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A NEW APPROACH FOR STABILIZATION OF NONSTANDARD SINGULARLY PERTURBED SYSTEMS

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Abstract: In this paper, we consider the stabilization problem of nonstandard singularly perturbed systems by using state feedback. Different from the existing sequential design procedures, we propose a parallel design method to construct the stabilizing controller. The method involves solving two completely independnet algebraic Riccati equations.

KeyWords: Nonstandard singularly perturbed systems, stabilizing controller, parallel design.

1. INTRODUCTION

Consider the linear time-invarant singularly perturbed system

$$\dot{x} = A_{11}x + A_{12}z + B_1u, \tag{1a}$$

$$\varepsilon\dot{z} = A_{21}x + A_{22}z + B_2u, \tag{1b}$$

$$y = C_1x + C_2z, \tag{1c}$$

where ε is a small positive parameter, $x(t) \in R^n$ and $z(t) \in R^m$ are states, $u(t) \in R^r$ is the control and $y(t) \in R^l$ is the output. The system (1) is called the nonstandard singularly perturbed system if the matrix A_{22} is singular.

In this paper, we consider the stabilization problem for the system (1) by applying state feedback control $u = -[F_1 \quad F_2]\begin{bmatrix} x \\ z \end{bmatrix}$. Such a problem has been studied in [1], where two methods were proposed. The first is via the use of prefeedback that puts the system into the standard form, and the second is via permutation of the state and control variables of the slow model that also puts the system into the standard form. Obviously, both of the methods are the sequential design procedure since they all involve the pre-transformation to change the nonstandard singularly perturbed system to the standard one. In this paper, we provide a parallel design method to solve the problem. Our method involves solving two completely independent algebraic Riccati equations. The solutions of the algebraic Riccati equations are used to construct the stabilizing controller.

2. SLOW AND FAST MODELS

Let $\varepsilon = 0$, we obtain the slow model

$$\dot{x}_s = A_{11}x_s + A_{12}z_s + B_1u_s, \tag{2a}$$

$$0 = A_{21}x_s + A_{22}z_s + B_2u_s, \tag{2b}$$

$$y_s = C_1x_s + C_2z_s. \tag{2c}$$

Subtracting (2b) from (1b), we obtain the fast model

$$\varepsilon\dot{z}_f = A_{22}z_f + B_2u_f, \tag{3a}$$

$$y_f = C_2z_f, \tag{3b}$$

where $z_f = z - z_s$ and $u_f = u - u_s$.

Assumption 1. The slow model (2) is stabilizable and detectable, that is, for all s with $Re[s] \geq 0$,

$$\textbf{Rank}\begin{bmatrix} sI_n - A_{11} & -A_{12} & B_1 \\ -A_{21} & -A_{22} & B_2 \end{bmatrix} = n + m, \tag{4a}$$

$$\textbf{Rank}\begin{bmatrix} sI_n - A_{11}^T & -A_{21}^T & C_1^T \\ -A_{12}^T & -A_{22}^T & C_2^T \end{bmatrix} = n + m. \tag{4b}$$

Assumption 2. The fast model (3) is stabilizable and detectable, that is, for all s with $Re[s] \geq 0$

$$\textbf{Rank}[sI_m - A_{22} \ B_2] = m, \tag{5a}$$

$$\textbf{Rank}[sI_m - A_{22}^T \ C_2^T] = m. \tag{5b}$$

From Assumption 2, it is obvious that there exists a unique positive semidefinite solution to the algebraic Riccati equation

$$A_{22}^TP_f + P_fA_{22} - P_fB_2B_2^TP_f + C_2^TC_2 = 0, \tag{6}$$

where the subscript 'f' denotes the items related to the fast model.

3. STABILIZING CONTROLLER

Designing stabilizing controller can be carried out by solving two algebraic Riccati equations. One is (6) and another is

$$A_s^T P_s + P_s A_s - P_s B_s B_s^T P_s + C_s^T C_s = 0, \quad (7)$$

where

$$A_s = A_{11} - (A_{12} - B_1 B_2^T P_f)(A_{22} - B_2 B_2^T P_f)^{-1} \times A_{21} + B_s B_2^T (A_{22} - B_2 B_2^T P_f)^{-T} \times (C_1^T C_2 + A_{21}^T P_f)^T, \quad (8)$$

$$B_s = B_1 - (A_{12} - B_1 B_2^T P_f) \times (A_{22} - B_2 B_2^T P_f)^{-1} B_2, \quad (9)$$

$$C_s = C_1 - C_2 (A_{22} - W_f C_2^T C_2)^{-1} \times (A_{21} - W_f C_2^T C_1), \quad (10)$$

and the subscript 's' means the items related to the slow model. W_f is the solution of the algebraic Riccati equation

$$A_{22} W_f + W_f A_{22}^T - W_f C_2^T C_2 W_f + B_2 B_2^T = 0, \quad (11)$$

which is the dual algebraic Riccati equation of (6), and admits a unique positive semidefinite solution according to Assumption 2.

When P_f and P_s are obtained, we can construct a state feedback controller as follows.

$$u = -[B_1^T \ B_2^T] \begin{bmatrix} P_s & 0 \\ P_m & P_f \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \quad (12)$$

where

$$P_m = -(A_{22} - B_2 B_2^T P_f)^{-T} [(A_{12} - B_1 B_2^T P_f)^T P_s + (C_1^T C_2 + A_{21}^T P_f)^T]. \quad (13)$$

The formulae (8)-(10) show the connections between the algebraic Riccati equation (7) and the algebraic Riccati equations (6),(11), where P_f and W_f are contained in the parameter matrices of (8)-(10). However, these formulae are only useful for the theoretic analysis. In the following, we will prove that the algebraic Riccati equation (7) is completely decoupled to the algebraic Riccati equations (6),(11). In order to find the the state feedback controller (12), we need to calculate the solutions P_f, P_s of two independent algebraic Riccati equations (6),(7), separately. That implies a parallel design procedure for stabilizing controller (12).

Define matrices

$$T_1 = \begin{bmatrix} A_{11} & -B_1 B_1^T \\ -C_1^T C_1 & -A_{11}^T \end{bmatrix},$$

$$T_2 = \begin{bmatrix} A_{12} & -B_1 B_2^T \\ -C_1^T C_2 & -A_{21}^T \end{bmatrix},$$

$$T_3 = \begin{bmatrix} A_{21} & -B_2 B_1^T \\ -C_2^T C_1 & -A_{12}^T \end{bmatrix},$$

$$T_4 = \begin{bmatrix} A_{22} & -B_2 B_2^T \\ -C_2^T C_2 & -A_{22}^T \end{bmatrix},$$

where T_4 is invertible under Assumption 2.

Lemma 1. The algebraic Riccati equation (7) is independent to the algebraic Riccati equations (6),(11).

Proof. Associated with the algebraic Riccati equation (7) is the $2n \times 2n$ Hamiltonian matrix

$$H_s = \begin{bmatrix} A_s & -B_s B_s^T \\ -C_s^T C_s & -A_s^T \end{bmatrix}. \quad (14)$$

It suffices the proof to show the relation $H_s = T_1 - T_2 T_4^{-1} T_3$ since the matrices $T_i, i = 1, 2, 3, 4$, contain no P_f and W_f. We leave it in Appendix.

Now, we will prove that the algebraic Riccati equation (7) admits a unique positive semidefinite stabilizing solution.

Lemma 2. Suppose that Assumptions 1, 2 are satisfied. The algebraic Riccati equation (7) admits a unique positive semidefinite stabilizing solution P_s.

Proof. Since

$$\begin{bmatrix} I_n & -\hat{A}_{12}\hat{A}_{22}^{-1} \\ 0 & -\hat{A}_{22}^{-1} \end{bmatrix} \begin{bmatrix} sI_n - A_{11} & -A_{12} & B_1 \\ -A_{21} & -A_{22} & B_2 \end{bmatrix} \times \begin{bmatrix} I_n & 0 & 0 \\ -\hat{A}_{22}^{-1} A_{21} & I_m & \hat{A}_{22}^{-1} B_2 \\ -B_2^T P_f \hat{A}_{22}^{-1} A_{21} & B_2^T P_f & I_r + B_2^T P_f \hat{A}_{22}^{-1} B_2 \end{bmatrix}$$
$$= \begin{bmatrix} sI_n - A_0 & 0 & B_s \\ 0 & I_m & 0 \end{bmatrix}, \quad (15)$$

where

$$A_0 = A_{11} - \hat{A}_{12}\hat{A}_{22}^{-1} A_{21},$$

$$\hat{A}_{12} = A_{12} - B_1 B_2^T P_f,$$

$$\hat{A}_{22} = A_{22} - B_2 B_2^T P_f,$$

we have $\mathbf{rank}[sI_n - A_0 \ B_s] = n$, for all $Re(s) \geq 0$ according to Assumption 1, that is, the matrix pair (A_0, B_s) is stabilizable. Furthermore, since

$$A_s = A_0 + B_s B_2^T \hat{A}_{22}^{-T} (C_1^T C_2 + A_{21}^T P_f)^T,$$

and that the feedback $B_2^T \hat{A}_{22}^{-T}(C_1^T C_2 + A_{21}^T P_f)^T$ does not change the stabilzability of (A_0, B_s), we arrive at the conclusion that the matrix pair (A_s, B_s) is stabilizable. Similarly, we can prove that the matrix pair (A_s, C_s) is detectable. Therefore, the algebraic Riccati equation (7) admits a unique positive semidefinite stabilizing solution P_s. That means

$$\mathbf{Re}\lambda(A_s - B_s B_s^T P_s) < 0. \tag{16}$$

Theorem 1. The state feedback controller (12) is a stabilizing controller of the nonstandard singularly perturbed system (1).

Proof. First, we prove that the state feedback controller

$$u_s = -[B_1^T \ \ B_2^T] \begin{bmatrix} P_s & 0 \\ P_m & P_f \end{bmatrix} \begin{bmatrix} x_s(t) \\ z_s(t) \end{bmatrix}, \tag{17}$$

is a stabilizing controller of the slow model (2). Substituting (17) into the slow model (2), we have

$$\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_s \\ \dot{z}_s \end{bmatrix} =$$

$$\begin{bmatrix} A_{11} - B_1 B_1^T P_s - B_1 B_2^T P_m & \hat{A}_{12} \\ A_{21} - B_2 B_1^T P_s - B_2 B_2^T P_m & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} x_s \\ z_s \end{bmatrix}. \tag{18}$$

(18) can be further transformed to a standard state space system and an algebraic equation

$$\dot{x}_s = [A_{11} - B_1 B_1^T P_s - B_1 B_2^T P_m$$

$$- \hat{A}_{12} \hat{A}_{22}^{-1}(A_{21} - B_2 B_1^T P_s - B_2 B_2^T P_m)] x_s, \tag{19a}$$

$$z_s = -\hat{A}_{22}^{-1}(A_{21} - B_2 B_1^T P_s - B_2 B_2^T P_m) x_s. \tag{19b}$$

Clearly, if we can prove that the system matrix of (19a) is $A_s - B_s B_s P_s$, then we have $x_s(t) \to 0$ as $t \to \infty$. That also implies $z_s(t) \to 0$ as $t \to \infty$. Now, substituting (13) into the system matrix of (19a) and making some manipulations yield

$$A_{11} - B_1 B_1^T P_s - B_1 B_2^T \{-\hat{A}_{22}^{-T} [\hat{A}_{12}^T P_s$$

$$+ (C_1^T C_2 + A_{21}^T P_f)^T]\} - \hat{A}_{12} \hat{A}_{22}^{-1}(A_{21} - B_2 B_1^T P_s$$

$$- B_2 B_2^T \{-\hat{A}_{22}^{-T} [\hat{A}_{12}^T P_s^+ + (C_1^T C_2 + A_{21}^T P_f)^T]\}$$

$$= A_0 + B_s B_2^T \hat{A}_{22}^{-T}(C_1^T C_2 + A_{21}^T P_f)^T - B_s B_s^T P_s$$

$$= A_s - B_s B_s^T P_s. \tag{20}$$

Therefore, the state feedback controller (17) is the stabilizing controller of the slow model (2). On the other hand, it is obvious that the state feedback controller

$$u_f = -B_2 P_f z_f, \tag{21}$$

is a stabilizing controller of the fast model (3). Therefore, the composite state feedback controller

$$u = u_s + u_f$$

$$= -[B_1^T \ \ B_2^T] \begin{bmatrix} P_s & 0 \\ P_m & P_f \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \tag{22}$$

is a stabilizing controller of the nonstandard singularly perturbed system (1) according to [1].

4. AN ILLUSTRATIVE EXAMPLE

The design procedure for the state feedback stabilizing controller (12) is summarized.

(1) Calculate the inverse of the Hamiltonian matrix T_4, and the Hamiltonian matrix $H_s = T_1 - T_2 T_4^{-1} T_3$.

(2) Find the solutions of two algebraic Riccati equations (6), (7) respectively.

(3) Construct the state feedback stabilizing controllers (12).

Example. Consider the following system

$$\dot{x}(t) = x(t) + 2z(t) + 2u(t), \tag{23a}$$

$$\varepsilon \dot{z}(t) = 2x(t) + u(t), \tag{23b}$$

$$y(t) = 2x(t) + z(t). \tag{23c}$$

For this example, we have

$$T_1 = \begin{bmatrix} 1 & -4 \\ -4 & -1 \end{bmatrix}, \ T_2 = \begin{bmatrix} 2 & -2 \\ -2 & -2 \end{bmatrix},$$

$$T_3 = \begin{bmatrix} 2 & -2 \\ -2 & -2 \end{bmatrix}, \ T_4 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Hence

$$H_s = T_1 - T_2 T_4^{-1} T_3 = \begin{bmatrix} -7 & -4 \\ -4 & 7 \end{bmatrix}.$$

From T_4 and H_s, we arrive at two completely decoupled algebraic Riccati equations

$$(i) \quad 1 - p_f^2 = 0,$$

$$(ii) \quad 4 - 14 p_s - 4 p_s^2 = 0,$$

where, the small letters are used to denote scalars. From (i), $p_f = 1$, a stabilizing solution. On the other hand, (ii) has a unique stabilizing solution $p_s = (2\sqrt{65} - 14)/8$. Hence, $p_m = 4$ from (13). The state feedback stabilizing controller is

$$u = -[2 \ 1] \begin{bmatrix} (2\sqrt{65} - 14)/8 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}.$$

5. CONCLUSIONS

This paper presents a new approach for designing state feedback stabilizing controllers of the nonstandard singularly perturbed system. The developed method involves solving two completely decoupled algebraic Riccati equations. Therefore, we called it a parallel method. A simple example is solved to show the designing procedure.

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Appendix. Proof of Lemma 1.

T_4^{-1} can be expressed as

$$\begin{bmatrix} A_{22} & -B_2B_2^T \\ -C_2^TC_2 & -A_{22}^T \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ P_f & I \end{bmatrix} \begin{bmatrix} \hat{A}_{22}^{-1} & -\hat{A}_{22}^{-1}B_2B_2^T\hat{A}_{22}^{-T} \\ 0 & -\hat{A}_{22}^{-T} \end{bmatrix} \begin{bmatrix} I & 0 \\ -P_f & I \end{bmatrix}. \tag{24}$$

Therefore,

$$T_1 - T_2T_4^{-1}T_3 = \begin{bmatrix} A_{11} & -B_1B_1^T \\ -C_1^TC_1 & -A_{11}^T \end{bmatrix} - \begin{bmatrix} A_{12} & -B_1B_2^T \\ -C_1^TC_2 & -A_{21}^T \end{bmatrix} \begin{bmatrix} I & 0 \\ P_f & I \end{bmatrix} \times \begin{bmatrix} \hat{A}_{22}^{-1} & -\hat{A}_{22}^{-1}B_2B_2^T\hat{A}_{22}^{-T} \\ 0 & -\hat{A}_{22}^{-T} \end{bmatrix} \begin{bmatrix} I & 0 \\ -P_f & I \end{bmatrix} \times \begin{bmatrix} A_{21} & -B_2B_1^T \\ -C_2^TC_1 & -A_{12}^T \end{bmatrix} = \begin{bmatrix} \hat{A}_s & -\hat{S}_s \\ -\hat{Q}_s & -\hat{A}_s^T \end{bmatrix}, \tag{25}$$

where

$$\hat{A}_s = A_{11} - N_1A_{21} + B_1B_2^TN_2^T - N_1B_2B_2^TN_2^T, \tag{26a}$$

$$\hat{S}_s = B_1B_1^T - N_1B_2B_1^T - B_1B_2^TN_1^T + N_1B_2B_2^TN_1^T, \tag{26b}$$

$$\hat{Q}_s = C_1^TC_1 - N_2A_{21} - A_{21}^TN_2^T - N_2B_2B_2^TN_2^T, \tag{26c}$$

and

$$N_1 = \hat{A}_{12}\hat{A}_{22},$$

$$N_2 = (C_1^TC_2 + A_{21}^TP_f)\hat{A}_{22}^{-1}.$$

It is easy to check, after expansions of N_1 and N_2, that

$$\hat{A}_s = A_0 + B_sB_2^TN_2^T = A_s, \tag{27}$$

$$\hat{S}_s = [B_1 - N_1B_2][B_1 - N_1B_2]^T = B_sB_s^T. \tag{28}$$

On the other hand, in order to obtain the expression for C_s, we use a different expression for T_4^{-1}, that is,

$$\begin{bmatrix} A_{22} & -B_2B_2^T \\ -C_2^TC_2 & -A_{22}^T \end{bmatrix}^{-1} = \begin{bmatrix} I & -W_f \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A}_{22}^{-1} & 0 \\ -\tilde{A}_{22}^{-T}C_2^TC_2\tilde{A}_{22}^{-1} & -\tilde{A}_{22}^{-T} \end{bmatrix} \begin{bmatrix} I & W_f \\ 0 & I \end{bmatrix}, \tag{29}$$

where

$$\tilde{A}_{22} = A_{22}^T - W_fC_2^TC_2. \tag{30}$$

Using the expression (29) to calculate $T_1 - T_2T_4^{-1}T_3$, we arrive at

$$\hat{Q}_s = C_s^TC_s, \tag{31}$$

where

$$C_s = C_1 - C_2(A_{22} - W_fC_2^TC_2)^{-1} \times (A_{21} - W_fC_2^TC_1). \tag{32}$$

We have thus completed the proof of Lemma 1.