\rho + \tilde{\mathcal{B}}^\gamma_\rho\,\tilde{\mathcal{U}}^\rho. \tag{10}$$

Theorem 1: If a linear system on the orthogonal straight coordinate axes $(\bar{x}^i), i = 1, \ldots, n$

$$\frac{d}{dt}\bar{\mathcal{X}}^\mu = \bar{\mathcal{A}}^\mu_\nu\,\bar{\mathcal{X}}^\nu + \bar{\mathcal{B}}^\mu_\nu\,\bar{\mathcal{U}}^\nu \tag{11}$$

is observed on the curvilinear axes $(\tilde{x}^i), i = 1, \ldots, n$, then the system is represented as

$$\frac{d\tilde{\mathcal{X}}^\gamma}{dt} + \widetilde{\{{}_\beta{}^\gamma{}_\lambda\}}\frac{d\tilde{x}^\lambda}{dt}\tilde{\mathcal{X}}^\beta = \tilde{\mathcal{A}}^\gamma_\rho\,\tilde{\mathcal{X}}^\rho + \tilde{\mathcal{B}}^\gamma_\rho\,\tilde{\mathcal{U}}^\rho \tag{12}$$

Proof: Proof is given as above.

As a special case, if $\tilde{\mathcal{A}}^\gamma_\rho$ and $\tilde{\mathcal{B}}^\gamma_\rho$ become two zero tensors and $\tilde{\mathcal{X}}^\rho$ is chosen as $\tilde{\mathcal{X}}^\rho = \frac{d\tilde{x}^\rho}{dt}$, the Riemannian geometric model (12) becomes a geodesic on the curvilinear coordinate system $(\tilde{x}^i)$ [4].

$$\frac{d^2\tilde{x}^\gamma}{dt^2} + \widetilde{\{{}_\beta{}^\gamma{}_\lambda\}}\frac{d\tilde{x}^\lambda}{dt}\frac{d\tilde{x}^\beta}{dt} = 0 \tag{13}$$

Now, the Riemannain geometric model has a dual model. The dual model is derived from the transpose expression of a linear model

$$\frac{d}{dt}\bar{\mathcal{X}}^t = \bar{\mathcal{X}}^t\bar{\mathcal{A}}^t + \bar{\mathcal{U}}^t\bar{\mathcal{B}}^t, \tag{14}$$

by observing on the curvilinear coordinate system instead of the orthogonal straight coordinate system. Using a Riemannian metric tensor $g_{\mu\nu}$, two (0,1)-tensors $\tilde{\mathcal{X}}_\nu$ and $\tilde{\mathcal{U}}_\nu$ are induced from $\tilde{\mathcal{X}}^\nu$ and $\tilde{\mathcal{U}}^\nu$.

$$\tilde{\mathcal{X}}_\nu = g_{\mu\nu}\tilde{\mathcal{X}}^\nu \tag{15}$$
$$\tilde{\mathcal{U}}_\nu = g_{\mu\nu}\tilde{\mathcal{U}}^\nu \tag{16}$$

From the definition of the matrix representation of tensors, (0,1)-tensors are expressed as the row vectors. Therefore the equation (14) is represented as a tensor equation

$$\frac{d}{dt}\bar{\mathcal{X}}_\mu = \bar{\mathcal{X}}_\nu\overline{{}^t\mathcal{A}}^\nu{}_\mu + \bar{\mathcal{U}}_\nu\overline{{}^t\mathcal{B}}^\nu{}_\mu, \tag{17}$$

where ${}^t\mathcal{A}^\nu{}_\mu$ and ${}^t\mathcal{B}^\nu{}_\mu$ are two tensors and have the following properties.

$$\overline{{}^t\mathcal{A}}^i{}_j = \bar{\mathcal{A}}^j_i \tag{18}$$
$$\overline{{}^t\mathcal{B}}^i{}_j = \bar{\mathcal{B}}^j_i \tag{19}$$

From the transformation formula of a tensor component, we have

$$\overline{{}^t\mathcal{A}}^\mu_\nu = \frac{\partial \bar{x}^\mu}{\partial \tilde{x}^\alpha}\frac{\partial \tilde{x}^\beta}{\partial \bar{x}^\nu}\widetilde{{}^t\mathcal{A}}^\alpha_\beta \tag{20}$$
$$\overline{{}^t\mathcal{B}}^\mu_\nu = \frac{\partial \bar{x}^\mu}{\partial \tilde{x}^\alpha}\frac{\partial \tilde{x}^\beta}{\partial \bar{x}^\nu}\widetilde{{}^t\mathcal{B}}^\alpha_\beta \tag{21}$$

Theorem 2: If a transposed linear system on the orthogonal straight coordinate axes $(\bar{x}^i), i = 1, \ldots, n$

$$\frac{d\bar{\mathcal{X}}_\mu}{dt} = \bar{\mathcal{X}}_\nu\overline{{}^t\mathcal{A}}^\nu{}_\mu + \bar{\mathcal{U}}_\nu\overline{{}^t\mathcal{B}}^\nu{}_\mu \tag{22}$$

is observed on the curvilinear axes $(\tilde{x}^i), i = 1, \ldots, n$, then the system is represented as

$$\frac{d\tilde{\mathcal{X}}_\rho}{dt} - \tilde{\mathcal{X}}_\beta\frac{d\tilde{x}^\delta}{dt}\widetilde{\{{}_\delta{}^\beta{}_\rho\}} = \tilde{\mathcal{X}}_\beta\widetilde{{}^t\mathcal{A}}^\beta{}_\rho + \tilde{\mathcal{U}}_\beta\widetilde{{}^t\mathcal{B}}^\beta{}_\rho \tag{23}$$

Proof: From the transformation formula of a tensor component, we have

$$\bar{\mathcal{X}}_\mu = \tilde{\mathcal{X}}_\beta\frac{\partial \tilde{x}^\beta}{\partial \bar{x}^\mu} \tag{24}$$
$$\bar{\mathcal{U}}_\mu = \tilde{\mathcal{U}}_\beta\frac{\partial \tilde{x}^\beta}{\partial \bar{x}^\mu} \tag{25}$$

Substituting these equations into (22), we have

$$\begin{aligned}\frac{d\tilde{\mathcal{X}}_\beta}{dt}\frac{\partial \tilde{x}^\beta}{\partial \bar{x}^\mu} &+ \tilde{\mathcal{X}}_\beta\frac{d\bar{x}^\lambda}{dt}\frac{\partial^2 \tilde{x}^\beta}{\partial \bar{x}^\lambda \partial \bar{x}^\mu} \\ &= \tilde{\mathcal{X}}_\beta\frac{\partial \tilde{x}^\beta}{\partial \bar{x}^\nu}\overline{{}^t\mathcal{A}}^\nu{}_\mu + \tilde{\mathcal{U}}_\beta\frac{\partial \tilde{x}^\beta}{\partial \bar{x}^\nu}\overline{{}^t\mathcal{B}}^\nu{}_\mu.\end{aligned} \tag{26}$$

Multiplying $\frac{\partial \bar{x}^\mu}{\partial \tilde{x}^\rho}$ into (26) on the right side, we have

$$\begin{aligned}\frac{d\tilde{\mathcal{X}}_\rho}{dt} &+ \tilde{\mathcal{X}}_\beta\frac{d\bar{x}^\lambda}{dt}\frac{\partial^2 \tilde{x}^\beta}{\partial \bar{x}^\lambda \partial \bar{x}^\mu}\frac{\partial \bar{x}^\mu}{\partial \tilde{x}^\rho} \\ &= \tilde{\mathcal{X}}_\beta\frac{\partial \tilde{x}^\beta}{\partial \bar{x}^\nu}\overline{{}^t\mathcal{A}}^\nu{}_\mu\frac{\partial \bar{x}^\mu}{\partial \tilde{x}^\rho} + \tilde{\mathcal{U}}_\beta\frac{\partial \tilde{x}^\beta}{\partial \bar{x}^\nu}\overline{{}^t\mathcal{B}}^\nu{}_\mu\frac{\partial \bar{x}^\mu}{\partial \tilde{x}^\rho}\end{aligned} \tag{27}$$

Replacing x^ν in the equation (12) of Appendix A2 with $\bar{x}^\nu$, and considering the relation

$$\frac{\partial^2 \bar{x}^\nu}{\partial \bar{x}^\lambda \partial \bar{x}^\mu} = 0, \tag{28}$$

we have

$$\frac{\partial^2 \bar{x}^\nu}{\partial \tilde{x}^i \partial \tilde{x}^j} = -\frac{\partial \bar{x}^\lambda}{\partial \tilde{x}^i}\frac{\partial \bar{x}^\mu}{\partial \tilde{x}^j}\widetilde{\{\lambda^\nu{}_\mu\}} \tag{29}$$

Substituting the equation (29) into (27) with (20) and (21), we have (23). Q.E.D.

3. Homeomorphism

In this section, the mapping is clarified between two integral monifolds of the Riemannian geometric model on the orthogonal straight coordinate system and the curvilinear coordinate system. Let $\bar{U}$ and $\tilde{U}$ be two coordinate neighborhoods, which have the coordinate system $(\bar{x}^i)$ and $(\tilde{x}^i)$, respectively. On $\bar{U} \cap \tilde{U}$, we have

$$d\bar{x}^\nu = \frac{\partial \bar{x}^\nu}{\partial \tilde{x}^\gamma} d\tilde{x}^\gamma \tag{30}$$

$$d\tilde{x}^\nu = \frac{\partial \tilde{x}^\nu}{\partial \bar{x}^\gamma} d\bar{x}^\gamma. \tag{31}$$

Let τ^ν_γ and T^γ_ν be

$$\tau^\nu_\gamma = \frac{\partial \bar{x}^\nu}{\partial \tilde{x}^\gamma} \tag{32}$$

$$T^\gamma_\nu = \frac{\partial \tilde{x}^\gamma}{\partial \bar{x}^\nu}. \tag{33}$$

Then, we have

$$\tau^\mu_\gamma \, T^\gamma_\nu = \delta^\mu_\nu, \tag{34}$$

where δ^μ_ν is Kronecker's δ. If τ^ν_γ and T^γ_ν are determined on all over $\bar{U} \cap \tilde{U}$, then the coordinate space $(\bar{x}^i)$ and $(\tilde{x}^i)$ are also determined on it by using the relations (30) and (31).
From the transformation formula of a tensor component on $\bar{U} \cap \tilde{U}$, we have

$$\bar{\mathcal{X}}^\nu = \frac{\partial \bar{x}^\nu}{\partial \tilde{x}^\gamma}\,\tilde{\mathcal{X}}^\gamma = \tau^\nu_\gamma\,\tilde{\mathcal{X}}^\gamma \tag{35}$$

$$\tilde{\mathcal{X}}^\gamma = \frac{\partial \tilde{x}^\gamma}{\partial \bar{x}^\nu}\,\bar{\mathcal{X}}^\nu = T^\gamma_\nu\,\bar{\mathcal{X}}^\nu \tag{36}$$

$$\bar{\mathcal{X}}_\nu = \frac{\partial \tilde{x}^\gamma}{\partial \bar{x}^\nu}\,\tilde{\mathcal{X}}_\gamma = \tau^\gamma_\nu\,\tilde{\mathcal{X}}_\gamma \tag{37}$$

$$\tilde{\mathcal{X}}_\gamma = \frac{\partial \bar{x}^\nu}{\partial \tilde{x}^\gamma}\,\bar{\mathcal{X}}_\nu = T^\nu_\gamma\,\bar{\mathcal{X}}_\nu \tag{38}$$

Taking a different view of these relations, τ and T can be regarded as the mappings between $\bar{\mathcal{X}}^\nu, \bar{\mathcal{X}}_\nu \in \bar{U}$ and $\tilde{\mathcal{X}}^\gamma, \tilde{\mathcal{X}}_\gamma \in \tilde{U}$. Considering the equation (34), T becomes the inverse mapping of τ. If τ and T are both continuous mappings, then τ becomes homeomorphism between two integral manifolds.
Substituting these τ and T into the equations (4) and (5), we have

$$\tilde{\mathcal{A}}^\gamma_\rho = T^\gamma_\mu \bar{\mathcal{A}}^\mu_\lambda \tau^\lambda_\rho \tag{39}$$

$$\tilde{\mathcal{B}}^\gamma_\rho = T^\gamma_\mu \bar{\mathcal{B}}^\mu_\lambda \tau^\lambda_\rho. \tag{40}$$

Differentiating (32) and (33) with t, we have

$$\frac{d\tau^\nu_\gamma}{dt} = \frac{\partial^2 \bar{x}^\nu}{\partial \tilde{x}^\lambda \partial \tilde{x}^\gamma}\frac{d\tilde{x}^\lambda}{dt} \tag{41}$$

$$\frac{dT^\gamma_\nu}{dt} = \frac{d\bar{x}^\lambda}{dt}\frac{\partial^2 \tilde{x}^\gamma}{\partial \bar{x}^\lambda \partial \bar{x}^\nu}. \tag{42}$$

Therefore, the Riemannian geometric model and its dual model become

$$\frac{d\tilde{\mathcal{X}}^\gamma}{dt} = (T^\gamma_\mu \bar{\mathcal{A}}^\mu_\lambda \tau^\lambda_\rho - T^\gamma_\mu \frac{d\tau^\mu_\rho}{dt})\,\tilde{\mathcal{X}}^\rho + \tilde{\mathcal{B}}^\gamma_\mu\,\tilde{\mathcal{U}}^\mu \tag{43}$$

$$\frac{d\tilde{\mathcal{X}}_\rho}{dt} = \tilde{\mathcal{X}}_\beta(T^\beta_\nu \overline{{}^t\mathcal{A}}^\nu{}_\mu \tau^\mu_\rho - \frac{dT^\beta_\mu}{dt}\tau^\mu_\rho) + \tilde{\mathcal{U}}_\beta\,\widetilde{{}^t\mathcal{B}}^\beta{}_\rho. \tag{44}$$

These equations are derived by substituting (32),(33),(39),(40),(41) and (42) into (9) and (27).

4. Nonlinear optimal regulator

Theorem 3: Consider the Riemannian geometric model (43), and consider the performance index

$$J = \frac{1}{2}\int_{t_0}^{T} (\tilde{\mathcal{X}}^i g_{ij} \tilde{\mathcal{X}}^j + \tilde{\mathcal{U}}^i g_{ij} \tilde{\mathcal{U}}^j) dt, \tag{45}$$

where g_{ij} is the Riemannian metric tensor. Suppose there exist the homeomorphism τ, then the control law which monimizes J is derived as

$$\tilde{\mathcal{U}}^\gamma = -T^\gamma_m \overline{{}^t\mathcal{B}}^m{}_\mu \bar{\mathcal{S}}^\mu_j \tau^j_k \tilde{\mathcal{X}}^k, \tag{46}$$

where the tensor $\bar{\mathcal{S}}^\nu_\mu$ is the unique solution of the pseudo Riccati equation

$$\frac{d\bar{\mathcal{S}}^m_\mu}{dt} + \bar{\mathcal{S}}^m_n \bar{\mathcal{A}}^n_\mu + \overline{{}^t\mathcal{A}}^m{}_n \bar{\mathcal{S}}^n_\mu - \bar{\mathcal{S}}^m_n \bar{\mathcal{B}}^n_\gamma \overline{{}^t\mathcal{B}}^\gamma{}_\nu \bar{\mathcal{S}}^\nu_\mu + \delta^m_\mu = 0, \tag{47}$$

satisfying the boundary condition

$$\bar{\mathcal{S}}^m_\mu(T) = 0. \tag{48}$$

Proof: Since the Hamiltonian function H is a scalar function and is invariant to the transformation of the coordinate system, the caluculus of variation to the control problem can be extended to our Riemannian geometric case. The proof is given by following the procedures of section 9.3 in [7]. The Hamiltonian H for the Riemannian geometric model with the cost J of (45) is

$$H = \frac{1}{2}(\tilde{\mathcal{X}}^i g_{ij} \tilde{\mathcal{X}}^j + \tilde{\mathcal{U}}^i g_{ij} \tilde{\mathcal{U}}^j) + \psi_j[(T^j_\mu \bar{\mathcal{A}}^\mu_\lambda \tau^\lambda_\rho - T^j_\mu \frac{d\tau^\mu_\rho}{dt})\,\tilde{\mathcal{X}}^\rho + \tilde{\mathcal{B}}^j_\rho\,\tilde{\mathcal{U}}^\rho], \tag{49}$$

where ψ_j is the costate covariant vector. The canonical equation is derived as

$$\frac{d\psi_\alpha}{dt} = -\frac{\partial H}{\partial \tilde{\mathcal{X}}^\alpha} = -\tilde{\mathcal{X}}^j g_{j\alpha} - \psi_j[T^j_\mu \bar{\mathcal{A}}^\mu_\lambda \tau^\lambda_\alpha - T^j_\mu \frac{d\tau^\mu_\alpha}{dt}], \tag{50}$$

with the boundary condition

$$\psi_\alpha(T) = 0. \tag{51}$$

Along the optimal trajectory, we must have

$$0 = \frac{\partial H}{\partial \tilde{\mathcal{U}}^\rho} = \tilde{\mathcal{U}}^j g_{j\rho} + \psi_j \tilde{\mathcal{B}}^j_\rho \tag{52}$$

It then follows that

$$\tilde{\mathcal{U}}_\rho = \tilde{\mathcal{U}}^j g_{j\rho} = -\psi_j \tilde{\mathcal{B}}^j_\rho. \tag{53}$$

Suppose that

$$\psi_\alpha = \tilde{X}_i \tilde{S}^i_{\ \alpha} = \tilde{X}_i T^i_{\ \nu} \tilde{S}^\nu_{\ \mu} \tau^\mu_{\ \alpha}, \tag{54}$$

then, by substituting (54) into (50), we have

$$\frac{d\psi_\alpha}{dt} = -\tilde{X}_\alpha - \tilde{X}_i T^i_{\ \nu} \tilde{S}^\nu_{\ \mu} \tilde{A}^\mu_{\ m} \tau^m_{\ \alpha} + \tilde{X}_i T^i_{\ \nu} \tilde{S}^\nu_{\ \mu} \frac{d\tau^\mu_{\ \alpha}}{dt}. \tag{55}$$

On the other hand, by differentiating (54) with t, and using the dual model (44) and the equations (53) and (54), we have

$$\begin{aligned} \frac{d\psi_\alpha}{dt} &= \tilde{X}_j \{ T^j_{\ m} \overline{{}^t\mathcal{A}}^m_{\ n} \tau^n_{\ i} - \frac{dT^j_{\ n}}{dt} \tau^n_{\ i} \\ &- T^j_{\ m} \tilde{S}^m_{\ n} \tau^n_{\ k} \tilde{B}^k_{\ \gamma} \widetilde{{}^t B}^\gamma_{\ i} \} T^i_{\ \nu} \tilde{S}^\nu_{\ \mu} \tau^\mu_{\ \alpha} + \tilde{X}_j \frac{dT^j_{\ \nu}}{dt} \tilde{S}^\nu_{\ \mu} \tau^\mu_{\ \alpha} \\ &+ \tilde{X}_j T^j_{\ \nu} \frac{d\tilde{S}^\nu_{\ \mu}}{dt} \tau^\mu_{\ \alpha} + \tilde{X}_j T^j_{\ \nu} \tilde{S}^\nu_{\ \mu} \frac{d\tau^\mu_{\ \alpha}}{dt}. \end{aligned} \tag{56}$$

Thus we have

$$\begin{aligned} \frac{d\psi_\alpha}{dt} &= \tilde{X}_j T^j_{\ m} \overline{{}^t\mathcal{A}}^m_{\ n} \tilde{S}^n_{\ \mu} \tau^\mu_{\ \alpha} - \tilde{X}_j T^j_{\ m} \tilde{S}^m_{\ n} \tau^n_{\ k} \tilde{B}^k_{\ \gamma} \widetilde{{}^t B}^\gamma_{\ i} T^i_{\ \nu} \tilde{S}^\nu_{\ \mu} \tau^\mu_{\ \alpha} \\ &+ \tilde{X}_j T^j_{\ \nu} \frac{d\tilde{S}^\nu_{\ \mu}}{dt} \tau^\mu_{\ \alpha} + \tilde{X}_j T^j_{\ \nu} \tilde{S}^\nu_{\ \mu} \frac{d\tau^\mu_{\ \alpha}}{dt}. \end{aligned} \tag{57}$$

Comparing (55) with (57), we have

$$\begin{aligned} 0 = \tilde{X}_j T^j_{\ m} [& \delta^m_{\ \mu} + \tilde{S}^m_{\ n} \tilde{A}^n_{\ \mu} + \overline{{}^t\mathcal{A}}^m_{\ n} \tilde{S}^n_{\ \mu} \\ & - \tilde{S}^m_{\ n} \tau^n_{\ k} \tilde{B}^k_{\ \gamma} \widetilde{{}^t B}^\gamma_{\ i} T^i_{\ \nu} \tilde{S}^\nu_{\ \mu} + \frac{d\tilde{S}^m_{\ \mu}}{dt}] \tau^\mu_{\ \alpha}. \end{aligned} \tag{58}$$

Thus we have the pseudo Riccati equation

$$\frac{d\tilde{S}^m_{\ \mu}}{dt} + \tilde{S}^m_{\ n} \bar{A}^n_{\ \mu} + \overline{{}^t\mathcal{A}}^m_{\ n} \tilde{S}^n_{\ \mu} - \tilde{S}^m_{\ n} \bar{B}^n_{\ \gamma} \overline{{}^t B}^\gamma_{\ \nu} \tilde{S}^\nu_{\ \mu} + \delta^m_{\ \mu} = 0. \tag{59}$$

Considering (54), the boundary condition (51) becomes

$$\tilde{S}^m_{\ \mu}(T) = 0. \tag{60}$$

Using (53) and (54), the optimal control law is given as

$$\tilde{U}_\rho = -\tilde{X}_i T^i_{\ \nu} \tilde{S}^\nu_{\ \mu} \tau^\mu_{\ j} \tilde{B}^j_{\ \rho} = -\tilde{X}_i T^i_{\ \nu} \tilde{S}^\nu_{\ \mu} \bar{B}^\mu_{\ n} \tau^n_{\ \rho}. \tag{61}$$

Thus we obtain

$$\begin{aligned} \tilde{U}^\gamma &= \tilde{U}_\rho g^{\gamma\rho} \\ &= -\tilde{X}_i T^i_{\ \nu} \tilde{S}^\nu_{\ \mu} \bar{B}^\mu_{\ n} \tau^n_{\ \rho} g^{\gamma\rho} \\ &= -\tilde{X}_j g^{jk} g_{ki} T^i_{\ \nu} \tilde{S}^\nu_{\ \mu} \bar{B}^\mu_{\ n} \tau^n_{\ \rho} g^{\gamma\rho} \\ &= -\tilde{X}^k g_{ki} T^i_{\ \nu} \tilde{S}^\nu_{\ \mu} \bar{B}^\mu_{\ n} \tau^n_{\ \rho} g^{\gamma\rho} \\ &= -\tilde{X}^k \tau^j_{\ k} \tau^j_{\ i} T^i_{\ \nu} \tilde{S}^\nu_{\ \mu} \bar{B}^\mu_{\ n} \tau^n_{\ \rho} T^\rho_{\ m} T^\gamma_{\ m} \\ &= -\tilde{X}^k \tau^j_{\ k} \delta^i_{\ \nu} \tilde{S}^\nu_{\ \mu} \bar{B}^\mu_{\ n} \delta^n_{\ m} T^\gamma_{\ m} \\ &= -\tilde{X}^k \tau^j_{\ k} \tilde{S}^j_{\ \mu} \bar{B}^\mu_{\ m} T^\gamma_{\ m}. \end{aligned} \tag{62}$$

The solution $\tilde{S}^\mu_{\ \nu}$ of the pseudo Riccati equation, which is the special case of the Riccati equation, has the symmetric property (see Lemma 9-5 in [7]). Therefore, by using (19), we obtain

$$\tilde{U}^\gamma = -T^\gamma_{\ m} \overline{{}^t B}^m_{\ \mu} \tilde{S}^\mu_{\ j} \tau^j_{\ k} \tilde{X}^k. \tag{63}$$

On the other hand, the sufficient condition of a local minimum of J is clear, because using theorem 5-1 and corollary 5-1 in [7], the following relations are always satisfied.

$$\frac{\partial^2 H}{\partial \tilde{U}^\gamma \partial \tilde{U}^\rho} = \delta^j_{\ \gamma} g_{j\rho} = g_{\gamma\rho} > 0 \tag{64}$$

$$\begin{pmatrix} \frac{\partial^2 H}{\partial \tilde{X}^\beta \partial \tilde{X}^\alpha} & \frac{\partial^2 H}{\partial \tilde{X}^\gamma \partial \tilde{U}^\rho} \\ \frac{\partial^2 H}{\partial \tilde{U}^\beta \partial \tilde{X}^\alpha} & \frac{\partial^2 H}{\partial \tilde{U}^\gamma \partial \tilde{U}^\rho} \end{pmatrix} = \begin{pmatrix} g_{\beta\alpha} & 0 \\ 0 & g_{\gamma\rho} \end{pmatrix} > 0 \tag{65}$$

Q.E.D.

5. Gauge field

In this section, the method to deform the coordinate frame of a nonlinear plant and adjust it to that of a fictitious linear plant is discussed. On the curvilinear coordinate frame, the Riemannian geometric model

$$\frac{d\tilde{X}^\gamma}{dt} = (\tilde{A}^\gamma_{\ \mu} - \widetilde{\{{}_\mu{}^\gamma{}_\lambda\}} \frac{d\tilde{x}^\lambda}{dt}) \tilde{X}^\mu + \tilde{B}^\gamma_{\ \mu} \tilde{U}^\mu \tag{66}$$

must be observed as the nonlinear plant

$$\frac{d\tilde{X}}{dt} = \tilde{\alpha}(\tilde{X})\, \tilde{X} + \tilde{B}\, \tilde{U}. \tag{67}$$

Since the nonlinear plant (67) is represented as the tensor equation

$$\frac{d\tilde{X}^\gamma}{dt} = \tilde{\alpha}^\gamma_{\ \mu}(\tilde{X})\, \tilde{X}^\mu + \tilde{B}^\gamma_{\ \mu} \tilde{U}^\mu, \tag{68}$$

we have the following relation by comparing (68) with (66).

$$\tilde{A}^\gamma_{\ \mu} - \widetilde{\{{}_\mu{}^\gamma{}_\lambda\}} \frac{d\tilde{x}^\lambda}{dt} = \tilde{\alpha}^\gamma_{\ \mu}(\tilde{X}) \tag{69}$$

This relation holds good regardless of the control $\tilde{U}^\gamma$.

Theorem 4: Let S_L be the integral manifold of a linear system

$$\frac{d\tilde{X}^\gamma}{dt} = \tilde{A}^\gamma_{\ \mu} \tilde{X}^\mu + \tilde{B}^\gamma_{\ \mu} \tilde{U}^\mu, \tag{70}$$

and S_{NL} be the integral manifold of a nonlinear system

$$\frac{d\tilde{X}^\gamma}{dt} = \tilde{\alpha}^\gamma_{\ \mu}(\tilde{X})\, \tilde{X}^\mu + \tilde{B}^\gamma_{\ \mu} \tilde{U}^\mu. \tag{71}$$

A homeomorphic mapping τ between S_L and S_{NL} satisfies the following partial differntial equation

$$\frac{\partial \tau^\beta_{\ \mu}}{\partial \tilde{X}^\lambda} [\tilde{\alpha}^\lambda_{\ \gamma}(\tilde{X})\, \tilde{X}^\gamma + \tilde{B}^\lambda_{\ \gamma} \tilde{U}^\gamma] = \bar{A}^\beta_{\ \nu} \tau^\nu_{\ \mu} - \tau^\beta_{\ \gamma} \tilde{\alpha}^\gamma_{\ \mu}(\tilde{X}). \tag{72}$$

Proof: Substituting (32) and (33) into the equation (10) of Appendix A2, the Christoffel symbols become

$$\widetilde{\{{}_\mu{}^\gamma{}_\lambda\}} = T^\gamma_{\ k} \frac{\partial \tau^k_{\ \mu}}{\partial \tilde{x}^\lambda}. \tag{73}$$

Using (73) and (39), the equation (69) become

$$T^\gamma_{\ \beta} \bar{A}^\beta_{\ \nu} \tau^\nu_{\ \mu} - T^\gamma_{\ k} \frac{d\tau^k_{\ \mu}}{dt} = \tilde{\alpha}^\gamma_{\ \mu}(\tilde{X}) \tag{74}$$

Since the relation (69) which determines the homeomorphism τ holds good regardless of the control $\tilde{U}^\gamma$, τ becomes the function of $\tilde{X}^\gamma$ only. Therefore we have

$$\frac{d\tau^\beta_{\ \mu}}{dt} = \frac{\partial \tau^\beta_{\ \mu}}{\partial \tilde{X}^\lambda} \frac{d\tilde{X}^\lambda}{dt} = \frac{\partial \tau^\beta_{\ \mu}}{\partial \tilde{X}^\lambda} [\tilde{\alpha}^\lambda_{\ \gamma}(\tilde{X})\, \tilde{X}^\gamma + \tilde{B}^\lambda_{\ \gamma} \tilde{U}^\gamma] \tag{75}$$

Substituting (75) into (74) and multiplying $\tau^\beta_{\ \gamma}$ into (74) on the left side, we have a partial differential equation (72). Q.E.D.

6. Coordinate condition

In this section the necessary and sufficient condition with respect to the existence and the uniqueness of the homeomorphism is discussed. The characteristic equations of the partial differential equation for the homeomorphism are given as

$$\frac{d\tilde{X}^\gamma}{dt} = \tilde{\alpha}^\gamma_\mu(\tilde{X})\,\tilde{X}^\mu + \tilde{B}^\gamma_\mu\,\tilde{U}^\mu \tag{76}$$
$$\frac{d\tau^\beta_\mu}{dt} = \tilde{A}^\beta_\nu\tau^\nu_\mu - \tau^\beta_\gamma\tilde{\alpha}^\gamma_\mu(\tilde{X}). \tag{77}$$

If the control $\tilde{U}^\mu$ and the system function $\tilde{\alpha}^\gamma_\mu(\tilde{X})$ are selected as smooth functions, then the homeomorphism exists uniquely by considering the conditions about the existence and uniqueness of the solution of an ordinary differential equation. When the orbits of a nonlinear system are mapped to those of a fictitious linear system, the center of mapping, which is not always a point, does not move, so that we have

$$\frac{d\tau^\beta_\mu}{dt} = \tilde{A}^\beta_\nu\tau^\nu_\mu - \tau^\beta_\gamma\tilde{\alpha}^\gamma_\mu(\tilde{X}_0) = 0. \tag{78}$$

This relation means that $\tilde{A}$ is similar to $\tilde{\alpha}\,(\tilde{X}_0)$.

Theorem 5: If the following relations are satisfied on the center of mapping

$$\tilde{A} = \tau\,\tilde{\alpha}_0\,\tau^{-1} \neq 0, \quad \det\tau \neq 0, \tag{79}$$

and $\tilde{\alpha}\,(\tilde{X})$, $\frac{\partial\tilde{\alpha}\,(\tilde{X})}{\partial\tilde{X}}$ are continuous, then the partial differential equation for the homeomorphism has the unique solution τ.

Proof: Proof is given as above.

7. Algorithm and numerical example

The optimal control law in Theorem 3 is constructed by using the solution of the partial differential equation for the homeomorphism and the solution of the pseudo Riccati equation. When the homeomorphism is seeked, it is important to specify the region where the integral manifold of a nonlinear model is in contact with that of a fictitious linear model. In this paper the origin O is regarded as the point on which two integral manifolds are contacted. Therefore, considering Theorem 5, the matrices $\tilde{A}$ and $\tilde{B}$ of a fictitious linear system and τ_0 are selected as

$$\tilde{A} = \tilde{\alpha}\,(O), \quad \tilde{B} = \tilde{B}\,(O), \quad \tau_0 = I. \tag{80}$$

By using these conditions, the homeomorphism is calculated directly from the partial differential equation. Namely, the difference equations are solved with the boundary condition $\tau_0 = I$.

Since a real nonlinear plant is represented as the form of (1) instead of (67), the control u must be constructed by the condition

$$x = \tilde{X}, \quad B\,u = \tilde{B}\,\tilde{U}. \tag{81}$$

Using Theorem 3, we have

$$\begin{aligned} u &= B^\dagger\,\tilde{B}\,\tilde{U} = -B^\dagger T\tilde{B}\,\tau\,T\,{}^t\overline{B}\,S\,\tau\,\tilde{X} \\ &= -B^\dagger T\tilde{B}\,{}^t\overline{B}\,S\,\tau\,x, \end{aligned} \tag{82}$$

where $B^\dagger$ is a generallized inverse of the matrix B.

Example Consider the nonlinear system with van der Pol type.

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\varepsilon\{(x_1)^2 - 1\} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$+\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad \varepsilon = 1.0 \tag{83}$$

Simulation results are shown in Fig.1.

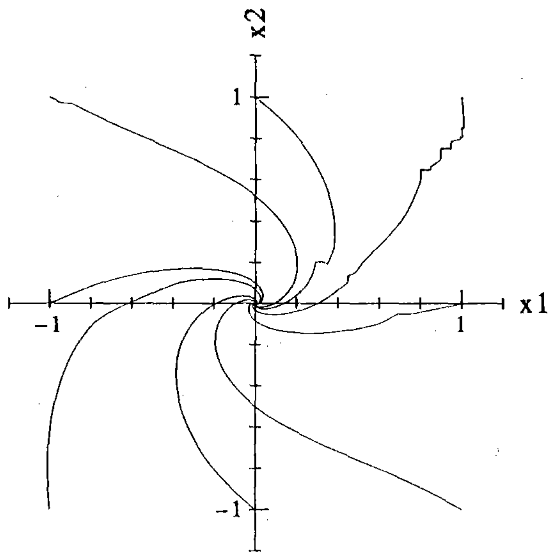

Fig.1: Trajectories of nonlinear regulator

8. Conclusion

By Choosing a suitable Riemannian space, a new lower dimensional geometric model (12) and its dual model (23) have been proposed. For this Riemannian geometric model, a nonlinear optimal regulator with the quadratic form performance index which contains the Riemannian metric tensor g_{ij} has been designed in Theorem 3. The integral manifold of this nonlinear regulator is homeomorphic to that of a fictitious linear regulator. Corresponding to the nonlinear plant, the homeomorphism is determined by a partial differential equation (72). Furthermore, in Theorem 5, the conditions with respect to the existence and the uniqueness of the homeomorphism have been shown. Since the dimension of the Riemannian space becomes lower than that in our previous papers, the computation time of the nonlinear regulator has been improved.

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Appendix

A1. Definition of tensor and its representation by matrix

Let U and U^* be an n-dimensional vector space and its dual space. An (r,s)-tensor F is defined as a multilinear map

$$F: \quad \underbrace{V^* \times \cdots \times V^*}_{r} \times \underbrace{V \times \cdots \times V}_{s} \longmapsto R,$$

where R is a set of numbers in which the addition and the product are naturally defined.

Let U and $\tilde{U}$ be two coordinate neighborhoods on an n-dimensional manifold M with the local coordinate $(x^1, \ldots, x^n)$ and $(\tilde{x}^1, \ldots, \tilde{x}^n)$, respectively. For each x on U, let $T_x(M)$ be a tangent space of M at x, then $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$ becomes a basis of $T_x(M)$. If $T_x(M)$ is selected as V^*, then the dual vector space V has a dual basis $(dx^1, \ldots, dx^n)$. Therefore an (r,s)-tensor F is redefined as

$$F = \sum_{\substack{i_1,\ldots,i_r \\ j_1,\ldots,j_s}} F^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}, \quad (1)$$

where $\otimes$ ia a tensor product, and r and s are a contravariant index and a covariant index, respectively.

Changing the coordinate system, the components of a tensor F on $U \cap \tilde{U}$ are transferred as

$$F^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} \quad (2)$$

$$= \sum_{\substack{\mu_1,\ldots,\mu_r \\ \lambda_1,\ldots,\lambda_s}} \frac{\partial x^{i_1}}{\partial \tilde{x}^{\mu_1}} \cdots \frac{\partial x^{i_r}}{\partial \tilde{x}^{\mu_r}} \frac{\partial \tilde{x}^{\lambda_1}}{\partial x^{j_1}} \cdots \frac{\partial \tilde{x}^{\lambda_s}}{\partial x^{j_s}} \tilde{F}^{\mu_1,\ldots,\mu_r}_{\lambda_1,\ldots,\lambda_s}.$$

Using the Einstein summation convention, the symbol Σ is usually omitted.

Since the tensor is a multilinear map, a tensor can be expressed as a matrix.

Definition: In an n-dimensional space an (r,s)-tensor is expressed as an $n^r \times n^s$ matrix whose

$$[i_r + n(i_{r-1} - 1) + n^2(i_{r-2} - 1) + \cdots + n^{r-1}(i_1 - 1),$$
$$j_s + n(j_{s-1} - 1) + n^2(j_{s-2} - 1) + \cdots + n^{s-1}(j_1 - 1)]$$

element is $F^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}$.

A2. Definitions and properties from Riemannian geometry

Let $\bar{U}$ be a coordinate neighborhood which has the orthogonal straight coordinate system $(\bar{x}^1, \ldots, \bar{x}^n)$ and $\tilde{U}$ be any coordinate neighborhood which has the general coordinate system $(\tilde{x}^1, \ldots, \tilde{x}^n)$. Let O and P be the origin and any point on $\bar{U} \cap \tilde{U}$, respectively. The coordinate of the vector $\overrightarrow{OP}$ becomes $(\bar{x}^1, \ldots, \bar{x}^n)$. The tangent vectors with the direction of the orthogonal straight coordinate axis are represented as

$$\bar{e}_i = \frac{\partial \overrightarrow{OP}}{\partial \bar{x}^i} = (\frac{\partial \bar{x}^1}{\partial \bar{x}^i}, \cdots, \frac{\partial \bar{x}^n}{\partial \bar{x}^i}) = \left(0, .. \overset{i}{,1,} ..,0\right). \quad (3)$$

Furthermore, the tangent vectors with the direction of the curvilinear coordinate axis are represented as

$$\tilde{e}_i = \frac{\partial \overrightarrow{OP}}{\partial \tilde{x}^i} = (\frac{\partial \bar{x}^1}{\partial \tilde{x}^i}, \cdots, \frac{\partial \bar{x}^n}{\partial \tilde{x}^i}). \quad (4)$$

The Riemannian metric tensor g_{kl} and the Christoffel symbols $\widetilde{\{\lambda{}^\nu{}_\mu\}}$ are defined as

$$g_{kl} = (\tilde{e}_k, \tilde{e}_l) = \sum_i \frac{\partial \bar{x}^i}{\partial \tilde{x}^k} \frac{\partial \bar{x}^i}{\partial \tilde{x}^l} \quad (5)$$

$$\widetilde{\{\lambda{}^\nu{}_\mu\}} = \frac{1}{2} g^{\nu k} (\frac{\partial g_{\lambda k}}{\partial \tilde{x}^\mu} + \frac{\partial g_{k\mu}}{\partial \tilde{x}^\lambda} - \frac{\partial g_{\mu\lambda}}{\partial \tilde{x}^k}), \quad (6)$$

where $g^{\lambda k}$ are defined as

$$g^{\lambda k} g_{kl} = \delta^\lambda_l = \frac{\partial \tilde{x}^\lambda}{\partial \tilde{x}^k} \frac{\partial \tilde{x}^k}{\partial \tilde{x}^l}. \quad (7)$$

Using the equation (5), we have

$$g^{\lambda k} = \sum_i \frac{\partial \tilde{x}^\lambda}{\partial \bar{x}^i} \frac{\partial \tilde{x}^k}{\partial \bar{x}^i} \quad (8)$$

$$\frac{\partial g_{kl}}{\partial \tilde{x}^\lambda} = \frac{\partial^2 \bar{x}^i}{\partial \tilde{x}^k \partial \tilde{x}^\lambda} \frac{\partial \bar{x}^i}{\partial \tilde{x}^l} + \frac{\partial \bar{x}^i}{\partial \tilde{x}^k} \frac{\partial^2 \bar{x}^i}{\partial \tilde{x}^l \partial \tilde{x}^\lambda}. \quad (9)$$

Thus the equation (6) is represented as

$$\widetilde{\{\lambda{}^\nu{}_\mu\}} = \frac{\partial \tilde{x}^\nu}{\partial \bar{x}^i} \frac{\partial^2 \bar{x}^i}{\partial \tilde{x}^\lambda \partial \tilde{x}^\mu}. \quad (10)$$

Let $(x^1, \ldots, x^n)$ be a curvilinear coordinate system that is different from $(\bar{x}^1, \ldots, \bar{x}^n)$ and $(\tilde{x}^1, \ldots, \tilde{x}^n)$. Then we have

$$\frac{\partial^2 \bar{x}^i}{\partial \tilde{x}^\lambda \partial \tilde{x}^\mu} = \frac{\partial}{\partial \tilde{x}^\lambda} (\frac{\partial \bar{x}^i}{\partial x^\delta} \frac{\partial x^\delta}{\partial \tilde{x}^\mu})$$
$$= \frac{\partial^2 \bar{x}^i}{\partial x^\epsilon \partial x^\delta} \frac{\partial x^\epsilon}{\partial \tilde{x}^\lambda} \frac{\partial x^\delta}{\partial \tilde{x}^\mu} + \frac{\partial \bar{x}^i}{\partial x^\delta} \frac{\partial^2 x^\delta}{\partial \tilde{x}^\lambda \partial \tilde{x}^\mu} \quad (11)$$

$$\frac{\partial^2 x^\nu}{\partial \tilde{x}^\lambda \partial \tilde{x}^\mu} = \frac{\partial}{\partial \tilde{x}^\lambda} (\frac{\partial x^\nu}{\partial \bar{x}^j} \frac{\partial \bar{x}^j}{\partial \tilde{x}^\mu})$$
$$= \frac{\partial^2 x^\nu}{\partial \bar{x}^i \partial \bar{x}^j} \frac{\partial \bar{x}^i}{\partial \tilde{x}^\lambda} \frac{\partial \bar{x}^j}{\partial \tilde{x}^\mu} + \frac{\partial x^\nu}{\partial \bar{x}^j} \frac{\partial^2 \bar{x}^j}{\partial \tilde{x}^\lambda \partial \tilde{x}^\mu}. \quad (12)$$

Using (10) and (11), we have

$$\widetilde{\{\lambda{}^\nu{}_\mu\}} = \frac{\partial \tilde{x}^\nu}{\partial x^k} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial^2 \bar{x}^i}{\partial x^\epsilon \partial x^\delta} \frac{\partial x^\epsilon}{\partial \tilde{x}^\lambda} \frac{\partial x^\delta}{\partial \tilde{x}^\mu} + \frac{\partial \tilde{x}^\nu}{\partial \bar{x}^i} \frac{\partial \bar{x}^i}{\partial x^\delta} \frac{\partial^2 x^\delta}{\partial \tilde{x}^\lambda \partial \tilde{x}^\mu}$$
$$= \frac{\partial \tilde{x}^\nu}{\partial x^k} \frac{\partial x^\epsilon}{\partial \tilde{x}^\lambda} \frac{\partial x^\delta}{\partial \tilde{x}^\mu} \{\delta{}^k{}_\epsilon\} + \frac{\partial \tilde{x}^\nu}{\partial x^\delta} \frac{\partial^2 x^\delta}{\partial \tilde{x}^\lambda \partial \tilde{x}^\mu}. \quad (13)$$

Multiplying $\frac{\partial x^k}{\partial \tilde{x}^\nu}$ into (13), we have

$$\frac{\partial x^k}{\partial \tilde{x}^\nu} \widetilde{\{\lambda{}^\nu{}_\mu\}} = \frac{\partial x^\epsilon}{\partial \tilde{x}^\lambda} \frac{\partial x^\delta}{\partial \tilde{x}^\mu} \{\delta{}^k{}_\epsilon\} + \frac{\partial^2 x^k}{\partial \tilde{x}^\lambda \partial \tilde{x}^\mu}. \quad (14)$$