Chaos in Nonlinear Control Systems

Joon-Suh Lee and Kun Soo Chang†

Automation Research Center and Department of Chemical Engineering
Pohang University of Science and Technology

Complicated dynamical behavior can occur in model reference adaptive control systems when two external sinusoidal signals are introduced although the plant and reference model are stable linear first order systems. The phase portrait plot and the power spectral analysis indicate chaotic behavior. In the system treated, a positive Lyapunov exponent and non-integer dimension clearly manifest chaotic nature of the system.

INTRODUCTION

In process industries, many chemical process systems possess complex nonlinear features, such as multiple steady states, simple bifurcation, Hopf bifurcation, torus bifurcation, period doubling bifurcation and nonlinear oscillation. Even complicated features such as chaos can occur.

In adaptive control systems, it has been shown that even though the controlled plant is completely linear, the overall system becomes nonlinear if adaptive controller is implemented. This can be caused by adaptive control law algorithms such as gradient algorithms or least-square adaptive algorithms. Therefore it is possible to result in nonlinear phenomena in the systems. This fact has been shown by a few previous workers: Rubio et al.(1985), for example, showed that a chaotic motion can occur when a simple adaptive controller is implemented to a nonlinear system. Mareels and Bitmead(1986, 1988) showed that under the presence of undermodeling error, nonlinear phenomena in the feedback gain such as limit cycle and even chaos arise in an discrete time adaptive control system. A similar study shown by Mossayebi and Hartely(1992) was that in discrete time indirect adaptive control systems, depending on the identification algorithms, chaos may appear. Salam and Bai showed continuous model reference adaptive control system with σ-modification scheme can give rise to a complicated behavior of motion when constant-plus-sinusoidal reference input or disturbance is introduced. In the present study we investigate chaotic behavior of a direct model reference adaptive control(MRAC) system where dual external periodic inputs are introduced.

† To whom all correspondences should be addressed.

MODEL REFERENCE ADAPTIVE CONTROL

We consider a linear plant described by the following ordinary differential equation,

$$\dot{y}_p(t) = a_p y_p(t) + u(t) + v(t) \tag{1}$$

where $y_p(t)$, a_p and v are plant output, unknown constant and a bounded disturbance, respectively. A reference model is given by

$$\dot{y}_m = -a_m y_m(t) + r(t)\ , \quad a_m > 0 \tag{2}$$

where $y_m(t)$ and *r(t)* is reference input. If we define control *u(t)* as

$$u(t) = \theta(t)y_p(t) + r(t) \quad (3)$$

where $\theta(t)$ is control parameter, and $v(t)$, $r(t)$ are bounded disturbance and reference input, respectively. When output error and parameter error are defined as

$$e_1 = y_p - y_m,\ \phi = \theta - \theta^* \quad (4)$$

and there exist no disturbance, the system can be reduced to

$$\dot{e}_1(t) = -a_m e_1(t) + \phi(t)y_p(t) \quad (5)$$

$$\dot{\phi}(t) = -g \cdot e_1(t)y_p(t) \quad (6)$$

where g is a positive adaptive gain that accelerates its convergence. For this ideal case, based on the Lyapunov second method, uniform stability is guaranteed since a Lyapunov function such as

$$V = (e_1, \phi) = (e_1^2 + \frac{1}{g}\phi^2)/2 \quad (7)$$

is obtained(Narendra and Annaswamy, 1989; Sastry and Bodson, 1989). Hence the output error converges to zero, i.e., $\lim_{t \to \infty} e_1(t) = 0$. Therefore any other attractors ,e.g., limit cycles or chaos does not exist. However, this stability proof does not hold when certain types of disturbance or reference input exist.

If $y_p(t)$ is expressed in terms of $y_m(t)$ and $e_1(t)$, the system is given by

$$\dot{e} = -a_m e_1 + \phi y_m + \phi e_1 + v \quad (8)$$

and a modified parameter updating rule(Narendra and Annaswamy, 1989) of the form,

$$\dot{\phi} = -e_1 y_m - e_1^2 \quad (9)$$

In this study, we consider two external inputs which are reference input and disturbance,

$$y_m(t) = p_1 \cos(w_1 t) \quad (10)$$

$$v(t) = p_2 \cos(w_2 t) \quad (11)$$

where p_1, p_2, w_1 and w_2 are some constant parameters. As shown in eq.(8) and (9), MRAC includes nonlinearity due to parameter adjusting algorithm (9). Hence, various nonlinear phenomena might happen in this class of systems if stability is not guaranteed. Next, several methods to distinguish deterministic chaos from uncertain random noise will be briefly described relevant to the present study.

CHARACTERIZATION OF CHAOS

Simply stated, chaos is a *bounded random behavior of motion* from a deterministic system. The attractor of a chaotic system is called *strange attractor.* To distinguish deterministic chaos from external random noise, several observation techniques and computation methods are frequently used. The easiest way to characterize a chaotic dynamics is observing its time series. But this method does not guarantee the presence of chaos, but if the time series shows random behavior, one might suspect that this system possesses chaotic nature. However this is not sufficient to show that it is chaos. If we plot 2D phase portrait of given chaotic time series, the flow of chaotic system shows tangled feature in a certain bounded area of a state space. Third method is to consider a hyperplane so that the flow of the system intersects the plane and to observe its piercing points. Here the hyperplane is often called *Poincar section.* If the system is chaos, intersection points on the plane are localized in a thin band. *Power spectrum* of the chaotic time series shows *broad band type continuous spectrum* with multiple peaks sitting on the of the broad band.

To qualify and quantify its behavior of motion more concretely, it is necessary to compute *Lyapunov exponent* or *dimension* of the attractor. We consider an n-dimensional state space where a trajectory of a certain dynamical system takes place. If we observe the long term evolution of an infinitesimal n-sphere whose center is a initial condition of the trajectory. After a certain period of time, the sphere will become an n-ellipsoid due to locally deforming nature of the flow. The ith one-dimensional Lyapunov exponent is then defined as

$$\lambda_i \equiv \lim_{t\to\infty}\frac{1}{t}\log_2\frac{p_i(t)}{p_i(0)} \tag{12}$$

where $p_i(t)$ is the length of principal axis at time t. This Lyapunov exponent is quantitative value of expansion or contraction of the given flow along linearly independent directions. The sum of the Lyapunov exponents is the time-averaged divergence of the state space velocity. Positive exponent implies divergence along a direction, while negative implies contraction of the nearby trajectory. Therefore any dissipative system has at least one negative exponent and sum of all Lyapunov exponents is negative unless it is unstable. In three dimensional continuous system, the limit set, attractor has only three possibilities of Lyapunov exponents' combination. Those are (-, -, -) which is asymptotically stable equilibrium point; (0,-,-), a limit cycle; (0,0,-), a two-frequency torus and (+,0,-), a strange attractor. Systems with one or more Lyapunov exponent is chaos. When a flow takes place in a 3-dimensional state space, the strange attractor or chaos will have one possible set of Lyapunov exponents, e.g., where its signs of the values are (+,0,-). In continuous 4-dimensional system, three possible types can exist which are (+,+,0,-),(+,0,0,-), and (+,0,-,-) (See Parker and Chua, 1987; Wolf et al., 1985). The first combination that has two positive Lyapunov exponents is defined as *hyper-chaos* that has two divergent directions.

Another characterization method of chaos is calculating the attractor's dimension. While non-chaotic attractors such as equilibrium point, limit cycle, or n-frequency torus, have dimension of 0, 1 and n, respectively, which are integer dimensions, chaotic attractor has non-integer dimension. One of the easiest definition of dimension is *fractal dimension* defined by

$$D_{fract} \equiv \lim_{\varepsilon\to 0}\frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} \tag{13}$$

where $N(\varepsilon)$ is the number of volume elements to cover an attractor with diameter ε volume elements. If the objective attractor is a manifold such as line, point, or surface, evidently D_{fract} is integer. However, a strange attractor has non-integer dimension. A definition of dimension, *information dimension* based on Lyapunov exponent is defined by

$$D_i = j + \frac{\sum_{i=1}^{j}\lambda_i}{\left|\lambda_{j+1}\right|} \tag{14}$$

where λj are ordered from largest to smallest, or

$$\sum_{i=1}^{j+1}\lambda > 0 \text{ and } \sum_{i=1}^{j+1}\lambda < 0 \tag{15}$$

Typical chaotic systems, e.g., Lorenz system has Lyapunov exponents of (2.16, 0.00, -32.4) and information dimension of 2.07. R ssler's 3 dimensional system has Lyapunov exponents of (0.13, 0.00, -14.1) and information dimension of 2.01. Other definitions of fractal dimension-like quantities are defined by numbers of references(See Parker and Chua,1987, 1989).

CHAOS FROM MRAC

We investigate the adaptive control system of eqs.(8) to (11) when constant parameters p_1, p_2, w_1 and w_2 are 20, 5, 1.2 and 13, respectively. We integrate from an initial condition (0,0) as time varies. Time series of eqs.(8) to (11), shows random behavior of motion(See Fig.1). Integration was done with relative error tolerance of 10^{-6} using LSODE subroutine in Trigem SDT-700.

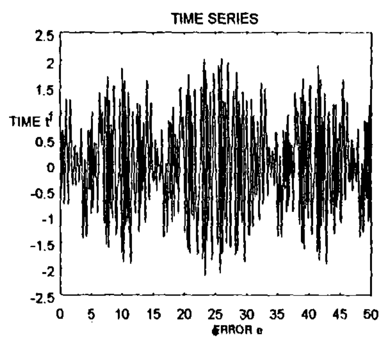

Fig.1 Time series of eqs.(8) - (11).

This result is not sufficient to prove that it is chaos, since a quasiperiodic flow on an n-frequency torus might show similar dynamical time series. This results initiated our supplementary investigation of the system. After post-transient response, the flow eventually falls on the strange attractor. The limit set does not show any equilibrium, limit cycle or n-periodic limit set. Two and three dimensional phase portrait is given by Figs.2 and 3 respectively. Power spectrum is computed with 10000 point of time sequences from integration of the system with MATLAB ver. 4.0(See Fig.4) on a 486DX2 PC. Power spectrum and 95% confidence interval were obtained. The spectrum shows continuous broad band noise with numbers of peaks. This results shows that the system is not a quasiperiodic dynamics from n-frequency torus which is discontinuous in power spectrum.

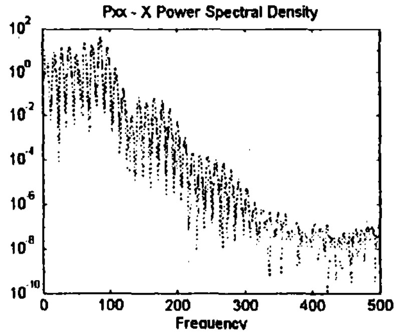

Fig.4 Power spectrum of the system.

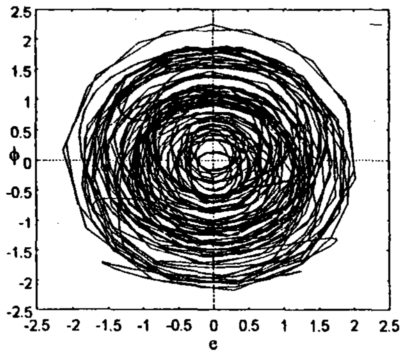

Fig.2 2-D phase portrait of the system.

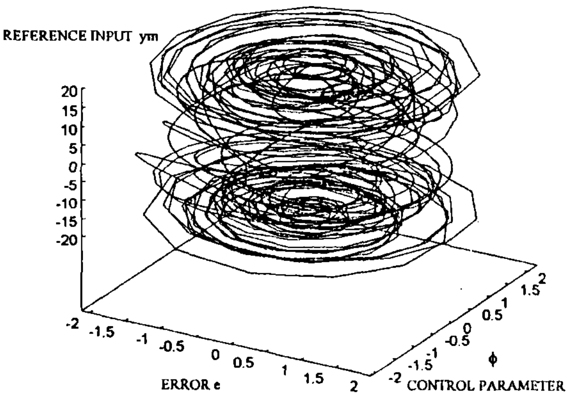

Fig.3 3-D phase portrait of the system.

The direct experimental results of time series, phase portrait, and power spectrum of Figs.1 to 4 are sufficient to prove that this system is chaos. But several questions still remain: how fast does a nearby orbit converge and what is the dimension of the attractor? Answers to these two quantitative questions can be obtained by calculation of Lyapunov exponents and fractal-like dimensions. A method of Lyapunov exponent is given by Wolf et al. (1985), which utilizes *fiducial trajectory* and orthonormal vectors of given system. The Lyapunov exponents obtained are [9.65126, -12.3986, 0, 0]. Two zero exponents are obviously from two periodic eqs.(10) and (11). One of the Lyapunov exponent, 9.65126 shows the rate of divergence while -12.3986 is rate of convergence. Summation of all Lyapunov exponents is -2.74734, which implies this system has an attracting limit set. From calculated exponents, information dimension is directly obtained by eq.(14), which is 3.7784. This value shows that the attractor is not a manifold such as n-torus, but a cantor-dust-like (See Parker and Chua, 1987 for details) fractal set.

CONCLUSIONS

In this study, a model reference adaptive control system was investigated when two non-autonomous factors and a modified control parameter adjustment algorithm are introduced. The plant and reference model were typical linear first order continuous systems. The system is asymptotically stable in the sense of Lyapunov if there is no non-autonomous element with straightforward gradient algorithm. However, if some periodic external inputs such as sinusoidal signals, it affects the system and generates chaos that is a bounded random motion with a certain degree of regularity.

The system was rigorously investigated through phase portrait, power spectrum analysis, and quantitative characterization such as Lyapunov exponents and information dimension that manifest the existence of strange attractor, chaos.

REFERENCE

[1] Golden M. P. and B. E. Ydstie, " Small Amplitude Chaos and Ergodicity in Adaptive Control," *Automatica*, **28** 1, pp. 11 - 25, 1992.

[2] Huberman and E. Lumer, " Dynamics of Adaptive Systems, " *IEEE Trans.* CAS -**37** 4, pp. 547 - 550, 1990.

[3] Lee. J. S. and K.S. Chang, "The Roles of Fractals and Chaos in Process Systems," *J. of Process Control*, accepted, 1994.

[4] Mareels M. Y. and R. R. Bitmead, " Nonlinear Dynamics in Adaptive Control: Chaotic and Periodic Stabilization," *Automatica*, **22** 6, pp 641 - 655, 1986.

[5] Mareels M. Y. and R. R. Bitmead, " Nonlinear Dynamics in Adaptive Control: Chaotic and Periodic Stabilization - II Analysis," *Automatica*, **24** 4, pp. 485 - 497, 1988.

[6] Mossayebi, F. and T. Hartley, " Chaos in an Adaptive Controller," *J. of Franklin Institute*, **329** 1, pp. 135 - 143, 1992.

[7] Narendra,K.S. and A. M. Annaswamy, *Stable Adaptive Systems,* Prentice Hall, Englewood Cliff, NJ., 1989

[8] Parker, T and L. O. Chua, "Chaos: a Tutorial for Engineers," *proc. IEEE 75* 8, pp. 982-1008, August 1987.

[9] Qammar, H. K., F. Mossayebi and L. Murphy, " Dynamical Complexity Arising in the Adaptive Control of chaotic Systems," *Phy. Lett. A*, **178**, pp. 279 - 283, 1993.

[10] Rubio, F. R., J. Aracil and E. F. Camacho," Chaotic Motion in an Adaptive Control System," *Int. J. of Control*, **42** 2, pp. 353 - 360, 1985.

[11] Salam, A. and S. Bai, " Complicated dynamics of Prototype Continuous - Time Adaptive Control System," IEEE Trans. CAS - 35 7, pp. 842, 1988.

[12] Sastry. S and M. Bodson, *Adaptive Control, Stability, Convergence, and Robustness*, Prentice Hall, Englewood Cliff, NJ., 1989.

[13] Sinha, R. Ramaswamy, and J. Rao, "Adaptive Control in Nonlinear Dynamics," *Physica D*, **43**, pp. 118 - 128, 1990.

[14] Wolf, A., J. B. Swift, H. L. Swinney and J. A. Vastano, "Determining Lyapunov Exponents from a Time Series," *Pysica 16D*, pp.285-317,1986.

[15] Ydstie, E. and M. P. Golden, " Chaos and Strange Attractors in Adaptive Control Systems, " *proceedings of IFAC 10th Trinnial World Congress*, Munich, FRG, pp 133 - 138, 1987,