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정규모우드를 활용한 비선형 대칭구조물의 강제진동해석

박 철 희*, °최 성 철**

On the Forced Vibrations in the Nonlinear Symmetric Structure by Using the Normal Modes

(Chol-Hui Pak , Sung-Chul Choi)

1. Summary

The forced vibration with the symmetric boundary condition in nonlinear structure is studied by utilizing the characteristic of the free vibration which have two modes with the similar natural frequency. Two linear modes exist to have no concern with the amplitude. It is found that the normal mode or elliptic orbit as the newly coupled modes is generated in accordance with changing the stability. It is also known that responses for forced vibration having the small external force and damping are near mode of free vibration and the stability for each response is determined according to the stability in mode of free vibration. Finally the stability and bifurcation are analyzed in proportion to increment of external force and damping.

2. Calculation of Normal Mode

A normal mode is also a periodic motion of the system which passes through the origin and which has two rest points.And the formal definition of normal mode was firstly introduced by Rosenberg. The concept of normal modes has a very sinificant meaning in that the resonance in forced vibrations occurs when the forcing frequency lies near to the natural frequencies and the system vibrates in normal modes in the neighborhood of resonance[5].

In this section the behaviors of normal modes are investigated by using the harmonic balance method. This methos is applicable to nonlinear systems and only first term approximation of harmonics in the Fourier series expansion can make a good results. Thus nonlinear normal modes are to be approximated by considering just first order harmonics. Now consider the following equations of motion:

$$\begin{aligned} \ddot{x} + P_1^2 x + \alpha x^3 + \beta x y^2 &= 0 \\ \ddot{y} + P_2^2 y + \gamma y^3 + \beta y x^2 &= 0 \end{aligned} \quad (1)$$

where the kinetic and potential energies are given by :

$$\begin{aligned} T &= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) \\ V &= \frac{1}{2}(P_1^2 x^2 + P_2^2 y^2) + \frac{1}{4}\alpha x^4 + \frac{1}{2}\beta x^2 y^2 + \frac{1}{4}\gamma y^4 \end{aligned} \quad (2)$$

Assuming that the solution is the first term in Fourier series,

$$x = A \sin\omega t, \qquad y = B \sin\omega t \quad (3)$$

and substituting equation (3) into (1), neglecting the third-harmonic components, eliminating ω^2-term and ignoring higher harmonic,

$$AB(P_1^2 - P_2^2) + \frac{3}{4}AB\,[\,(\alpha-\beta)A^2 + (\beta-\gamma)B^2] = 0 \quad (4)$$

Transforming equation (4) into polar coordinates,

$$A = R\cos\theta \ , \quad B = R\sin\theta$$

where θ is the angle between modal line and x-axis and R is the amplitude of normal mode. After representing equation (4) in polar coord., dividing by $\cos^4\theta$ and supposing p = $\tan\theta$, where p is mode shape.

$$p[\,(P_1^2-P_2^2)(1+p^2) + \frac{3}{4}\{(\alpha-\beta) + (\beta-\gamma)p^2\}R^2] = 0 \quad (5)$$

When R = 0 in eq.(5), p = 0, θ = 0 (x-mode) and θ = π/2 (y-mode) are obtained. These modes correspond to the normal modes in the linearized system. Since the amplitude R increases (the total energy h increases), θ change from 0 to π/2. So mode shape p has the following form.

$$p = 0 \quad \text{and} \quad p^2 = \frac{(P_2^2-P_1^2) + \frac{3}{4}(\beta-\alpha)R^2}{(P_1^2-P_2^2) + \frac{3}{4}(\beta-\gamma)R^2}$$

According to the increment of energy, there are bifurcations and 4 cases in this problem.

Case 1 : $\beta > \alpha$ and $\beta > \gamma$: $R_1 = \sqrt{\dfrac{P_2^2-P_1^2}{\frac{3}{4}(\beta-\gamma)}}$

Bifurcation exists continuously if R is larger than R_1 .

Case 2 : $\beta < \alpha$ and $\beta < \gamma$: $R_2 = \sqrt{\dfrac{P_2^2-P_1^2}{\frac{3}{4}(\alpha-\beta)}}$

Bifurcation exists continuously if R is larger than R_2 .

Case 3 : $\beta < \alpha$ and $\beta > \gamma$

In this case bifurcation exists in some range of R but if R exceeds that range, bifurcation disappears.

Case 4 : $\beta > \alpha$ and $\beta < \gamma$

Bifurcation doesn't occur in any range of R .

3. Justification of Existence of Elliptic Orbit by Perturbation Analysis

3-1.Perturbation Analysis

* 인하대학교 기계공학과
** 인하대학교 기계공학과 대학원

To use the perturbation method, let's consider the systems with the potential energy V :

$$V = \frac{1}{2}(x^2+P^2y^2) + \varepsilon(\frac{1}{4}\alpha x^4+\frac{1}{2}\beta x^2y^2+\frac{1}{4}\gamma y^4)$$

where P^2 is the linearized frequency ratio between P_1 and P_2. And we suppose $P^2 = 1 + \varepsilon\Delta$ to utilize the perturbation method. Equations of motions in this system is

$$\begin{aligned} \ddot{x} + x + \varepsilon(\beta xy^2 + \alpha x^3) &= 0 \\ \ddot{y} + (1+\varepsilon\Delta)y + \varepsilon(\beta yx^2 + \gamma y^3) &= 0 \end{aligned} \tag{6}$$

By using the two variable expansion perturbation method [2], we obtain derivatives on $R_1, R_2, \theta_1, \theta_2$ which naturally written in terms of the variable ϕ : $\phi = \theta_2-\theta_1$,

$$\begin{aligned} \frac{dR_1}{d\eta} &= \frac{1}{8}\beta R_1R_2^2\sin 2\phi \\ \frac{dR_2}{d\eta} &= -\frac{1}{8}\beta R_2R_1^2\sin 2\phi \\ \frac{d\theta_1}{d\eta} &= -\frac{1}{8}\beta R_2^2[\cos 2\phi+2] - \frac{3}{8}\alpha R_1^2 \\ \frac{d\theta_2}{d\eta} &= -\frac{\Delta}{2}-\frac{1}{8}\beta R_1^2[\cos 2\phi+2] - \frac{3}{8}\gamma R_2^2 \end{aligned} \tag{7}$$

where R_i is amplitude and θ_i is the phase difference.
Finally we transform to polars R, ψ in the R_1-R_2 plane,

$$R_1 = R\cos\psi , \quad R_2 = R\sin\psi$$

which replaces Eq.(7) by the following :

$$\begin{aligned} \frac{dR}{d\eta} &= 0 \\ \frac{d\psi}{d\eta} &= -\frac{1}{16}\beta R^2\sin 2\psi\cdot\sin 2\phi \\ \frac{d\phi}{d\eta} &= -\frac{\Delta}{2}-\frac{1}{8}\beta R^2\cos 2\psi(\cos 2\phi+2)+\frac{3}{8}\alpha R^2\cos^2\psi-\frac{3}{8}\gamma R^2\sin^2\psi \end{aligned}$$

The above equation can be integrated exactly and the first integral of the motion is what as follows :

$$\begin{aligned} K(\psi, \phi) &= -\frac{\Delta}{R^2}\cos 2\psi + \frac{3}{32}(\alpha+\gamma)\cos 4\psi + \frac{3}{8}(\alpha-\gamma)\cos 2\psi \\ &\quad + \frac{1}{8}\beta\cos^2\phi - \frac{\beta}{16}(2+\cos 2\phi)\cos 4\psi \\ &= constant \end{aligned} \tag{8}$$

Because the orbit of a periodic motion is an ellipse on the configuration space, we defined such a periodic motion as an elliptic orbit(EO).(R. Rand et. al., 1992) EO's correspond to singularities in the $\phi-\psi$ plane for which $\phi \neq 0, \pi$, cf. Fig.2. In addition to revealing the existence of periodic motions, the first integral (8) also disclose their stability. The stability of the periodic motion in the original system (6) is the same as that of the singular point in the slow flow eq.(8). Since the system is conservative, only centers and saddles can generically occur, and so their stability is easily discerned. A elliptic orbits is stable and two normal

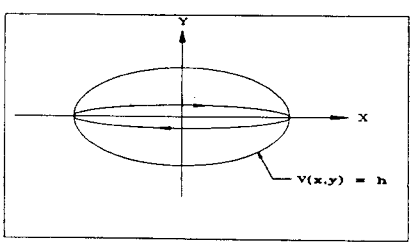


Fig.1 Elliptic orbit in configuration space

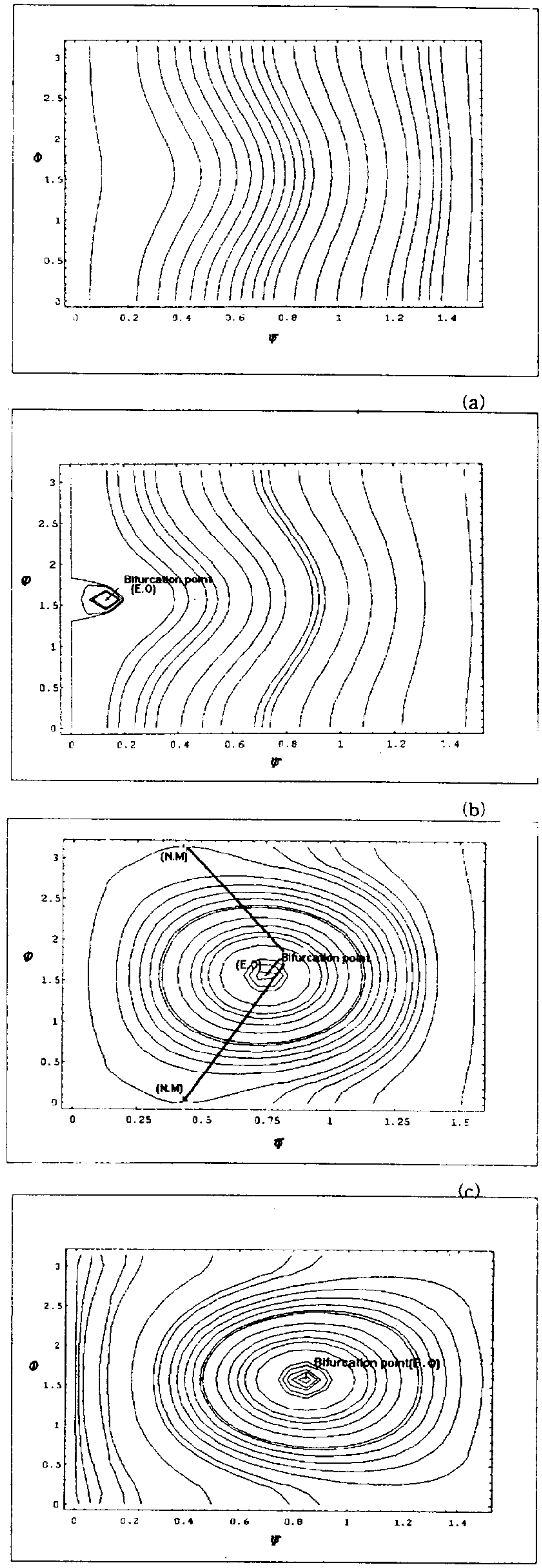


Fig.2 Flow of the vector field on the two-Torus (d)

modes are unstable.

Figure 2 is the picture of flow changing as energy is increased, where the energy is selected when the bifurcation is about to be generated. This result corresponds to the results of the calculation of bifurcation energy for normal mode and elliptic orbit. To illustrate the stability and bifurcation for elliptic orbit, consider a system which has the following parameters[1].

$$\{\Delta, \alpha, \beta, \gamma\} = \{1.25, 0.195762, 0.164874, 0.160992\}$$

Fig.(a) has no bifurcation at R = 2. Fig.(b) show a bifurcation of elliptic orbit at $\phi = \pi/2$ and R = 3.5. Fig.(c) has three bifurcation points which is two bif.pts for normal mode at ϕ = 0, π and R = 8 and one for elliptic orbit at $\phi = \pi/2$ and R = 8. Fig.(d) has only bifurcation point for elliptic orbit at $\phi = \pi/2$ and R =21.

3-2. Calculation for Elliptic Orbit

Assuming that the solution is the first term in Fourier series,

$$x = A\sin\omega t$$
$$y = B\cos\omega t$$

and if we claculate the elliptic orbit with same procedure in normal mode, one can obtain the following equation and mode shape p has the following form.

$$\dot{p} = 0 \quad \text{and} \quad p^2 = \frac{(P_2^2 - P_1^2) + \frac{1}{4}(\beta - 3\alpha)R^2}{(P_1^2 - P_2^2) + \frac{1}{4}(\beta - 3\gamma)R^2}$$

According to the increment of energy, bifurcations occur and 4 cases in this problem.

Case 1 : $\beta > 3\alpha$ and $\beta > 3\gamma$: $R_1 = \sqrt{\dfrac{P_2^2 - P_1^2}{\frac{1}{4}(\beta - 3\gamma)}}$

Bifurcation exists continuously if R is larger than R_1 .

Case 2 : $\beta < 3\alpha$ and $\beta < 3\gamma$: $R_2 = \sqrt{\dfrac{P_2^2 - P_1^2}{\frac{1}{4}(3\alpha - \beta)}}$

Bifurcation exists continuously if R is larger than R_2 .

Case 3 : $\beta < 3\alpha$ and $\beta > 3\gamma$

In this case bifurcation exists in some range of R but if R exceeds that range, bifurcation disappear.

Case 4 : $\beta > 3\alpha$ and $\beta < 3\gamma$

Bifurcation isn't occur in any range of R .

Synge's concept of the stability of periodic motion is treted. Synge has introduced a somwhat different concept about stability of periodic motion, called stability in the kinematico-statical sense. This concept is equivalent to the orbital stability and The concept of Synge's stability in the kinematico-statical sense seems to be stronger than that of Liapunov stability because it is based on a concept reminicent of orbital stability defined in the configuration space. Moreover Synge's procedure for determining the stability is likely to have some advantages in comparision to others. For x-mode,thus,we have

$$\therefore \quad \ddot{\beta} + (\delta + \varepsilon\cos 2\tau)\beta = 0 \quad \cdots\cdots (9)$$

where

$$\delta = \frac{P_2^2}{\omega^2} + \frac{\beta A^2}{2\omega^2}, \quad \varepsilon = \frac{\beta A^2}{2\omega^2}, \quad \omega^2 = P_1^2 + \frac{3}{4}\alpha A^2$$

The equation (9) is standard mathieu equation whose Strutt chart. is well-known. $\vec{A}$ and $\vec{B}$ are the direction which the bifurcated modes proceed to as energy increases. Point C is a bifurcating point for elliptic orbit and point D is a bifurcating point for normal modes.

4. Stability in the Lyapunov Sense

To study the stabilities of normal modes and elliptic orbit in the sense of Lyapunov, the slowly changing phase and amplitude(SCPA) method will be utilized. We find the autonomous system by averaging method,

$$\dot{a} = -\frac{\omega^2 - P_1^2}{2\omega}c + \frac{3\alpha}{8\omega}(a^2 + c^2)c + \frac{\beta}{8\omega}(b^2c + 3cd^2 + 2abd)$$
$$\dot{b} = -\frac{\omega^2 - P_2^2}{2\omega}d + \frac{3\gamma}{8\omega}(b^2 + d^2)d + \frac{\beta}{8\omega}(a^2d + 3dc^2 + 2abc)$$
$$\dot{c} = \frac{\omega^2 - P_1^2}{2\omega}a - \frac{3\alpha}{8\omega}(a^2 + c^2)a - \frac{\beta}{8\omega}(d^2a + 3ab^2 + 2bcd)$$
$$\dot{d} = \frac{\omega^2 - P_2^2}{2\omega}a - \frac{3\gamma}{8\omega}(b^2 + d^2)b - \frac{\beta}{8\omega}(c^2b + 3ba^2 + 2acd)$$

For the stationary solution, let $\{\dot{a}, \dot{b}, \dot{c}, \dot{d}\} = 0$ and a = A, b = B, c = C, d = D in above equation.Finding backbone

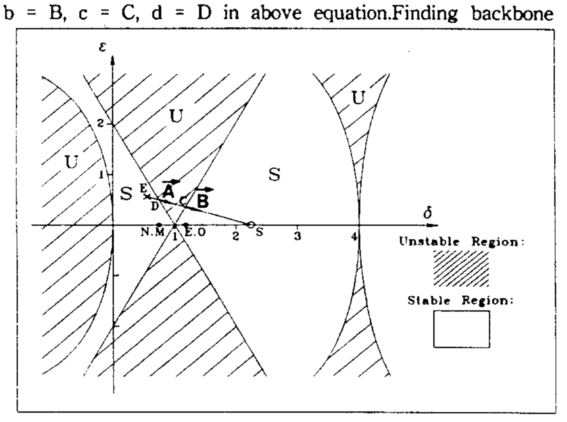


Fig. 3 Change of Stability for x-mode in Strutt-Chart

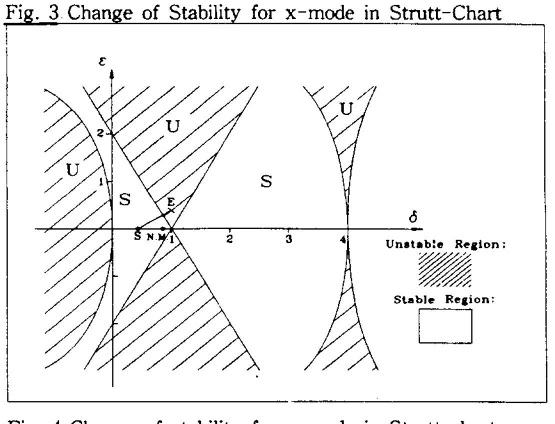


Fig. 4 Change of stability for y-mode in Strutt chart

curve for normal mode, let c = d = 0. We can get the above equations

$$\omega^2 = P_1^2 + \frac{3}{4}\alpha A^2 + \frac{3}{4}\beta B^2$$
$$\omega^2 = P_2^2 + \frac{3}{4}\gamma B^2 + \frac{3}{4}\beta A^2$$

Finding the backbone curve for x-mode, B = 0 and otherwise for y-mode, A = 0. Determining the stability for normal mode , let's suppose that a = A + η_1, b = B + η_2, c = η_3, d = η_4 . Substituting the supposition to the equations, one can obtain the linearized variational equation.

$$\{\dot\eta\} = [\ G\ ]\ \{\eta\}$$

where

$$\{\eta\}=\{\eta_1,\ \eta_2,\ \eta_3,\ \eta_4\}^T,\ [\ G] = [\ g_{ij}]\ ,\ (i,j=1,2,3,4)$$

The Jacobian matrix is composed of the following elements.

$$g_{13_0} = -\frac{\omega^2-P_1^2}{2\omega} + \frac{3\alpha A^2+\beta B^2}{8\omega}$$
$$g_{14_0} = g_{23_0} = \frac{\beta AB}{4\omega}$$
$$g_{24_0} = -\frac{\omega^2-P_2^2}{2\omega} + \frac{3\gamma A^2+\beta B^2}{8\omega}$$
$$g_{31_0} = \frac{\omega^2-P_1^2}{2\omega} - \frac{9\alpha A^2+3\beta B^2}{8\omega}$$
$$g_{32_0} = g_{41_0} = -\frac{\beta AB}{4\omega}$$
$$g_{42_0} = \frac{\omega^2-P_2^2}{2\omega} - \frac{9\gamma B^2+3\beta A^2}{8\omega}$$

In this case the stability for normal modes is determined by types and positions of eigenvalues of Jacobian matrix [G]. And eigenvalues can be obtained in the following characteristic equation.

$$\lambda^4 - K_0\lambda^2 + \Delta_{1_0}\cdot\Delta_{2_0} = 0$$

where,

$$K_0 = g_{13_0}g_{31_0} + g_{14_0}g_{41_0} + g_{23_0}g_{32_0} + g_{24_0}g_{42_0}$$
$$\Delta_{1_0} = g_{13_0}g_{24_0} - g_{14_0}g_{23_0}$$
$$\Delta_{2_0} = g_{31_0}g_{42_0} - g_{32_0}g_{41_0}$$

In account of satisfying the equations of backbone curve, $\Delta_{1_0}=0$.

Then characteristic equation becomes : $\lambda^4 - K_0\lambda^2 = 0$

Because B = 0 for x-mode, eigenvalues are

$$\lambda_{1,2} = 0,$$
$$\lambda_{3,4} = \pm\sqrt{\left(-\frac{\omega^2-P_2^2}{2\omega} + \frac{\beta A^2}{8\omega}\right)\left(\frac{\omega^2-P_2^2}{2\omega} - \frac{3\beta A^2}{8\omega}\right)}$$

According to the above results, calculating the eigenvalues, we can get the same result in comparison to the different methods which determine the stability. For example if

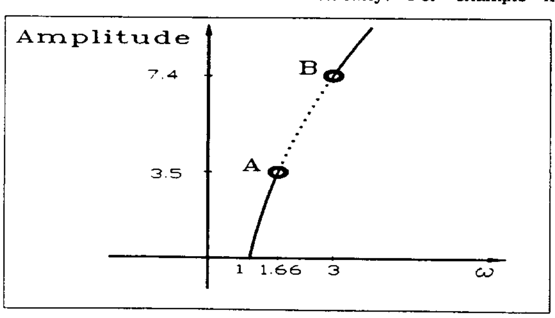


Fig.5 backbone curve for x-mode

eigenvalues are purely imaginary, this case is stable in this frequency region and if eigenvalues are real, this case is unstable in this frequency region. The point A where the stability is changed for unstable is bifurcating point for elliptic orbit and the point B where the stability is changed for stable is bifurcating point for normal mode.

5. Undamped Forced Vibration

The steady-state solution of the system can be approximately obtained by using the SCPA method. The frequency response curnes can be obtained by solving the statioinary solutions of equations :

$$\dot a = -\frac{\omega^2-p_1^2}{2\omega}c + \frac{1}{8\omega}[\ 3\alpha(a^2c+c^3)+\beta(b^2c+2abd+3cd^2)]$$
$$\dot b = -\frac{\omega^2-p_2^2}{2\omega}d + \frac{1}{8\omega}[\ 3\gamma(b^2d+d^3)+\beta(a^2d+2abc+3dc^2)] \qquad (10)$$
$$\dot c = \frac{\omega^2-p_1^2}{2\omega}a - \frac{1}{8\omega}[\ 3\alpha(c^2a+a^3)+\beta(d^2a+2bcd+ab^2)] + \frac{\varepsilon f_1}{2\omega}$$
$$\dot d = \frac{\omega^2-p_2^2}{2\omega}b - \frac{1}{8\omega}[\ 3\gamma(d^2b+b^3)+\beta(c^2b+2acd+3ba^2)] + \frac{\varepsilon f_2}{2\omega}$$

But we will consider the case when f_2 = 0 and $f_1 \neq$ 0.

5-1. The Stationary Solutions and Frequency Response Curves for Single Mode (x-mode)

The stationary solutions correspond to single mode response of the form $a = A\cos\omega t$ or $-A\cos\omega t$ and b = c = d = 0. Thus response curve of single mode is the following form. This form corresponds to the upper branch (out of phase) and the lower branch (in phase)

$$\dot a = \dot b = \dot d = 0$$
$$\dot c = \frac{1}{2\omega}(\omega^2-P_1^2)A - \frac{1}{8\omega}(3\alpha A^3) \pm \frac{\varepsilon f_1}{2\omega} = 0$$

5-2. The Stationary Solutions and Frequency Response Curves for Normal Mode

The stationary solutions correspond to normal modes are

$$a = A\cos\omega t \quad or \quad a = -A\cos\omega t,$$
$$b = B\cos\omega t \quad or \quad b = -B\cos\omega t, \qquad (11)$$
$$\text{and} \quad c = d = 0.$$

Substitutig eq.(11), c and d into eq.(10), equations of upper branch and the equations of lower branch are

$$\dot a = \dot b = 0$$
$$\dot c = \frac{1}{2\omega}(\omega^2-p_1^2)A - \frac{1}{8\omega}[\ 3\alpha A^3+3\beta AB^2] \pm \frac{\varepsilon f_1}{2\omega} = 0$$
$$\dot d = \frac{1}{2\omega}(\omega^2-p_2^2)B - \frac{1}{8\omega}[\ 3\gamma B^3+3\beta BA^2] = 0 \qquad \text{--------}(*)$$

Equation (*) is a condition for amplitude B to exist.

5-3. The Stationary Solutions and Frequency Response Curves for Elliptic Orbit

The stationary solutions correspond to elliptic modes is

$$a = A\cos\omega t \quad or \quad a = -A\cos\omega t,$$
$$d = D\sin\omega t \quad or \quad d = -D\sin\omega t, \qquad (12)$$
$$\text{and} \quad b = c = 0.$$

Putting eq.(12), b and c into eq.(10), we get the equations of upper branch and lower branch are

$$\dot{a} = \dot{d} = 0$$

$$\dot{b} = -\frac{1}{2\omega}(\omega^2-P_2^2)D+\frac{1}{8\omega}\{3\gamma D^3+\beta A^2 D\} = 0 \quad \text{-------(**)}$$

$$\dot{c} = \frac{1}{2\omega}(\omega^2-P_1^2)\frac{A-1}{8\omega}\{3\alpha A^3+\beta D^2 A\}\pm\frac{\varepsilon f_1}{2\omega} = 0$$

Equation (**) is a condition for amplitude D to exist.

5-4. Stability Analysis of Undamped Forced Vibration and Investication for Bifurcation

To analyze stability for single mode response, normal mode and elliptic orbit, we calculate the Jacobian matrix by the perturbation method and investigate the changing position of eigenvalues. Firstly to investigate the stability of stationary solutions, let

$$a=A+\eta_1,\quad b=B+\eta_2,\quad c=C+\eta_3,\quad d=D+\eta_4. \tag{13}$$

Substituting the introduced equation (21) into (17) and then composing the Jacobian matrix, we can get the following variational equation.

$$\dot{\eta} = [\ G\ ]\ \eta,$$

where [G] is

$$e_{11} = \frac{(6A\alpha C + 2B\beta D)}{8\omega}$$
$$e_{12} = \frac{\beta(2BC + 2AD)}{8\omega}$$
$$e_{13} = \frac{3\alpha(A^2 + 3C^2) + \beta(B^2 + 3D^2)}{8\omega} - \frac{(-P_1^2 + \omega^2)}{2\omega}$$
$$e_{14} = \frac{\beta(2AB + 6CD)}{8\omega}$$
$$e_{21} = \frac{\beta(2BC + 2AD)}{8\omega}$$
$$e_{22} = \frac{(2A\beta C + 6B\gamma D)}{8\omega}$$
$$e_{23} = \frac{\beta(2AB + 6CD)}{8\omega}$$
$$e_{24} = \frac{\beta(A^2 + 3C^2) + 3\gamma(B^2 + 3D^2)}{8\omega} - \frac{(-P_2^2 + \omega^2)}{2\omega}$$
$$e_{31} = -\frac{3\alpha(3A^2 + C^2) + \beta(3B^2 + D^2)}{8\omega} + \frac{(-P_1^2 + \omega^2)}{2\omega}$$
$$e_{32} = -\frac{\beta(6AB + 2CD)}{8\omega}$$
$$e_{33} = -\frac{(6A\alpha C + 2B\beta D)}{8\omega}$$
$$e_{34} = -\frac{\beta(2BC + 2AD)}{8\omega}$$
$$e_{41} = -\frac{\beta(6AB + 2CD)}{8\omega}$$
$$e_{42} = -\frac{\beta(3A^2 + C^2) + 3\gamma(3B^2 + D^2)}{8\omega} + \frac{(-P_2^2 + \omega^2)}{2\omega}$$
$$e_{43} = -\frac{\beta(2BC + 2AD)}{8\omega}$$
$$e_{44} = -\frac{(2\beta AC + 6\gamma BD)}{8\omega}$$

The stability of the undamped forced vibration can be determined by the types of the corresponding eigenvalue λ's of the characteristic equations

$$Det\,|\,[\ G\ ] - \lambda[\ I\ ]\ | = 0 \tag{14}$$

or

$$\lambda^4 - K\lambda^2 + \Delta_1(\varepsilon)\cdot\Delta_2(\varepsilon) = 0$$

where

$K(\varepsilon) = e_{13}(\varepsilon)e_{31}(\varepsilon) + e_{14}(\varepsilon)e_{41}(\varepsilon) + e_{23}(\varepsilon)e_{32}(\varepsilon) + e_{24}(\varepsilon)e_{42}(\varepsilon)$

$\Delta_1(\varepsilon) = e_{13}(\varepsilon)e_{24}(\varepsilon) - e_{14}(\varepsilon)e_{23}(\varepsilon)$

$\Delta_2(\varepsilon) = e_{31}(\varepsilon)e_{42}(\varepsilon) - e_{32}(\varepsilon)e_{41}(\varepsilon)$

Composing the Jacobian matrix to find eigenvalues for single mode,

$$e_{11} = e_{12} = e_{14} = e_{21} = e_{22} = e_{23} = e_{32} = e_{33} = e_{34} = e_{41} = e_{43} = e_{44} = 0$$

$$e_{13} = \frac{3\alpha}{8\omega}A^2 - \frac{\omega^2 - P_1^2}{2\omega}$$
$$e_{24} = \frac{\beta}{8\omega}A^2 - \frac{\omega^2 - P_2^2}{2\omega}$$
$$e_{31} = -\frac{9\alpha}{8\omega}A^2 + \frac{\omega^2 - P_1^2}{2\omega}$$
$$e_{42} = -\frac{3\beta}{8\omega}A^2 + \frac{\omega^2 - P_1^2}{2\omega}$$

Calculating the eigenvalues,one can obtain the below equation.

$$\frac{1}{4096\omega^4}(\lambda^2 + E)(\lambda^2 + F) = 0$$

where

$$E:\ 27\alpha^2A^4 + 48\alpha A^2(P_1^2-\omega^2) - 32P_1^2\omega^2 + 16(\omega^4+P_1^2)$$
$$F:\ 3\beta A^4 + 16\beta A^2(P_2^2-\omega^2) - 32P_2^2\omega^2 + 16(\omega^4+P_2^2)$$

According to the changes of E and F, stability of the single mode response is determined. Jacobian matrix for normal mode is made up of the following elements.

$$e_{11} = e_{12} = e_{21} = e_{22} = e_{33} = e_{34} = e_{43} = e_{44} = 0$$

$$e_{13} = \frac{(3\alpha A^2 + \beta B^2)}{8\omega} - \frac{(-P_1^2+\omega^2)}{2\omega}$$
$$e_{14} = \frac{\beta AB}{4\omega}$$
$$e_{23} = \frac{\beta AB}{4\omega}$$
$$e_{24} = \frac{(\beta A^2 + 3\gamma B^2)}{8\omega} - \frac{(-P_2^2 + \omega^2)}{2\omega}$$
$$e_{31} = -\frac{(9\alpha A^2 + 3\beta B^2)}{8\omega} + \frac{(-P_1^2 + \omega^2)}{2\omega}$$
$$e_{32} = \frac{-3\beta AB}{4\omega}$$
$$e_{41} = -\frac{3\beta AB}{4\omega}$$
$$e_{42} = -\frac{(9\gamma B^2 + 3\beta A^2)}{8\omega} + \frac{(-P_2^2 + \omega^2)}{2\omega}$$

We can get eigenvalues through the followed equation but the characteristic equation for normal modes is very complex.

5-4-1. For Stable Normal Mode

Let's suppose $K(\varepsilon) = -\Omega^2$, where Ω is a real constant. It is range of backbone curve that is stable. Then eq.(14) becomes

$$\lambda^4 + \Omega^2\lambda^2 + \varepsilon\Delta = 0$$

where $\Delta = f(\Delta_1,\Delta_2)$.

Therefore the eigenvalues are

$$\lambda_{1,2} = \pm\sqrt{\frac{-\Omega^2 + \sqrt{\Omega^4-4\varepsilon\Delta}}{2}}$$
$$\lambda_{3,4} = \pm\sqrt{\frac{-\Omega^2 - \sqrt{\Omega^4-4\varepsilon\Delta}}{2}}$$

If

$$-\Omega^2 + \sqrt{\Omega^4-4\varepsilon\Delta} > 0$$

then $4\varepsilon\Delta < 0$. This means that two eigenvalues are real and two eigenvalues are imaginary. This implies that if $\varepsilon\Delta < 0$, the response is unstable.

And if

$$-\Omega^2 + \sqrt{\Omega^4 - 4\varepsilon\Delta} < 0$$

then $4\varepsilon\Delta > 0$. This means that four eigenvalues are imaginary. This implies that if $\varepsilon\Delta > 0$, the response is stable.
Therefore in this case, the stability of upper branch is opposite to that of lower branch of frequency response curve.

5-4-2. For unstable normal mode

Let's suppose $K(\varepsilon) = \Omega^2$, where Ω is a real constant. It is range of backbone curve that is unstable. Then eq(14) becomes

$$\lambda^4 - \Omega^2\lambda^2 + \varepsilon\Delta = 0$$

where $\Delta = f(\Delta_1, \Delta_2)$.
Therefore the eigenvalues are

$$\lambda_{1,2} = \pm\sqrt{\frac{\Omega^2 + \sqrt{\Omega^4 - 4\varepsilon\Delta}}{2}}$$
$$\lambda_{3,4} = \pm\sqrt{\frac{\Omega^2 - \sqrt{\Omega^4 - 4\varepsilon\Delta}}{2}}$$

If

$$\Omega^2 - \sqrt{\Omega^4 - 4\varepsilon\Delta} < 0$$

then $4\varepsilon\Delta < 0$. This means that two eigenvalues are real and two eigenvalues are imaginary. This implies that if $\varepsilon\Delta < 0$, the response is unstable.
And if

$$\Omega^2 - \sqrt{\Omega^4 - 4\varepsilon\Delta} > 0$$

then $4\varepsilon\Delta > 0$. This means that four eigenvalues are real. This implies that if $\varepsilon\Delta > 0$, the response is unstable.
Therefore the frequency responses of the upper and lower branch of freqeucy response curve are all unstable.
Jacobian matrix for elliptic orbit is composed of the following element.

$$e_{11} = e_{14} = e_{22} = e_{23} = e_{32} = e_{33} = e_{41} = e_{44} = 0$$

$$e_{12} = \frac{\beta AD}{4\omega}$$
$$e_{13} = \frac{3\alpha A^2 + 3\beta D^2}{8\omega} - \frac{-P_1^2 + \omega^2}{2\omega}$$
$$e_{21} = \frac{\beta AD}{4\omega}$$
$$e_{24} = \frac{\beta A^2 + 9\gamma D^2}{8\omega} - \frac{-P_2^2 + \omega^2}{2\omega}$$
$$e_{31} = -\frac{9\alpha A^2 + \beta D^2}{8\omega} + \frac{-P_1^2 + \omega^2}{2\omega}$$
$$e_{34} = -\frac{\beta AD}{4\omega}$$
$$e_{42} = -\frac{3\beta A^2 + 3\gamma D^2}{8\omega} + \frac{-P_2^2 + \omega^2}{2\omega}$$
$$e_{43} = -\frac{\beta AD}{4\omega}$$

We can get eigenvalues through the followed equation but the characteristic equation for normal modes is also complex. Therefore eigenvalues for elliptic orbit is calculated by the computer package mathematica. Because the characteristic equation of this case is equivalent to that of normal modes, the stability of frequency response curve for elliptic orbit is equal to the case for normal mode.

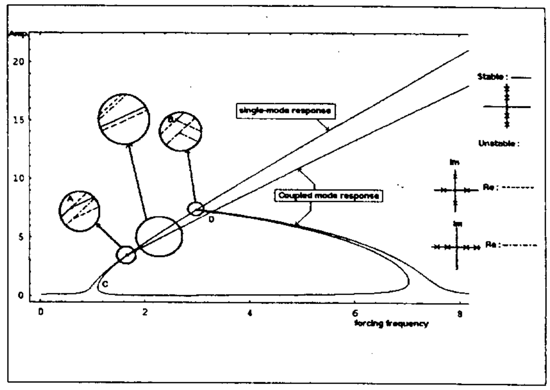


Fig. 6 response curves for undamped forced vibration

The section AB, that is range of the forcing frequency, is the unstable region of the backbone curve in free vibration. In this range the upper branch which have two real eigenvalues and two imaginary eigenvalues is unstable and lower branch having four real eigenvalues is perfectly unstable.
The points A,B are concerned with the response equations of the normal modes and elliptic orbit. The pitchfork bifurcation is generated in this points. Thus pitchfork bifurcation from single-mode response to coupled-mode response constant solutions occurs only because of the coupled-mode disturbances. The point C is associated with the response equation for single mode response and a saddle-node bifurcation occurs at this point where the two characteristic exponents are zero at these jumping points. Hence these saddle-node bifurcation results in the familiar "jump"phenomenon in the single mode response.
To illustrate the stability relation between free and forced vibration,consider a system which has the following parameters.

$$\{\omega_1, \omega_2\} = \{1, 1.5\}, \{\alpha, \beta, \gamma\} = \{0.195762, 0.164874, 0.160992\}$$
$$\{\varepsilon f_1, \varepsilon f_2\} = \{0.1,\ 0.1\}, \{c_1, c_2\} = \{0,\ 0\}$$

6. Damped Forced Vibration.

6-1. Derivation of Autonomous System by the Method of Multiple Scales.

To analyze the following equations for the damped forced vibration, we employ the method of multiple scales.

$$\begin{aligned} \ddot{x} + P_1^2 x + \bar{\alpha} x^3 + \bar{\beta} x y^2 + \bar{c}_1 \dot{x} &= \bar{f}_1 \cos\omega t \\ \ddot{y} + P_2^2 y + \bar{\gamma} x^3 + \bar{\beta} y x^2 + \bar{c}_2 \dot{y} &= \bar{f}_2 \cos\omega t \end{aligned} \qquad (15)$$

where,

$$\omega^2 = P_1^2 + \varepsilon\Delta_1$$
$$\omega^2 = P_2^2 + \varepsilon\Delta_2$$
$$\{\bar{\alpha}, \bar{\beta}, \bar{\gamma}\} = \{\varepsilon\alpha, \varepsilon\beta, \varepsilon\gamma\},\ \{\bar{c}_1, \bar{c}_2\} = \{\varepsilon c_1, \varepsilon c_2\}$$
$$\{\bar{f}_1, \bar{f}_2\} = \{\varepsilon f_1, \varepsilon f_2\}$$

One begins by introducing new independent variables according to

$$T_n = \varepsilon^n t \ , \qquad n=1,2,\cdots \tag{16}$$

It follows that the derivatives with respect to t become expansion in terms of the partial derivatives with respect to the T_n according to

$$\begin{aligned} \frac{d}{dt} &= D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots \\ \frac{d^2}{dt^2} &= D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 D_1^2 + 2\varepsilon^2 D_0 D_2 + \cdots \end{aligned} \tag{17}$$

One assumes that the solution of eq.(15) can be represented by an expansion having the form

$$\begin{aligned} x(t;\varepsilon) &= x_0(T_0, T_1, T_2, \cdots) + \varepsilon x_1(T_0, T_1, T_2, \cdots) + \varepsilon^2 x_2(T_0, T_1, T_2, \cdots) \\ y(t;\varepsilon) &= y_0(T_0, T_1, T_2, \cdots) + \varepsilon y_1(T_0, T_1, T_2, \cdots) + \varepsilon^2 y_2(T_0, T_1, T_2, \cdots) \end{aligned}$$

Substituting eq.(16) and eq.(17) into eq.(15) and equating the coefficients of ε^0 , ε^1 to zero, we can get the following equations.

$$\varepsilon^0: \ D_0^2 x_0 + \omega^2 x_0 = 0 \tag{18}$$

$$D_0^2 y_0 + \omega^2 y_0 = 0 \tag{19}$$

$$\begin{aligned} \varepsilon^1: \ & D_0^2 x_1 + 2D_0 D_1 x_0 + \omega^2 x_1 - \Delta_1 x_0 + \alpha x_0^3 + \beta x_0 y_0^2 + c_1 D_0 x_0 = f_1 \cos\omega t \\ & D_0^2 y_1 + 2D_0 D_1 y_0 + \omega^2 y_1 - \Delta_2 y_0 + \gamma y_0^3 + \beta y_0 x_0^2 + c_2 D_0 y_0 = f_2 \cos\omega t \end{aligned}$$

With this approach it turns out to be convenient to write the solutions of (18) in the form

$$\begin{aligned} x_0 &= A e^{i\omega T_0} + \overline{A} e^{-i\omega T_0} \\ y_0 &= B e^{i\omega T_0} + \overline{B} e^{-i\omega T_0} \end{aligned} \tag{20}$$

where A and B are an unknown complex function and $\overline{A}$ and $\overline{B}$ are the complex conjugate of A and B. Substituting the eq.(20) into the equations of ε^1-order,

$$\begin{aligned} D_0^2 x_1 + \omega^2 x_1 = \ & f_1 \cos\omega t - 2D_1(A i\omega e^{i\omega T_0} - \overline{A} i\omega e^{-i\omega T_0}) \\ & + \Delta_1(A e^{i\omega T_0} + \overline{A} e^{-i\omega T_0}) \\ & - \alpha(A^3 e^{3i\omega T_0} + 3A^2 \overline{A} e^{i\omega T_0} + 3A\overline{A}^2 e^{-i\omega T_0} + \overline{A}^3 e^{-3i\omega T_0}) \\ & - \beta(A e^{i\omega T_0} + \overline{A} e^{-i\omega T_0})(B e^{i\omega T_0} + \overline{B} e^{-i\omega T_0})^2 \\ & - c_1(A i\omega e^{i\omega T_0} - \overline{A} i\omega e^{-i\omega T_0}) \end{aligned}$$

and a similar equation on y_1. Elliminatig the secular terms from x_1 and y_1 ,

$$\begin{aligned} & \frac{f_1}{2} - 2A' i\omega + \Delta_1 A - 3\alpha A^2 \overline{A} - 2\beta A B \overline{B} - \beta \overline{A} B^2 - c_1 A i\omega = 0 \\ & \frac{f_2}{2} - 2B' i\omega + \Delta_2 B - 3\gamma B^2 \overline{B} - 2\beta A B \overline{A} - \beta \overline{B} A^2 - c_2 B i\omega = 0 \end{aligned} \tag{21}$$

In solving equations having the form of (21), we find it convenient to write A in the polar form.

$$A = \frac{R_1}{2} e^{-i\gamma_1} \ , \quad B = \frac{R_2}{2} e^{-i\gamma_2} \tag{22}$$

where R_1, R_2, γ_1 and γ_2 are real functions of T_1 .

Putting the eq.(22) on eq.(21) , we can take the following equations.

$$\begin{aligned} & \frac{f_1}{2}(\cos\gamma_1 + \sin\gamma_1) - iR_1'\omega - R_1\gamma_1'\omega + \Delta_1 \frac{R_1}{2} - \frac{3}{8}\alpha R_1^3 - \frac{\beta}{4} R_1 R_2^2 \\ & \quad - \frac{\beta}{8} R_1 R_2^2(\cos 2(\gamma_1 - \gamma_2) + i\sin 2(\gamma_1 - \gamma_2)) - c_1 i\omega \frac{R_1}{2} = 0 \\ & \frac{f_2}{2}(\cos\gamma_2 + \sin\gamma_2) - iR_2'\omega - R_2\gamma_2'\omega + \Delta_2 \frac{R_2}{2} - \frac{3}{8}\gamma R_2^3 - \frac{\beta}{4} R_2 R_1^2 \\ & \quad - \frac{\beta}{8} R_2 R_1^2(\cos 2(\gamma_2 - \gamma_1) + i\sin 2(\gamma_2 - \gamma_1)) - c_2 i\omega \frac{R_2}{2} = 0 \end{aligned}$$

Separating the result into real and imaginary parts,

$$\begin{aligned} (a)\ & R_1' = -\frac{c_1}{2} R_1 + \frac{\beta}{8\omega} R_1 R_2^2 \sin 2(\gamma_2 - \gamma_1) + \frac{f_1}{2\omega}\sin\gamma_1 \\ (b)\ & R_1\gamma_1' = -\frac{\Delta_1}{2\omega} R_1 + \frac{3}{8\omega}\alpha R_1^3 + \frac{\beta}{8\omega} R_1 R_2^2 [\ 2 + \cos 2(\gamma_2 - \gamma_1)] - \frac{f_1}{2\omega}\cos\gamma_1 \\ (c)\ & R_2' = -\frac{c_2}{2} R_2 - \frac{\beta}{8\omega} R_2 R_1^2 \sin 2(\gamma_2 - \gamma_1) + \frac{f_2}{2\omega}\sin\gamma_2 \\ (d)\ & R_2\gamma_2' = -\frac{\Delta_2}{2\omega} R_2 + \frac{3}{8\omega}\gamma R_2^3 + \frac{\beta}{8\omega} R_2 R_1^2 [\ 2 + \cos 2(\gamma_2 - \gamma_1)] - \frac{f_2}{2\omega}\cos\gamma_2 \end{aligned} \tag{23}$$

If these equations are transformed into cartesian coordinates, those are same to the equation derived by harmonic balance method if damping term is ignored. Therefore methods of derivation is different but each result is alike. In next section we'll obtain the frequency response equations and determine the stability with the derived equations.

6-2. Derivation and Analysis of the Frequency Response Equations for Single and Coupled Modes

6-2-1. For Single Mode

This case is that f_2 and R_2 are zero.Therefore in eq.(23-a) and (23-b) one can take the following equations.

$$\begin{aligned} R_1' &= -\frac{c_1}{2} R_1 + \frac{f_1}{2\omega}\sin\gamma_1 = 0 \\ R_1\gamma_1' &= \frac{w^2 - P_1^2}{2\omega} R_1 - \frac{3}{8\omega}\alpha R_1^3 + \frac{f_1}{2\omega}\cos\gamma_1 = 0 \end{aligned} \tag{24}$$

Using Eq.(24), we find the frequency response equation in the form.

$$\left(\frac{c_1}{2} R_1\right)^2 + \left(\frac{\omega^2 - P_1^2}{2\omega} R_1 - \frac{3\alpha}{8\omega} R_1^3\right)^2 = \frac{f_1^2}{4\omega^2}$$

6-2-2. For Coupled Modes

This case is that f_2 is zero and R_2 is not zero. Then in Eq.(23-c) and (23-d),

$$\begin{aligned} R_2' &= -\frac{c_2}{2} R_2 - \frac{\beta}{8\omega} R_1^2 R_2 \sin 2(\gamma_2 - \gamma_1) = 0 \\ R_2\gamma_2' &= \frac{\omega^2 - P_2^2}{2\omega} R_2 - \frac{1}{8\omega}[\ 3\gamma R_2^2 + \beta R_1^2 R_2(2 + \cos 2(\gamma_2 - \gamma_1))] = 0 \end{aligned} \tag{25}$$

Rearranging the sine and cosine terms to the right parts and squaring on both sides, we have

$$\frac{c_2^2}{4} + \left(\frac{\omega^2 - P_2^2}{2\omega} - \frac{3\gamma}{8\omega} R_2^2 - \frac{\beta R_1^2}{4\omega}\right)^2 = \left(\frac{\beta R_1^2}{8\omega}\right)^2$$

R_2 is calculated in above equation. A calculated equation for R_2 substitutes into the Eq.(31-a) nad (31-b) and rearranges,

$$\begin{aligned} & \left[-\frac{\omega^2 - P_1^2}{2\omega} R_1 + \frac{1}{8\omega}\{3\gamma R_1^2 + \frac{4(\omega^2 - P_2^2)}{R_1} R_2^2 - \frac{3\gamma}{R_1} R_2^4\}\right] \\ & \quad + \left(\frac{c_1}{2} R_1 + \frac{c_2}{2R_1} R_2^2\right)^2 = \frac{f_1^2}{4\omega^2} \end{aligned} \tag{26}$$

Eq.(26) is the frequency response equation for coupled modes.

6-2-3. Analysis for Relation between Forcing Amplitude and Damping Constant

The energy dissipated in a complete cycle is given by

$$W_{out} = \oint \varepsilon c_1 \dot{x}\, dx + \oint \varepsilon c_2 \dot{y}\, dy$$

where

$$\dot{x} = -\omega R_1 \sin(\omega t - \gamma_1)$$
$$\dot{y} = -\omega R_2 \sin(\omega t - \gamma_2)$$

Therefore we can rewrite equation of the dissipated energy,

$$W_{out} = \varepsilon \left(\int_0^{\frac{2\pi}{\omega}} c_1 \omega^2 R_1^2 \sin^2(\omega t - \gamma_1)\, dt + \int_0^{\frac{2\pi}{\omega}} c_2 \omega^2 R_2^2 \sin^2(\omega t - \gamma_2)\, dt \right)$$
$$= \varepsilon c_1 \omega^2 R_1^2 \left(\frac{\pi}{\omega} + \sin 2\gamma_1\right) + \varepsilon c_2 \omega^2 R_2^2 \left(\frac{\pi}{\omega} + \sin 2\gamma_2\right)$$

And the total excited energy is

$$W_{in} = \oint \varepsilon f_1 \cos \omega t\, dx + \oint \varepsilon f_2 \cos \omega t\, dy$$
$$= -\int_0^{\frac{2\pi}{\omega}} \varepsilon f_1 \omega R_1 \cos \omega t \sin(\omega t - \gamma_1) dt - \int_0^{\frac{2\pi}{\omega}} \varepsilon f_2 \omega R_2 \cos \omega t \sin(\omega t - \gamma_2) dt$$
$$= \varepsilon f_1 R_1 \pi \sin \gamma_1 + \varepsilon f_2 R_2 \pi \sin \gamma_2$$

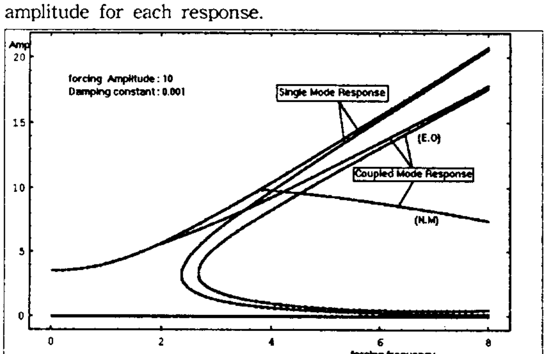
For the case of single-mode response it has the conditions, $f_2 = 0$ and $R_2 = 0$. The excited eneryg is equal to the dissipated energy. Thus we can get the relation force amplitude and damping constant.

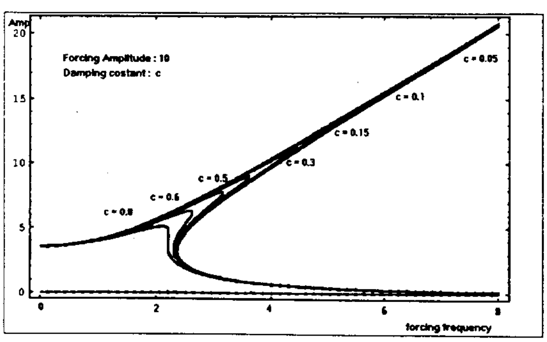
$$W_{in} = W_{out}$$
$$R_1 = \frac{f_1 \pi \sin \gamma_1}{c_1 \omega^2 \left(\frac{\pi}{\omega} + \sin 2\gamma_1\right)}$$

There is a phase difference between the applied force and the response and if the phase difference is $\pi/2$, the following relation is taken.

$$R_1 = \frac{f_1}{c_1 \omega}$$

Using the introduced method, we can obtain the concern of amplitude, damping constant,frequency and forcing amplitude for each response.





7. Conclusion.

1. It is ascertained that another periodic motion which is ellipse in the configuration space exists.
2. The change of the stability in force vibration is deeply concerned with that of free vibration.
3. The stability for single mode response changes at the points where the single mode response meets the coupled modes response and reason for changing the stability is the disturbance of coupled modes and it is known that a bifurcation occurs when the stability for each mode changes.
4. It is known that the equations for the stationary solution obtained by the averaging methods are same to that derived by the method of multiple scales.

References

1. Chang S. I.,1993. "Studies in Nonlinear Multi-mode Response of Harmonically Exicted Rectangular Plates with Internal Resonances." Ph.D Thesis,Purdue University.
2. Rand R. H., Pak C. H., Vakakis A. F., 1992. "Bifurcation of Nonlinear Normal Modes in a Class of Two Degree of Freedom Systems." Acta Mechanics 3, pp. 129-145.
3. Synge J. L., 1926. "On the Geometry of Dynamics." Philosofical Transcations A, Vol. 226, pp. 31-106.
4. Pak C. H., 1989. "On the Stability Behavior of Bifurcated Normal Modes in Coupled Nonlinear Systems." J.Applied Mechanics, Vol. 56, pp. 155-161.
5. Rosenberg R. M., 1966. "On Nonlinear Vibrations of Systems with Many Degrees of Freedom." Advances in Applied Mechanics, Vol. 9, pp. 155-242.
6. Nayfeh A. H., Mook D. T., 1979. *Nonlinear Oscillations.* John Wiley & Sons, New York.
7. Singiresu S. Rao, 1990. *Mechanical Vibration(2nd ED.).*, Addison-Wesley Publishing Company.
8. Yun Y. S., 1990. "On the Stability Relation Between Free and Forced Vibrations in Nonlinear Coupled Oscillators." Ph.D Thesis, Inha University.
9. Vakakis A. F.,1991. "Analysis and Identification of Linear and Nonlinear Normal Modes in Vibrating Systems." Ph.D Thesis, California Institute of Technology.
10. Guckenheimer J., Holmes P., 1983. *Nonlinear and Oscillations,Dynamical Systems,and Bifurcations of Vector Fields.* Spinger-Verlag,New York.
11. Rand R. H.,Pak C. H., Moon F. C., 1992. "Free Vibrations of a Thin Elastica by Normal Modes." Nonlinear Dynamics 3, pp.347-364.
12. Lee K. S., 1994. "On the Subharmonic Melnikov Analysis and Chaotic Behaviors in a Two-Degree-of-Freedom Hamiltonian System." M. S. Thesis, Inha Univ.