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Non-linear Vibration of Rectangular Plates

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1 Introduction

One of the important characteristics of the response of nonlinear systems is the existence of subharmonic resonances [1]. When some conditions in parameter space are satisfied, it is possible, even in the presence of damping, for a periodically excited nonlinear system to possess a response which is the combination of a contribution at the excitation frequency and a component at the system natural frequency. The system natural frequency being a submultiple of the excitation frequency implies that the resulting response is a subharmonic oscillation. In general, there also co-exists, for the system, a response at the excitation frequency, and initial conditions determine which of the steady-state responses is achieved in an experiment or a numerical simulation. In single-degree-of-freedom systems with harmonic excitation, depending on the type of the nonlinearity, e.g., cubic or quadratic, the frequency of subharmonic response is respectively, one-third or one-half of that of the excitation frequency.

Although subharmonic resonance is one of the principal characteristics of a nonlinear system, the subharmonic responses of structures in the presence of internal resonances, have been studied very rarely. In this work, we consider subharmonic responses in the two-mode approximation of the plate equations. Averaged equations for the subharmonic response of order three are obtained for the two-mode approximation of the plate equations. It is assumed that the two modes are in one-to-one internal resonance. Constant and periodic steady-state solutions of the averaged equations are studied. Finally, the results of direct time integration of the original equations of motion are presented and compared with those obtained from the averaged equations.

2 Formulation of the problem

Consider an isotropic rectangular plate of thickness h, and edge lengths a and b. Let $Oxyz$ be a Cartesian coordinate system with Oxy in the midplane of the plate and the origin at a corner. The plate is subjected to a uniform stretching force N_0 (in x- and y-directions). Under these conditions, the von Karman-type equations of motion for the plate, in nondimensional form, are as follows:

$$\begin{aligned} w_{,tt} &- \frac{1}{\pi^2}(w_{,xx} + \kappa^2 w_{,yy}) \\ &+ D(w_{,xxxx} + 2\kappa^2 w_{,xxyy} + \kappa^4 w_{,yyyy}) \\ &= \epsilon(F_{,yy}w_{,xx} - 2F_{,xy}w_{,xy} + F_{,xx}w_{,yy}) \\ &\quad - c\, w_{,t} + q. \end{aligned} \tag{1}$$

$$F_{,xxxx} + 2\kappa^2 F_{,xxyy} + \kappa^4 F_{,yyyy} = w_{,xy}^2 - w_{,xx} w_{,yy}. \tag{2}$$

where $w(x,y,t)$, $F(x,y,t)$ and $q(x,y,t)$ are the nondimensional transverse deflection, the stress function, and the external force normal to the plate, respectively. The dimensionless parameters ϵ, κ, D and c represent the thickness parameter, the aspect ratio, the ratio of bending stiffness to uniform stretching force and the damping coefficient, respectively. Furthermore, the subscript x, y or t denotes a partial differentiation with respect to the nondimensional variable. The boundary conditions considered here are that all the edges are simply supported and immovable. Applying Galerkin's procedure with two mode approximation gives the following two discretized equations of motion:

$$\begin{aligned} \ddot{X}_1 &+ \Omega_1^2 X_1 = \epsilon(A_1 X_1^2 + A_2 X_2^2)X_1 - c\dot{X}_1 \\ &+ Q_1 \cos \omega t, \\ \ddot{X}_2 &+ \Omega_2^2 X_2 = \epsilon(A_2 X_1^2 + A_3 X_2^2)X_2 - c\dot{X}_2 \\ &+ Q_2 \cos \omega t. \end{aligned} \tag{3}$$

where A_1, A_2 and A_3 are the constant non-linear coefficients determined for the specific mode combinations, and Ω_1 and Ω_2 are the corresponding natural frequencies of the two linear modes. Here X_1 is the amplitude of some (m,n) mode and X_2 is the amplitude of some other (r,s) mode which is in 1:1 internal resonance with the (m,n) mode.

3 Averaged equations and local bifurcation analysis

It has been shown [2] that the method of averaging is a quite effective tool for the analysis of weakly nonlinear system, especially for primary resonance case. However, there are some reports saying that the method of averaging doesn't approximate the original system appropriately in some secondary resonance cases. In this work, the method of averaging [3, 4] is applied to approximate the subharmonic responses of the original system (equations (3)) and local bifurcation analysis of the averaged system is performed.

In order to study subharmonic resonance of order-three, let us assume solutions to equations (3) in the form

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$$X_i = R_i \cos(\frac{1}{3}\omega t - \gamma_i) + D_i \cos \omega t,$$
$$= u_i \cos \frac{1}{3}\omega t + v_i \sin \frac{1}{3}\omega t + D_i \cos \omega t, \quad i = 1,2 \tag{4}$$

where

$$D_i = \frac{Q_i}{\Omega_i^2 - \omega^2}, \; i = 1,2, \tag{5}$$

and where (R_i, γ_i) (or equivalently (u_i, v_i)) are slowly varying functions of time. Then by using a variation of constants procedure and the method of averaging [5, 6] in the spirit of the harmonic balance method and noting that the excitation frequency ω is nearly three times the two close natural frequencies, Ω_1 and Ω_2, we obtain the following averaged equations for the amplitudes R_i and the phases γ_i:

$$\begin{aligned}
\dot{R}_1 = {} & \frac{3\epsilon A_2 \sin(-2\gamma_2 + 2\gamma_1) R_1 R_2^2}{8\omega} + \frac{9\epsilon A_1 D_1 \sin(3\gamma_1) R_1^2}{8\omega} \\
& + \frac{3\epsilon A_2 D_2 \sin(\gamma_2 + 2\gamma_1) R_1 R_2}{4\omega} \\
& + \frac{3\epsilon A_2 D_1 \sin(2\gamma_2 + \gamma_1) R_2^2}{8\omega} \\
& - \frac{cR_1}{2} + \frac{3\epsilon A_2 D_2 D_1 \sin(-\gamma_2 + \gamma_1) R_2}{2\omega},
\end{aligned}$$

$$\begin{aligned}
\dot{\gamma}_1 = {} & \frac{9\epsilon A_1 R_1^2}{8\omega} + \frac{3\epsilon A_2 R_2^2}{4\omega} + \frac{3\epsilon A_2 \cos(-2\gamma_2 + 2\gamma_1) R_2^2}{8\omega} \\
& + \frac{9\epsilon A_1 D_1 \cos(3\gamma_1) R_1}{8\omega} + \frac{3\epsilon A_2 D_1 \cos(2\gamma_2 + \gamma_1) R_2^2}{8\omega R_1} \\
& + \frac{3\epsilon A_2 D_2 \cos(\gamma_2 + 2\gamma_1) R_2}{4\omega} \\
& + \frac{3\epsilon A_2 D_2 D_1 \cos(-\gamma_2 + \gamma_1) R_2}{2\omega R_1} \\
& + \frac{9\epsilon A_1 D_1^2}{4\omega} + \frac{3\epsilon A_2 D_2^2}{4\omega} + \frac{\sigma_1(\omega)}{6\omega}
\end{aligned}$$

$$\begin{aligned}
\dot{R}_2 = {} & -\frac{3\epsilon A_2 \sin(-2\gamma_2 + 2\gamma_1) R_2 R_1^2}{8\omega} + \frac{9\epsilon A_3 D_2 \sin(3\gamma_2) R_2^2}{8\omega} \\
& + \frac{3\epsilon A_2 D_1 \sin(2\gamma_2 + \gamma_1) R_2 R_1}{4\omega} \\
& + \frac{3\epsilon A_2 D_2 \sin(\gamma_2 + 2\gamma_1) R_1^2}{8\omega} \\
& - \frac{cR_2}{2} - \frac{3\epsilon A_2 D_2 D_1 \sin(-\gamma_2 + \gamma_1) R_1}{2\omega},
\end{aligned}$$

$$\begin{aligned}
\dot{\gamma}_2 = {} & \frac{9\epsilon A_3 R_2^2}{8\omega} + \frac{3\epsilon A_2 R_1^2}{4\omega} + \frac{3\epsilon A_2 \cos(-2\gamma_2 + 2\gamma_1) R_1^2}{8\omega} \\
& + \frac{9\epsilon A_3 D_2 \cos(3\gamma_2) R_2}{8\omega} + \frac{3\epsilon A_2 D_2 \cos(\gamma_2 + 2\gamma_1) R_1^2}{8\omega R_2} \\
& + \frac{3\epsilon A_2 D_1 \cos(2\gamma_2 + \gamma_1) R_1}{4\omega} \\
& + \frac{3\epsilon A_2 D_1 D_2 \cos(-\gamma_2 + \gamma_1) R_1}{2\omega R_2} \\
& + \frac{3\epsilon A_2 D_1^2}{4\omega} + \frac{9\epsilon A_3 D_2^2}{4\omega} + \frac{\sigma_2(\omega)}{6\omega},
\end{aligned} \tag{6}$$

where $\sigma_1(\omega) = \omega^2 - (3\Omega_1)^2$ and $\sigma_2(\omega) = \omega^2 - (3\Omega_2)^2$. These parameters $\sigma_1(\omega)$ and $\sigma_2(\omega)$ can be interpreted as external detuning parameters for the (m, n) and (r, s) modes, respectively. It is easy to observe from equations (6) that, even when $Q_1 \neq 0$, and/or $Q_2 \neq 0$, steady-state zero solutions, that is, $R_1 = R_2 = 0$, always exist. Thus, the linear harmonic response at the excitation frequency always exists. When there also exist solutions for which at least one of the amplitudes, R_1 or R_2, is non-zero, the response of the system has a period which is three times that of the excitation. Then, for parameter values for which both types of solutions are stable, depending on the initial conditions, the response of the system can be periodic with either a period which is three times that of excitation, or a period which is the same as that of excitation.

We now present numerical results corresponding to the above analysis, and for solutions obtained by use of the numerical tools AUTO [7] and dstool [8]. We consider the response of the plate when (3,1) mode is directly excited, that is, $Q_2 \neq 0$ and $Q_1 = 0$. For $Q_2 \leq 500.0$, both the saddle-node and pitchfork bifurcation sets (denoted by PF_1 and PF_2) are found to exist for the single-mode response of the plate. These sets, as obtained by an analysis are shown in Figure 1.

A response diagram for subharmonic response, for $Q_2 = 200.0$, $c = 0.19$, is shown in Figure 2. A stable single-mode solution (response in (3,1) mode) exists over the frequency interval (SNS_1, PF_1). It undergoes a subcritical pitchfork bifurcation at PF_1 into a coupled-mode subharmonic response. The coupled-mode response is stable only over a small frequency interval (SNC, HB) and joins the unstable branch of single-mode solutions at PF_2 via a supercritical pitchfork bifurcation. Thus, in the frequency interval (SNC, HB), a single-mode subharmonic, a coupled-mode subharmonic, and a single-mode periodic response are stable and coexist.

At frequency near the Hopf point HB, periodic solutions of the averaged equations arise via a supercritical Hopf bifurcation. This periodic solutions branch, continued by using AUTO, is shown in Figure 3 (a). This branch exhibits oscillatory behavior with the period becoming unbounded as the frequency ω is varied. The solutions in this branch also exhibit a period-doubling cascade. In Figure 3 (b) is shown a representative phase plot for the limit cycle solution which closely approximates the homoclinic orbit. The homoclinic orbit is biasymptotic to the saddle-type coupled-mode constant solution of the averaged equations.

In Figure 4 are shown the response curves for the case when both Q_1 and Q_2 are nonzero, e.g., when $Q_1 = 300.0$ and $Q_2 = 300.0$. All the subharmonic solutions now are of the coupled-mode type. There exist stable constant solutions over the whole frequency interval (SNC_1, SNC_2) except for the interval (HB_1, HB_2) in which the solutions are unstable by a Hopf bifurcation. The left and right Hopf points correspond to sub and super critical Hopf bifurcations, respectively. The periodic solutions branch of period one is also shown in this figure. The solutions in the supercritical periodic branch exhibit a period-doubling cascade process. Numerical study by direct time integration shows, for frequency near the Hopf point HB_2, the period-doubling sequence to a chaotic attractor via averaging theory, the existence of amplitude-modulated subharmonic motions of the plate.

In Figure 5 and Figure 6 are shown the response of the original system of ordinary differential equations, (3), variation of the two solution components X_1 and X_2, and the response in the configuration phase space are plotted in these figures. The results in Figure 5 (a) show a stable representative subharmonic response which coexists with the stable small amplitude harmonic response, c.f., Figure 5 (b) at a frequency ($\omega = 14.5$) for which the averaged equations predict stable constant amplitude solutions. In Figure 6 (a), (b) and (c) are shown the response in the configuration space, the time response, and the Poincaré section of the response, respectively. These correspond to a frequency, $\omega = 22.0$, for which the analysis of averaged equations, presented earlier in this section, clearly shows the possibility of the existence of an amplitude-modulated subharmonic response of the plate. In Figure 7 is shown the Poincaré section of the response at decreased damping, $c = 0.18$. It shows torus doubling processes.

4 Conclusion

In this work, the one-third subharmonic response of the plate in the presence of 1-to-1 internal resonance has been studied. The analytical solutions of various bifurcation sets and their numerical results have been presented. It has been shown that quantitative as well as qualitative differences in the bifurcation sets and diagrams result depending on which mode is directly excited.

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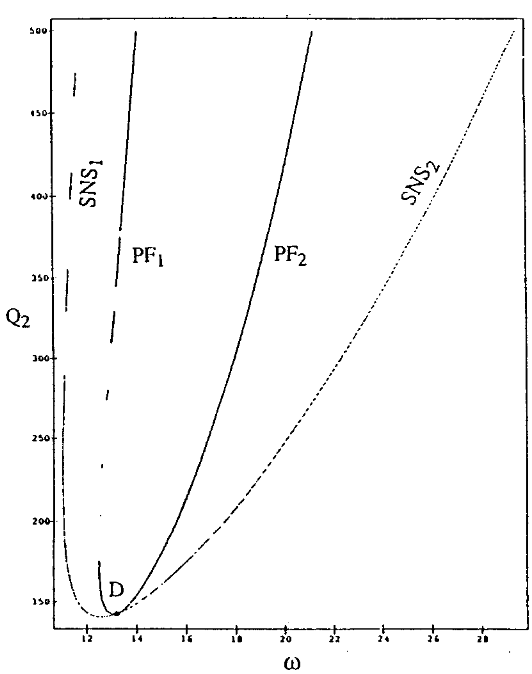


Figure 1: Saddle-node and pitchfork bifurcation sets for the single-mode solutions; $Q_1 = 0.0$, $c = 0.19$.

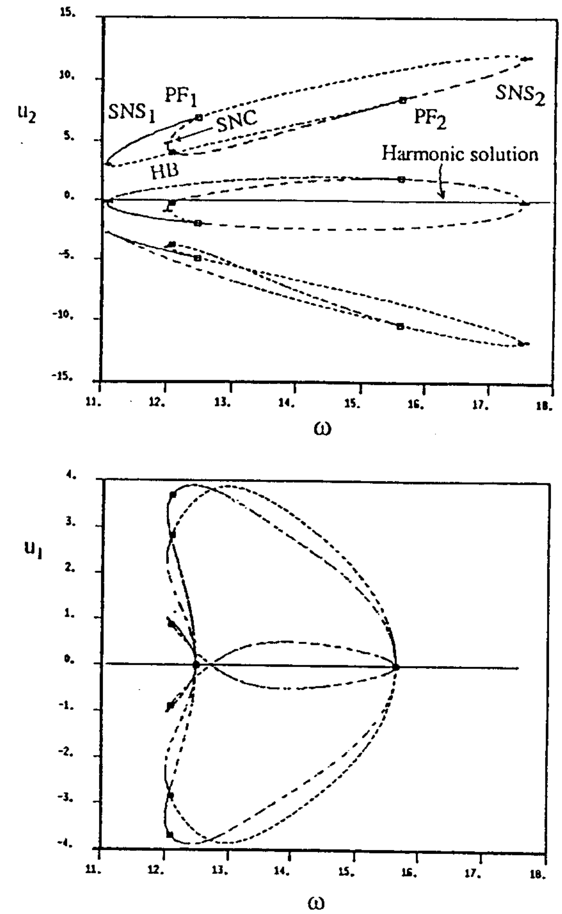
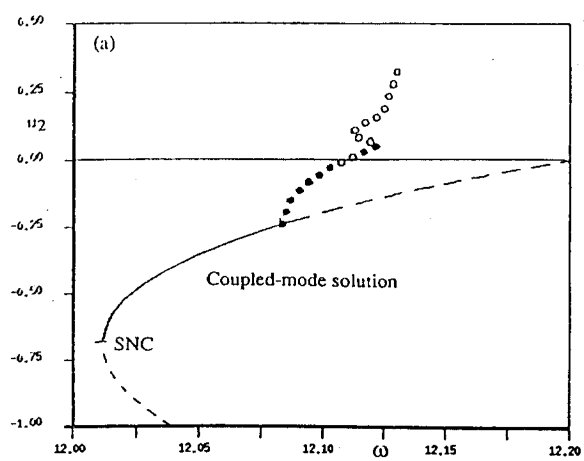


Figure 2: Response amplitude u_i as a function of the excitation frequency; $Q_1 = 0.0$, $Q_2 = 200.0$, $c = 0.19$.



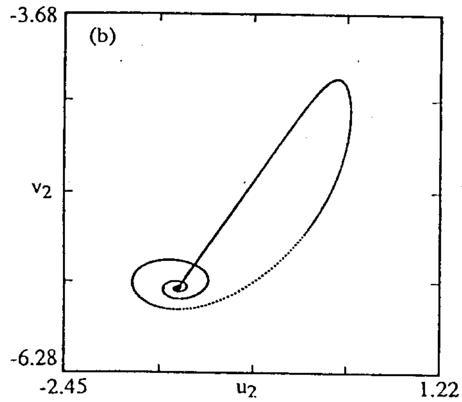


Figure 3: The periodic solution branch continued from the Hopf point; $Q_1 = 0.0$, $Q_2 = 200.0$, $c = 0.19$. a) amplitude response, b) phase plot of approximate homoclinic orbit for $\omega = 12.1313$.

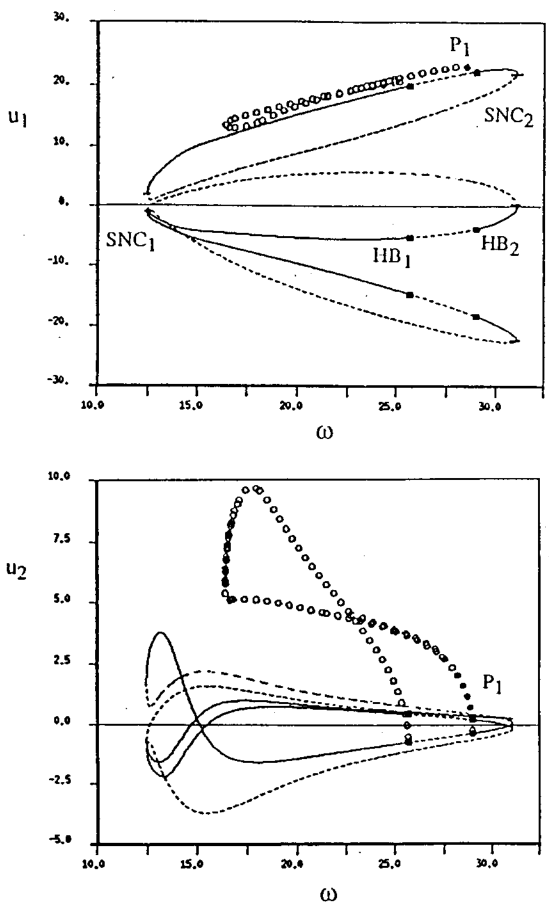


Figure 4: Bifurcation diagrams for the case when $Q_1 = 300.0$, $Q_2 = 300.0$, and $c = 0.19$.

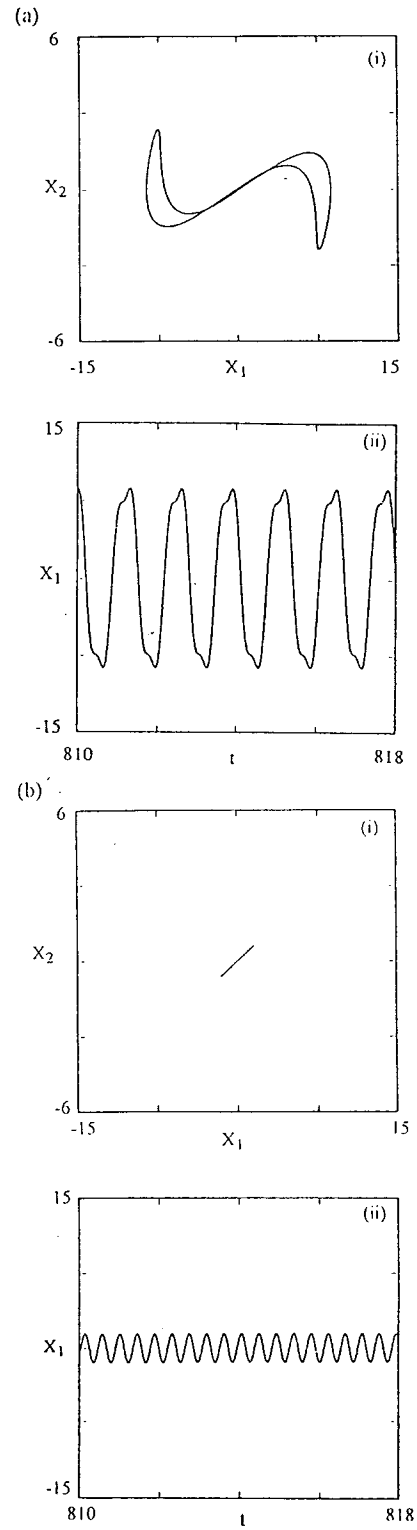


Figure 5: (a) subharmonic response, (b) coexisting harmonic response; (i)Phase plots for a steady-state solution of the original system of ODE when $Q_1 = 300.0$, $Q_2 = 300.0$, and $c = 0.19$, $\omega = 14.50$, (ii)corresponding response X_1 versus time.

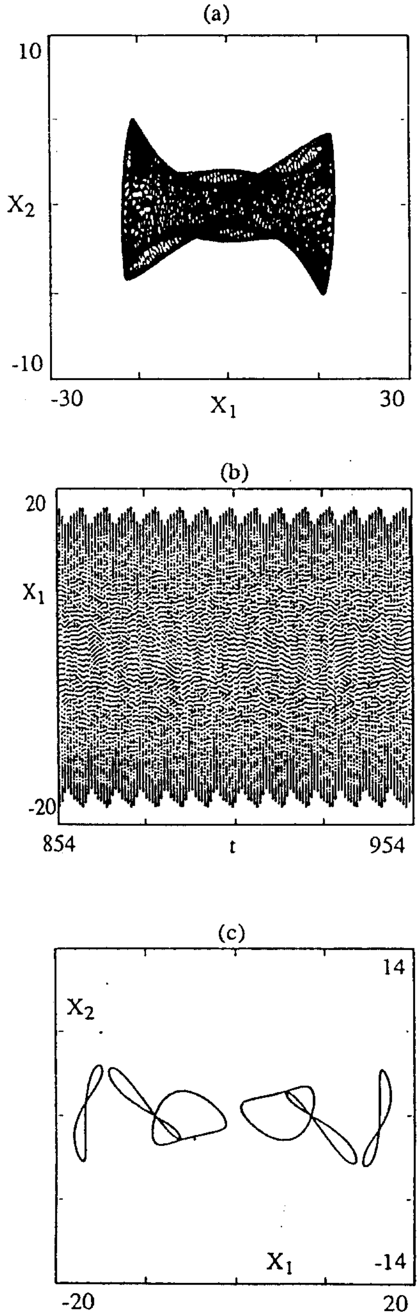


Figure 6: **(a)Phase plots for a periodic solution of the original system of ODE when Q_1 = 300.0, Q_2 = 300.0 c = 0.19 and ω = 22.0 , (b)corresponding response X_1 versus time, (c)corresponding Poincaré section.**

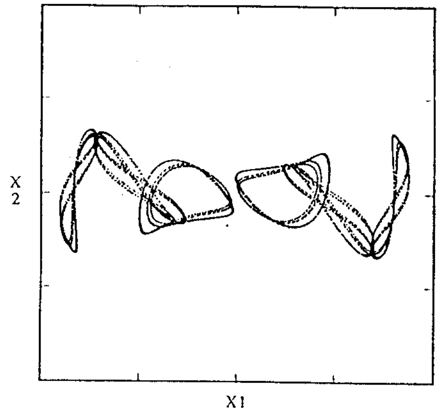


Figure 7: Poincaré section for a periodic solution of the original system of ODE when Q_1 = 300.0, Q_2 = 300.0 c = 0.18 and ω = 22.0.