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# Statically Compensated Modal Approximation of a class of Distributed Parameter Systems

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**Abstract:** A finite-dimensional approximation technique is developed for a class of spectral systems with input and output operators which are unbounded. A corresponding bounding technique on the frequency-response error is also established for control system design. Our goal is to construct an uncertainty model including a nominal plant and its error bounds so that the results from robust linear control theory can be applied to guarantee a closed loop control performance. We demonstrate by numerical example that these techniques are applicable, with a modest computational burden, to a wide class of distributed parameter system plants.

**Keywords:** finite-dimensional approximation, spectral systems, static compensation, error bounds, diffusion systems

## I. Introduction

A distributed parameter system described by a partial differential equation has been extensively used for modeling many types of physical plants. They are generally so-called infinite-dimensional, and some sort of finite dimensional approximations are necessary in most control system design. With recent developed robust linear control theory, we may guarantee stability and performance of the closed loop system if appropriate modeling error bounds are available. In model based control, a higher order model will generally results in higher order compensators, so the problem is how a lower order model with sufficient accuracy can be obtained.

It is well-known that conventional simulation techniques for distributed parameter systems are roughly classified into the finite difference method and the methods of weighted residuals. The latter include orthogonal polynomial expansions, eigenfunction expansion and Finite Element Methods, and usually they are said to generate a lower order model than by the finite difference method. But in most cases, orders of the model by the method of weighted residuals are even too high for control system design. A common approach to cope with this problem is to exploit lower order approximation techniques such as the balanced truncation method and Hankel norm approximation method. But then, such a model reduction will corrupt correspondence between states or parameters to ones for real physical plant. This causes difficulties in problems such as adjusting controller parameters and fault diagnosis in operations.

In this paper we propose an approach to finite dimensional approximation problems for a class of spectral systems. This is based on the modal truncation while a direct term is considered with which the static characteristic of the model coincide to that of the original system. The corresponding error bounds in frequency domain are also given in this paper. By static characteristics we mean dc component of the input-output response for stable systems. The static error has effects in all frequency range, and in many cases the number of terms has to be increased to obtain a sufficient static characteristics, and this seems to be one of the main reasons why approximation models are frequently to be of high orders. Static characteristics can be represented without any dynamic elements (integrator), so compensating the static error has been considered and several methods has been proposed so far in the model reduction problem for lumped parameter system[3].

The novelty of this paper is twofold. The one is that the main results are readily computable formulas, consisting of an inverse of the system operators and of a finite number of eigenvalues and eigenfunctions. This direction is partly inspired by Erickson, Smith and Laub [4]. The other one is that the results can be applied to systems with input and output operators which are unbounded, and we hope this enhances much utility of the results.

## II. System Formulation

Physical plants containing heat conduction, diffusion, strings, beam can be modeled as spectral systems described by countable number of modes. We first assume that a system contains a spectral operator which satisfies

**Assumption 1. (simple spectral operator [5])** *Let $\mathcal{A}$ be a closed operator with domain $D(\mathcal{A})$ dense in a separable Hilbert space $Z$ with compact resolvent, and its eigenvalues $\{\lambda_i,\ i = 1, 2, \ldots\}$ are simple and their real* *part be bounded above. Further, the normalized eigenfunctions $\{\phi_i,\ i = 1, \ldots\}$ for $\mathcal{A}$ can be selected to form a basis of $Z$, and the eigenfunctions $\{\psi_i,\ i = 1, \ldots\}$ of the conjugate $\mathcal{A}^*$ of $\mathcal{A}$ be normalized with respect to the inner product of $Z$ so that $\langle \phi_i, \psi_j \rangle = \delta_{ij}$ (Kronecker delta).*

It is known that $\mathcal{A}$ generate a strongly continuous semigroup $\{e^{t\mathcal{A}}\}$ on $Z$ if Assumption 1 is verified. We then consider a system as in the following

**Formulation.** (system with bounded input-output map [8])
*For $\mathcal{A}$ satisfying the Assumption 1, we consider Hilbert spaces $V$, $W$ such that $D(\mathcal{A}) \subset W \subset Z \subset V \subset D(\mathcal{A}^*)'$ where $D(\mathcal{A}^*)'$ is a dual space of $D(\mathcal{A}^*)$ with the graph norm of $\mathcal{A}^*$, and $Z'$ is identified with $Z$. Let $\mathcal{B} : \mathbf{R} \to V$, $\mathcal{C} : W \to \mathbf{R}$ and let an equation*

$$\frac{dz}{dt} = \mathcal{A}z + \mathcal{B}u \quad \text{on } V, \quad z(0) = z_0 \in W, \tag{1.a}$$

*and observation*

$$y = \mathcal{C}z. \tag{1.b}$$

*We assume that these define an bounded input-output map $L^2(0,T;\mathbf{R}) \to L^2(0,T;\mathbf{R}) : u(t) \mapsto y(t)$.*

The relations $W \subset Z \subset V$ are with dense embeddings, and a closed operator $\mathcal{A}$ on $V$ and its restrictions to $W$ and $Z$ generate a strongly continuous semigroup, respectively on each space. Let the state be $z(t) = \sum_{n=1}^{\infty} z_n(t)\phi_n$ where $z_n(t) = \langle z(t), \psi \rangle_Z$, $n = 1, 2, \ldots$ are called as modes. The equations are rewritten as

$$\begin{aligned} \dot{z}_n(t) &= \lambda_n z_n(t) + b_n u(t),\ n = 1, 2, \ldots \\ y(t) &= \sum_{n=1}^{\infty} c_n z_n(t), \end{aligned} \tag{2}$$

and the transfer function $G(s)$ from $u$ to $y$ can be written as [4]

$$G(s) = \sum_{n=1}^{\infty} \frac{c_n b_n}{s + \tau_n} \tag{3}$$

where $c_n := \mathcal{C}\phi_n$, $b_n := \mathcal{B}^*\psi_n$ and $\tan_n = -\lambda_n$ for $n = 1, 2, \ldots$. $\mathcal{B}$ and $\mathcal{C}$ can be represented by sequences$\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$, respectively.

In this paper, we consider exclusively a system satisfying the following

**Assumption 2. (stable self-adjoint systems)**
*i) $\mathcal{A}$ is a self-adjoint, i.e., $D(\mathcal{A}) = D(\mathcal{A}^*)$ and $\mathcal{A} = \mathcal{A}^*$.*
*ii) $\mathcal{A}$ is negative-definite.*
*iii) $W = D((-\mathcal{A})^{\alpha})$, $V = D((-\mathcal{A})^{\alpha-1})$ where $0 \le \alpha \le 1$.*

Distributed parameter systems of parabolic type such as thermal conductivity and diffusion systems are reduced to this kind of ones.

Since $\mathcal{A}$ is self-adjoint and negative-definite, its eigenvalue is all negative real number, and we assume consequently $0 < \tau_1 \le \tau_2 \le \cdots$. Note that $Z = D((-\mathcal{A})^0)$.

$V$ and $W$ are characterized as follows:

$$\begin{aligned} W &= \left\{ \psi \in Z \,\middle|\, \sum_{n=1}^{\infty} \{\tau_n^{\alpha} \langle \psi, \phi_n \rangle_Z\}^2 < \infty \right\} \\ V' &= \left\{ \psi \in Z \,\middle|\, \sum_{n=1}^{\infty} \{\tau_n^{1-\alpha} \langle \psi, \phi_n \rangle_Z\}^2 < \infty \right\} \end{aligned} \tag{4}$$

For example, for one-dimensional parabolic distributed parameter system, $\alpha = 0$ corresponds to the case with Dirichlet boundary input and bounded measurement operators, and $\alpha = 1/2$ with Neumann boundary input and point observation.

We note that finite numbers of $\{\tau_n\}$, $\{\phi_n\}$, $\{b_n\}$, $\{c_n\}$, $\langle \cdot, \cdot \rangle$ and $\mathcal{A}^{-1}$ are readily computable even in more general setting by using, e.g., finite element method.

## III. Modal Approximation

We mean by the $N$-th modal approximation model of $G(s)$, the $N$-th partial sum

$$G_N^{\circ}(s) = \sum_{n=1}^{N} \frac{c_n b_n}{s + \tau_n} \tag{5}$$

of infinite series (3). The modal approximation can be represented by state space form

$$G_N^{\circ}(s) = C_N(sI_N - A_N)^{-1}B_N \tag{6}$$

where $A_N = \mathrm{diag}(-\tau_1, \ldots, -\tau_N)$, $B_N = [b_1 \ \cdots \ b_N]^T$, $C_N = [c_1 \ \cdots \ c_N]$.

We now give an error bound for the modal approximation of spectral systems.

**Theorem 1.** (error bound for truncated model [7])
*An upper bound for the error between the modal approximation (5) and the system (1) is given as*

$$\|G - G_N^{\circ}\|_{\infty} \le d \tag{7}$$

*where*

$$d = \left(\|\mathcal{A}^{\alpha-1}c\|_2^2 - |A_N^{\alpha-1}C_N^T|_2^2\right)^{\frac{1}{2}} \left(\|\mathcal{A}^{-\alpha}b\|_2^2 - |A_N^{-\alpha}B_N|_2^2\right)^{\frac{1}{2}} \tag{8}$$

*which is shown in Table 1 for the case $\alpha = 0, 1/2, 1$.*

**Sketch of Proof:** It is easily shown that

$$\begin{aligned} \|G - G_N\|_{\infty} &\le \sum_{n=N+1}^{\infty} \tau_n^{-1}|c_n||b_n| \\ &\le \left[\sum_{n=N+1}^{\infty} \tau_n^{2(\alpha-1)}|c_n|^2\right]^{\frac{1}{2}} \left[\sum_{n=N+1}^{\infty} \tau_n^{-2\alpha}|b_n|^2\right]^{\frac{1}{2}}. \end{aligned}$$

The Theorem can be proved by seeing

$$\begin{aligned} \sum_{n=1}^{\infty} \left[\tau_n^{-\alpha}c_n\right]^2 &= \langle (-\mathcal{A})^{-\alpha}c, (-\mathcal{A})^{-\alpha}c \rangle = \|(-\mathcal{A})^{-\alpha}c\|_2^2 \\ &= \begin{cases} \|c\|_2^2, & \alpha = 0 \\ \langle c, -\mathcal{A}c \rangle, & \alpha = \frac{1}{2} \\ \|\mathcal{A}^{-1}c\|_2^2, & \alpha = 1. \end{cases} \end{aligned}$$

**Table 1** Error bounds for modal approximation.

| $\alpha$ | $V$ | $W$ | $d$ |
|---|---|---|---|
| 0 | $Z$ | $D(\mathcal{A}^{-1})$ | $\left(\lVert b\rVert_2^2-\lvert B_N\rvert_2^2\right)^{1/2}\left(\lVert\mathcal{A}^{-1}c\rVert_2^2-\lvert A_N^{-1}C_N^T\rvert_2^2\right)^{1/2}$ |
| 1/2 | $D((-\mathcal{A})^{-\frac{1}{2}})$ | $D((-\mathcal{A})^{-\frac{1}{2}})$ | $\left(\langle b,-\mathcal{A}^{-1}b\rangle-B_N^T(-A)_N^{-1}B_N\right)^{\frac{1}{2}}\left(\langle c,-\mathcal{A}^{-1}c\rangle-C_N(-A)_N^{-1}C_N^T\right)^{\frac{1}{2}}$ |
| 1 | $D(\mathcal{A}^{-1})$ | $Z$ | $\left(\lVert\mathcal{A}^{-1}b\rVert_2^2-\lvert A_N^{-1}B_N\rvert_2^2\right)^{1/2}\left(\lVert c\rVert_2^2-\lvert C_N\rvert_2^2\right)^{1/2}$ |

Note that in case of modal approximation for a system of parabolic type, we may not have generally any error bounds which correspond to strictly proper frequency weights; we can not say that there exist $d > 0$, $\tau > 0$ such that

$$|G_N^o(j\omega) - G(j\omega)| \le |d/(1+\tau\cdot j\omega)|, \quad \forall\omega.$$

Therefore, the multiplicative error is hard to evaluate above by using some proper function. One choice of upper bound for the multiplicative error may be

$$|W_2(j\omega)| = d\cdot|G_N^o(j\omega)|^{-1}. \tag{9}$$

As we have seen before, additive error for the truncation model is relatively large in low frequency range, so higher order model is used to obtain a model with higher precision in low frequency. If we truncate the series in low order, some limitation would be imposed in control performance for large error in low frequency range. Usually we can apply some other model reduction techniques to the high order truncation model, but the model reduction will corrupt the correspondence of physical meanings of states or parameters and some inconvenience may occur in practical application.

## IV. Static Compensation

In this chapter we consider a compensation of the dc gain difference between the model and the system using direct term.

*Suppose that a system satisfy Assumptions 1 and 2, we mean by a statically compensated model*

$$\begin{aligned} G_N(s) &= D_N + C_N(sI_N - A_N)^{-1}B_N \\ &= G_N^o(s) + (G(0) - G_N^o(0)) \end{aligned} \tag{10}$$

*where* $D_N = G(0) - G_N^o(0)$.

This type of models has been used for a distributed parameter system with boundary inputs. The statically compensated model can be also expressed as

$$G_N(s) = \sum_{n=1}^{N}\frac{c_n b_n}{s+\tau_n} + \sum_{n=N+1}^{\infty}\frac{c_n b_n}{\tau_n}. \tag{11}$$

A corresponding error bound is given as

**Theorem 2. (error bound for the compensated model)**
*An upper bound for the error between the statically compensated model (10) and the system (1) is given as*

$$|G(j\omega) - G_N(j\omega)| \le d\cdot\left|\frac{\tau_{N+1}^{-1}\cdot j\omega}{1+\tau_{N+1}^{-1}\cdot j\omega}\right| \tag{12}$$

*for all* $\omega \in \mathbf{R}$ *where* $d$ *is given in (8) and Table 1.*

**Sketch of Proof:** Using

$$G(s) - G_N(s) = \sum_{n=N+1}^{\infty}\frac{c_n b_n}{\tau_n}\frac{-s/\tau_n}{1+s/\tau_n},$$

we have

$$\begin{aligned} &|G(j\omega) - G_N(j\omega)| \\ &\le \sum_{n=N+1}^{\infty}\tau_n^{-1}|c_n||b_n|\left|\frac{j\omega/\tau_n}{1+j\omega/\tau_n}\right| \\ &\le \sum_{i=N+1}^{\infty}\tau_n^{-1}|c_n||b_n|\left|\frac{j\omega/\tau_{N+1}}{1+j\omega/\tau_{N+1}}\right| \end{aligned}$$

and the rest is similar to the proof of Theorem 1.

## V. Examples

For clarity of exposition we consider the following one-dimensional constant parameter parabolic distributed parameter system.

$$\frac{\partial z(t,x)}{\partial t} = \frac{\partial^2 z(t,x)}{\partial x^2} + b(x)u(t),\ x\in(0,1),\ t>0 \tag{13.a}$$

$$z(t,0) = z(t,1) = 0 \tag{13.b}$$

$$y(t) = \int_0^1 c(x)z(x)dx \tag{13.c}$$

$$b(x) = \begin{cases} 1, & \frac{1}{3}\le x\le\frac{2}{3} \\ 0, & \text{otherwise}\end{cases} \tag{13.d}$$

$$c(x) = \begin{cases} 1, & 0\le x\le\frac{1}{4} \\ 0, & \frac{1}{4}< x\le 1\end{cases} \tag{13.e}$$

The eigenvalues and eigenfunctions are $\lambda_n = -1/(n\pi)^2$, $\phi_n(x) = \sqrt{2}\sin n\pi x$ and $\langle\phi_n, b\rangle = (\sqrt{2}/n\pi)(\cos n\pi/3 - \cos n\pi 2/3)$, $\langle c,\phi_n\rangle = (\sqrt{2}/n\pi)(1-\cos n\pi/4)$. Furthermore, since

$$(-\mathcal{A}^{-1}b)(x) = \begin{cases} \frac{1}{6}x, & 0\le x<\frac{1}{3} \\ -\frac{1}{2}\left(x-\frac{1}{2}\right)^2+\frac{5}{27}, & \frac{1}{3}\le x\le\frac{2}{3} \\ \frac{1}{6}(1-x), & \frac{2}{3}<x\le 1\end{cases}$$

$$(-\mathcal{A}^{-1}c)(x) = \begin{cases} -\frac{1}{2}\left(x-\frac{7}{32}\right)^2+\frac{49}{2\cdot 32^2}, & 0\le x\le\frac{1}{4} \\ \frac{1}{32}(1-x), & \frac{1}{4}<x\le 1\end{cases}$$

$\langle b, -\mathcal{A}^{-1}b\rangle = 5/6^3 - 2/6^4 \approx 2.16 \times 10^{-2}$, $\langle c, -\mathcal{A}^{-1}c\rangle = (1/128)(17/12) \approx 1.11 \times 10^{-2}$.

In this example, the transfer function of the system (13) is given as closed form as follows:

$$G(s) = -\frac{\cosh\sqrt{s}2/3 - \cosh\sqrt{s}/3}{s^{3/2}\sinh\sqrt{s}} \cdot (\cosh\sqrt{s}/4 - 1) \quad (14)$$

while in general case such as multi-dimensional system with complex boundary conditions or containing spatially varying parameters, we have to compute error bounds from Theorem 1 and 2 and utilize it in design procedure.

The Bode plots of $G(j\omega)$, $G_N^{\circ}(j\omega)$ and $G_N(j\omega)$ are shown in Fig. 1, and the magnitude for additive errors $|\Delta_N^{\circ}(j\omega)| = |G(j\omega) - G_N^{\circ}(j\omega)|$, $|\Delta_N(j\omega)| = |G(j\omega) - G_N(j\omega)|$, for multiplicative error $|\Delta_N^{\circ}(j\omega)/G_N^{\circ}(j\omega)|$, $|\Delta_N(j\omega)/G_N(j\omega)|$ and their upper bound derived from (7),(12) are plotted in Fig. 2.

The statically compensated model has larger additive error in high frequency than for the truncated one, but we can see that the multiplicative error for the compensated model is better than the other in almost all frequency range.

We check achievable nominal performance of the closed loop. Let take the approximation model as a nominal plant, and formulate the design problem as a mixed sensitivity problem [1]. For the upper bound of the admissible additive perturbation, absolute value of (12) are used, and performance objective can be represented by the weight $W_1(s)$, which bound the magnitude of the sensitivity function $S(j\omega) = 1/(1 + G_N(j\omega)F(j\omega))$ which represent the effect from reference input to deviation. We take the weight as

$$W_1(s) := \frac{s/(10 \cdot a) + 1/b}{s/a + 1}. \quad (15)$$

This will bound the sensitivity by $b$ in a frequency range from 0 to $a$, and by 10 in range above $a/b$. We assume to give a specification that observation output tracks the reference with the error less than 1%, so we take $b = 1/100$. Then how we can make the band $a$ large without destabilizing the closed loop is determined by how wide range the model expresses the system precicely taking into account the approximation error. This can be obtained numerically as the upper limit of $a$ with which the mixed sensitivity problem is solvable as in table 2.

**Table 2** Supremum of $a$ for which the mixed sensitivity problem is solvable.

| $N$ | $a_{\text{sup}}$ |
|---|---|
| 1 | $5.04 \times 10^{-2}$ |
| 2 | $7.85 \times 10^{-2}$ |
| 3 | $1.05 \times 10^{-1}$ |
| 4 | $1.29 \times 10^{-1}$ |

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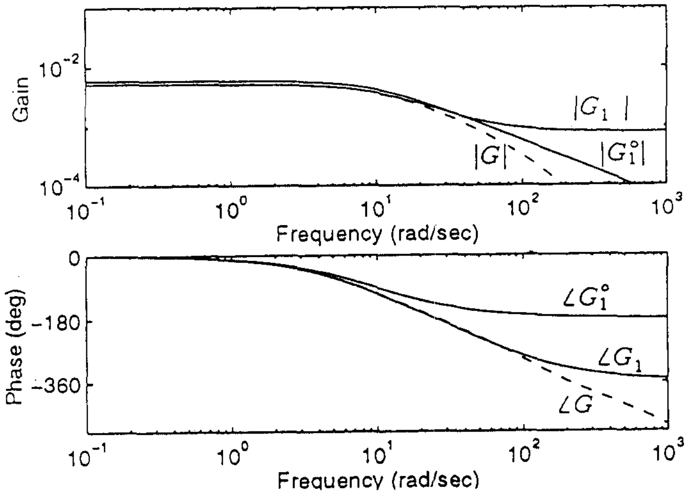


**Fig. 1** Bode plot for $G(j\omega)$, $G_1(j\omega)$ and $G_1^{st}(j\omega)$.

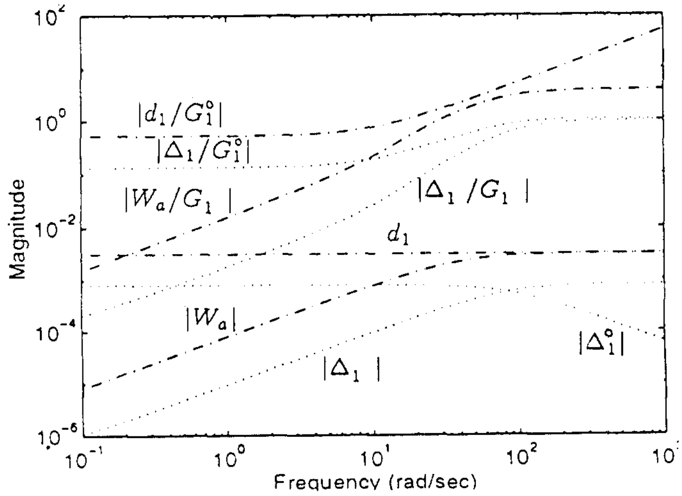


**Fig. 2** Magnitudes of additive and multiplicative error for each models.