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A Dynamic Game Approach to Robust Stabilization of Time-Varying Discrete Linear Systems via Receding Horizon Control Strategy

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Abstract

In this paper, a control law based on the receding horizon concept which robustly stabilizes time-varying discrete linear systems, is proposed. A dynamic game problem minimizing the worst case performance, is adopted as an optimization problem which should be resolved at every current time. The objective of the proposed control law is to guarantee the closed loop stability and the infinite horizon H^∞ norm bound. It is shown that the objective can be achieved by selecting the proper terminal weighting matrices which satisfy the inequality conditions proposed in this paper. An example is included to illustrate the results.

I. Introduction

The receding horizon control, which is equivalent to model predictive control, has received a lot of attentions, because it presents many advantages over infinite horizon control in some respects such as tracking property, simple computation mechanism, and I/O constraints handling [1]-[6], and it has been widely used in practical application to industrial systems. Especially, it presents a proper control strategy for time-varying systems. While we need the whole time-varying parameters throughout future time in an infinite horizon control law, we need time-varying parameters only for finite future time in a receding horizon control law.

Originally the receding horizon concept was established in the early 1970's and LQ performance index was adopted as an optimization criteria [1]. But generalization and stability were firstly given in the late 1970's by Kwon and Pearson who involved a fixed terminal equality constraint $x(t+T) = 0$ on a finite LQ optimization problem [2] [3]. Thereafter a lot of articles have been published related to the receding horizon control with an LQ optimization criteria [4]-[6].

It is well known that the dynamic game problem is naturally connected to H^∞ control problem. By employing this problem, the robustness property in the sense of disturbance attenuation and other robustness properties of H^∞ control can be achieved. In the very recent years, there have been a few attempts to adopt a dynamic game problem of minimizing worst case performance even in the receding horizon control strategy [14] [15]. In those articles, two methods were proposed for time-varying continuous linear systems. One is to obtain the receding horizon control law via involving the terminal equality constraint $x(t+T) = 0$, and the other is to obtain the control law via selecting the terminal weighting matrices which satisfy a matrix differential inequality. However there has not been such an attempt in the case of time-varying discrete linear systems.

In this paper, we propose conditions under which the receding horizon control law with game problem guarantee the closed loop stability and the infinite horizon H^∞ norm bound for time-varying discrete linear systems. Especially, we propose a new method to prove the closed loop stability in which the cost monotonicity property is utilized. By using the method, we can allow a lot of flexibilities in choosing controllers if the objective is only to guarantee the closed loop stability, and make it simple to show the closed loop stability in the receding horizon control.

II. Preliminaries

In this section, we introduce brief preliminaries on the dynamic games. The theory of dynamic game has been developed by [7] and extended to a method of H^∞ norm minimization in [8] and [9]. The theory is based on the idea that the disturbance tries to maximize while the controller is trying to minimize the performance index. In the sense that the control signal acts against the worst possible disturbances, there is a close link between the H^∞ minimization and the dynamic game approaches [8]. Such a link has been pointed out in a lot of literatures for both the case of continuous-time linear systems [11]-[13] and the case of discrete-time linear systems [8]-[10]. In this section, we introduce the finite horizon H^∞ optimization problem for discrete-time linear systems, in which the dynamic game approach is utilized to obtain a solution.

Now, we consider the following time-varying discrete linear system throughout this paper :

$$
\begin{aligned}
x_{k+1} &= A_k x_k + B_k u_k + G_k w_k, \\
z_k &= \begin{bmatrix} C_k x_k \\ u_k \end{bmatrix}
\end{aligned} \tag{1}
$$

where $x_k \in R^n$ is the state, $u_k \in R^m$ the control, $w_k \in R^p$ the disturbance, $y \in R^l$ the output, and the upper case letters denote matrices with appropriate dimensions. Now, the aim is to find u_k in $l_2[0, N-1]$ such that

$$||\tau|| < \gamma, \tag{2}$$

where $||\tau||$ is the induced 2-norm from w_k in $l_2[0, N-1]$ to z_k in $l_2[0, N-1]$ and N is a positive integer. The norm of z_k in $l_2[0, N-1]$ is defined by

$$||z||_2^2 = \sum_{k=0}^{N-1} z_k' z_k.$$

Without the loss of generality, we assume that $\gamma = 1$ throughout this paper. If this is not the case, we replace G_k in (1) by $D_k = \gamma^{-1} G_k$.

Now we consider the following performance index with a finite horizon $[0, N-1]$:

$$
\begin{aligned}
J &= \sum_{k=0}^{N-1} (z_k' z_k - w_k' w_k) + x_N' Q_N x_N \\
&= \sum_{k=0}^{N-1} (x_k' C_k' C_k x_k + u_k' u_k - w_k' w_k) + x_N' Q_N x_N
\end{aligned} \tag{3}
$$

The problem is to find the sequences u_k^* and w_k^* such that brings J to a saddle-point equilibrium, in other words, minimize and maximize J. If we find such a saddle-point equilibrium, (2) is guaranteed with the control sequences u_k^* [10]. The solution for this problem with the state feedback strategy is given in the following theorem.

THEOREM 1 *[10] The two-person zero sum dynamic game described by (1) and (3) with closed loop perfect state information pattern has a unique saddle-point solution if, and only if,*

$$I - D_k' M_{k+1} D_k > 0, \quad k \in [0, N-1], \tag{4}$$

where the sequence of nonnegative definite matrices M_{k+1}, $k \in [0, N-1]$ is generated by

$$
\begin{aligned}
M_k &= C_k' C_k + A_k' M_{k+1} \Lambda_k^{-1} A_k, \quad M_N = Q_N \quad (5) \\
\Lambda_k &= I + (B_k B_k' - D_k D_k') M_{k+1}. \quad (6)
\end{aligned}
$$

Then the saddle point solutions are given by

$$u_k^* = -B_k M_{k+1} \Lambda_k^{-1} A_k x_k, \text{ and } w_k^* = D_k' M_{k+1} \Lambda_k^{-1} A_k x_k. \tag{7}$$

With the saddle point policies in (7), the saddle point value for the performance index (3) is given by

$$J^*(x_0) = x_0' M_0 x_0. \tag{8}$$

Now, based on the preliminaries in this section, we will derive main results in the following section.

III. Receding Horizon Control

We consider the following cost at the current time k with the moving horizon $[k, k+N-1]$ for the system (1)

$$J(u, w, x_k, k) = \sum_{i=k}^{k+N-1} (z_i' z_i - w_i' w_i) + x_{k+N}' Q_{k+N} x_{k+N}, \tag{9}$$

where $u \in R^{m \times N}$ and $w \in R^{p \times N}$ are inputs and disturbances over the finite horizon $[k, k+N-1]$, respectively.

We now consider the following dynamic game problem :

$$\Phi(x_k, k) \ : \min_u \max_w \ J(u, w, x_k, k). \tag{10}$$

We denote that $u^*(x_k, k) \in R^{m \times N}$, $w^*(x_k, k) \in R^{p \times N}$ are the saddle point solutions of (10), $x^*(x_k, k) \in R^{n \times N}$ is the corresponding state trajectories, and $J^*(x_k)$ is the receding horizon saddle point value, where x_k is the current state and k is the current time. And we denote that $u_j^*(x_k, k)$ and $w_j^*(x_k, k)$ for $j \in [0, N-1]$ are the $(j+1)th$ vectors of $u^*(x_k, k)$ and $w^*(x_k, k)$, respectively. If the saddle point solution of the problem (10) exists, we can obtain from (7) that

$$
\begin{aligned}
u_{i-k}^*(x_k, k) &= -B_i M_{i+1} \Lambda_i^{-1} A_i x_i, \\
w_{i-k}^*(x_k, k) &= D_i' M_{i+1} \Lambda_i^{-1} A_i x_i, \forall i \in [k, k+N-1],
\end{aligned}
$$

where M_{i+1} is obtained from (5) with $i \in [k, k+N-1]$ and $M_{k+N} = Q_{k+N}$. In this case, the saddle point value $J^*(x_k)$ at every current state is given by $x_k' M_k x_k$. The basic concept of the receding horizon control is to calculate the optimal control $u^*(x_k, k)$ with the current state x_k at every current time k, and implement $u_k = u_0^*(x_k, k)$ repeatedly. In this paper, we are interested in the state feedback strategy. Hence we just only calculate the state feedback optimal gain at every current time. Note that if the system and the terminal weighting matrix are time-invariant, the receding horizon feedback gain will be constant.

Now we assume the following in order to make the problem feasible.

Assumption 1 *The dynamic game problem $\Phi(x_k, k)$ admits the unique state feedback saddle-point policies for all k.*

Under the following assumption, the saddle point values $J^*(x_k)$ at every current state will be positive, because positive definiteness and invertability of the matrices M_k will be guaranteed [10].

Assumption 2 *The pairs* $(A_k' \; C_k')'$, $k \in [0, \infty)$ *are injective, in other words* $rank(A_k' \; C_k')' = n$, $k \in [0, \infty)$, *and* Q_{k+N} *is positive definite.*

In the following two subsections, we will propose conditions under which the closed loop stability is guaranteed and the infinite horizon H^∞ norm bound is also guaranteed, respectively.

A. Closed Loop Stability

In this subsection we show that the receding horizon control obtained via the dynamic game problem in the previous section stabilizes the system (1) under the certain conditions. In order to obtain the closed loop stability the following assumptions are included.

Assumption 3 *The terminal weighting matrix* Q_k *satisfies the following matrix difference inequality for some* $K_k \in R^{m \times n}$ *:*

$$Q_k \geq \Psi_k + C_k'C_k + K_k'K_k, \tag{11}$$

where

$$\begin{aligned} \Psi_k &= F_k'Q_{k+1}D_k(I - D_k'Q_{k+1}D_k)^{-1}D_k'Q_{k+1}F_k \\ &\quad + F_k'Q_{k+1}F_k, &(12) \\ F_k &= A_k - B_kK_k &(13) \end{aligned}$$

Assumption 4 *The system (1) is observable. In other words,* $C_i\Phi(i,k)p \equiv 0$, $\forall k \leq i < k+n$ *implies* $p \equiv 0$, *where* $\Phi(i,k) = \prod_{j=k}^{i-1} A_j$.

Now, the following theorem will show that if we select the terminal weighting matrices such that they satisfy the condition (11) and the problem $\Phi(x_k, k)$ admits the saddle point policies, the closed loop stability for the system (1) with the receding horizon control is guaranteed.

THEOREM 2 [1] *Suppose that the assumption 1,2,3,4 are satisfied. Then the receding horizon control* $u_k = u_0^*(x_k, k)$ *stabilizes the system (1).*

In [3] and [14], the terminal state constraint $x(k+N) = 0$ is involved in order to guarantee the closed loop stability. In that case, the terminal constraint can be satisfied by setting the terminal weighting matrix as $Q_{k+N} = \infty$. However, the terminal equality constraint makes the dynamic game problem infeasible in the case of the discrete time systmes, because the disturbance at the one step ahead terminal time, *i.e.* $i = k+N-1$ may be arbitrary. Moreover, $Q_{k+N} = \infty$ does not satisfy the saddle point condition (4) in the dynamic game problem. Hence we proposed an inequality condition of terminal weighting matrices. In [15], there are similar conditions for continuous time systems. However, there are some differences in the procedure of the stability proof between [15] and this paper. While the monotonicity of the Riccati equation solution and the Lyapunov function are used to show stability for the continuous-time linear systems in [14] [15], the cost monotonicity property is used to show the closed loop stability for the discrete-time linear systems in this paper. Especially, in the condition (11), there are some flexibilities, because K_k is free parameter. We guess that the procedure in this paper can be easily utilized to show stability for the case of nonlinear discrete systems under some mild conditions such as existence of dynamic game solutions at every current time for nonlinear systems and positiveness of saddle point values.

Now we propose a practical way of obtaining the terminal weighting matrices Q_k in the assumption 3 because it is somewhat difficult to obtain the matrices which satisfy the inequality condition (11) directly.

LEMMA 1 *If there exists positive definite* $Q_k, \forall k \in [N, \infty)$ *which satisfies the following recursive equation for some* $K_k \in R^{m \times n}$

$$Q_{k+1} = [F_k(Q_k - C_k'C_k - K_k'K_k)^{-1}F_k' + D_kD_k']^{-1}, \tag{14}$$

where $F_k = A_k - B_kK_k$, *then* Q_k *satisfies the condition (11).*

Now, in the following subsection, we will propose a condition under which the infinite horizon H^∞ norm bound is guaranteed.

B. Infinite Horizon H^∞ Norm Bound

Before stating the results on infinite horizon H^∞ norm bound, we derive a Riccati equation from (5) in order to obtain the similar one as in LQ optimal control theory. Now we assume that $D_kD_k' = D_{k+1}D_{k+1}'$, $\forall k \in [0, \infty)$ throughout this subsection, and introduce the following notation:

$$P_k =: (M_k^{-1} - D_kD_k')^{-1}. \tag{15}$$

Then we obtain from (5)

$$\begin{aligned} P_k - P_kD_k(I + D'P_kD)^{-1}D_k'P_k &= A_k'P_{k+1}A_k \\ -A_k'P_{k+1}B_k(I + B_k'P_{k+1}B_k)^{-1}B_k'P_{k+1}A_k &+ C_k'C_k \quad (16) \end{aligned}$$

and the following saddle point policies from (7)

$$\begin{aligned} u_k^* &= -(I + B_k'P_{k+1}B_k)^{-1}B_k'P_{k+1}A_kx_k \\ w_k^* &= D_k(I + P_{k+1}B_kB_k')^{-1}P_{k+1}A_kx_k. &(17) \end{aligned}$$

For details, see the reference [8]. Now we denote the optimal state feedback gain and closed loop system matrix as

$$\begin{aligned} K_k^* &= (I + B_k'P_{k+1}B_k)^{-1}B_k'P_{k+1}A_k \\ F_k^* &= A_k - B_kK_k^*. &(18) \end{aligned}$$

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[1] All the proofs in this paper are skipped for lack of the space

Then the Riccati difference equation (16) will be

$$P_k - P_k D_k (I + D' P_k D)^{-1} D_k' P_k$$
$$= C_k' C_k + F_k^{*\prime} P_{k+1} F_k^* + K_k^{*\prime} K_k^* \qquad (19)$$

We note that the left side of the above equation (19) is equal to M_k from the definition (15).

Now, we will show that the solutions of (5) have the monotonicity property under the following assumption.

Assumption 5 *The terminal weighting matrix $Q_k (> 0)$ satisfies the following matrix difference inequality :*

$$Q_k \geq C_k' C_k + \bar{F}_k' (Q_{k+1}^{-1} - D_k D_k')^{-1} \bar{F}_k + \bar{K}_k' \bar{K}_k, \qquad (20)$$

where

$$\bar{K}_k = (I + B_k' \bar{P}_{k+1} B_k)^{-1} B_k' \bar{P}_{k+1} A_k$$
$$\bar{F}_k = A_k - B_k \bar{K}_k$$
$$\bar{P}_k = (Q_k^{-1} - D_k D_k')^{-1}$$

LEMMA 2 *Under the assumption 5, the solutions of (5) have the following monotonicity property.*

$$M(k, \sigma + 1) \leq M(k, \sigma) \quad \forall\, k \leq \sigma \qquad (21)$$

where $M(i, j)$ denotes the solution M_i with the terminal time j.

The following lemma will show that the set of Q_k which satisfies the assumption 5 is a subset of the set which satisfies the assumption 3.

LEMMA 3 *The terminal weighting matrix Q_k such that satisfies the assumption 5, also satisfies the assumption 3.*

Now we will show that the infinite horizon H^∞ norm bound is guaranteed under the above assumption.

THEOREM 3 *Suppose that the assumption 1,2,4,5 are satisfied. Then with the receding horizon control $u_k = u_0^*(x_k, k)$, the closed loop system of (1) is stable and the bound of the H^∞ norm is guaranteed, i.e. $\|\tau\| < 1$, where τ is the closed mapping $\tau : w \rightarrow z(x_0 = 0) : l_2[0, \infty) \rightarrow l_2[0, \infty)$.*

Since the proposed control law guarantee the infinite horizon H^∞ norm bound, the robustness property in the sense of disturbance attenuation and other robustness properties of H^∞ control can be achieved.

IV. Conclusion

In this paper, we proposed a control law to guarantee the closed loop stability and guarantee the infinite horizon H^∞ norm bound for time-varying discrete linear systems. The control law is based on the receding horizon control strategy in which the finite horizon H^∞ performance index is employed as an optimization criteria on the finite horizon, and the dynamic game approach is employed to construct an H^∞ controller on the finite horizon. The closed loop stability and H^∞ norm bound can be guaranteed under the proposed conditions on the terminal weighting matrices in the finite horizon H^∞ performance index.

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