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Motion Control of Nonholonomic System with Rolling Constraint

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Abstract In this paper, we propose a control strategy for a class of nonholonomic systems. A system with nonholonomic constraint is called a nonholonomic system, and as Brockett showed, the equilibrium of such systems can not be stabilized with any continuous static state feedbacks even though the system is controllable in the sense of nonlinear. A control strategy we propose is transforming this system into time-state control form by coordinate transformation and input transformation. We will apply this control strategy to the motion control of a rigid ball that is held between two parallel plates.

Key Words Nonholonomic system, Rolling constraint, Time-state control form

1 Introduction

A system with nonholonomic constraint is called a nonholonomic system. Space robots and cars and falling cats are good examples of nonholonomic systems. The interesting properties of drift-less nonholonomic systems is that a point in the n dimensional state space can be driven by less than n inputs. This is because the nonholonomic constraint does not reduce the degree of freedom of the system. This property makes the motion control of nonholonomic systems difficult and interesting. As Brockett showed [1], the equilibrium of such systems can not be stabilized with any continuous static state feedbacks even though the system is controllable in the sense of nonlinear.

In this paper, we propose a control strategy for a class of nonholonomic systems. A control strategy we propose is transforming a nonholonomic system into a time-state control form by coordinate transformation, input transformation, and time scale transformation. We apply this control strategy to the motion control of a rigid ball that is held between two parallel plates. By the simulation, we will show the efficiency of this method.

2 Nonholonomic System

Holonomic and Nonholonomic are the terms to classify the dynamical constraint. If the motion constraint is described by the equation

$$\mathbf{h}(\boldsymbol{x}, t) = \mathbf{O} \in \Re^m \qquad (1)$$

where $\boldsymbol{x}$ is generalized coordinates and t is the time, the constraint is said to be holonomic constraint. Suppose m equations in (1) are independent, then you can solve m generalized coordinates and the whole system can be described by $n-m$ independent generalized coordinates.

If the constraint is not holonomic, it is nonholonomic, i.e.if the constraint is described as differential equations

$$\mathbf{g}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = \mathbf{O} \in \Re^m \qquad (2)$$

and can not be reduced into the form of (1). In this case, you can not reduce the generalized coordinates using the constraint. The motion constraints such as rolling constraint or the constraint by the conservative law of angular momentum is this kind of constraint. These constraints are described by differential equations of states. Systems with such constraint can be described by drift-less nonlinear state equations

$$\dot{\boldsymbol{x}} = \mathbf{P}(\boldsymbol{x})\mathbf{u}, \quad \mathbf{u} \in \Re^{n-m}, \qquad (3)$$

where $\mathbf{P}(\boldsymbol{x}) = (\mathbf{p}_1(\boldsymbol{x}) \cdots \mathbf{p}_{n-m}(\boldsymbol{x})) \in \Re^{n \times (n-m)}$.

3 Control strategy using time-state control form

Controllable linear systems can be stabilized by state feedback of the form $\mathbf{u} = \mathbf{F}\boldsymbol{x}$, but controllability of nonlinear systems does not ensure the existence of the continuous state feedback of the form $\mathbf{u} = \alpha(\boldsymbol{x})$.
Consider an n dimensional drift-less nonlinear state equation

$$\frac{d\boldsymbol{x}}{dt} = p_1(\boldsymbol{x})u_1 + p_2(\boldsymbol{x})u_2 + \cdots + p_m(\boldsymbol{x})u_m, \qquad (4)$$

where $\boldsymbol{x} \in \Re^n$ is the state and $\mathbf{u} = (u_1, \cdots, u_m)^T$ is the input and $m < n$.

Brockett showed that for such systems, there does not exist a static continuous state feedback that asymptotically stabilizes the equilibrium even if $p_i(\boldsymbol{x})$ is independent at $\boldsymbol{x} = \mathbf{0}$ [1].

The method of transformation into time-state control form is proposed to design a controller for this kind of

systems [4]- [6].
First, to the system (4), apply a coordinate transformation

$$\begin{pmatrix} \xi \\ \tau \end{pmatrix} = T(\boldsymbol{x}), \quad T(\mathbf{0}) = 0 \tag{5}$$

where $\boldsymbol{\xi} \in \Re^{n-1}and\tau \in \Re$. And now, τ is going to be a new time axis.

Generally, time axis of a state equation is the time t, but talking about stability of the system, it is shown that any monotonically increasing functions can be chosen as a time axis [5] [6]. Thus, a monotonically increasing function τ that increases as time t increases can be chosen as a new time axis. Next, apply an input transformation

$$\mu_i = V_i(\boldsymbol{x}, u_1, \cdots, u_m),\ i = 1, 2, \cdots, m. \tag{6}$$

With these transformations, the system(4) will be transformed into the following two equations.

$$\frac{d\boldsymbol{\xi}}{d\tau} = f_0(\boldsymbol{\xi}) + f_1(\boldsymbol{\xi})\mu_1 + \cdots + f_{m-1}(\boldsymbol{\xi})\mu_{m-1} \tag{7}$$

$$\frac{d\tau}{dt} = h(\boldsymbol{\xi}, \tau)\mu_m \tag{8}$$

Equation (7) is called a state control equation and (8) is called a time control equation.

If a linear approximation around an equilibrium of the system (7)

$$\begin{aligned} \frac{d\boldsymbol{\xi}}{d\tau} &= \left.\frac{\partial f_0}{\partial \boldsymbol{\xi}}\right|_{\boldsymbol{\xi}=0} \boldsymbol{\xi} + f_1(0)\mu_1 + \cdots + f_{m-1}(0)\mu_{m-1} \\ &= \mathbf{A}\boldsymbol{\xi} + b_1\mu_1 + \cdots + b_{m-1}\mu_{m-1} \end{aligned} \tag{9}$$

is controllable, there exists a static continuous state feedback that stabilizes the equilibrium [1].

The system (7)-(8) are together called a time-state control form.

Since, for the time-state control form the time axis can be designed, we can use the following strategy to converge $\boldsymbol{\xi}$ and τ to 0.

1. Give an input μ_m to make time axis increase, and as time axis increase, make state $\boldsymbol{\xi}$ sufficiently close to 0.
2. Give an input μ_m to make time axis decrease, and as time axis decrease, stabilize $\boldsymbol{\xi}$ at 0 with continuous state feedback until τ is sufficiently close to 0.

Go over this step 1 and step 2 until both $\boldsymbol{\xi}$ and τ are sufficiently close to 0.

The controller is not continuous in this strategy, but we can design controller using ordinary design method of continuous feedback.

4 Motion control of a rigid ball between two pararel plates

4.1 Model of the system

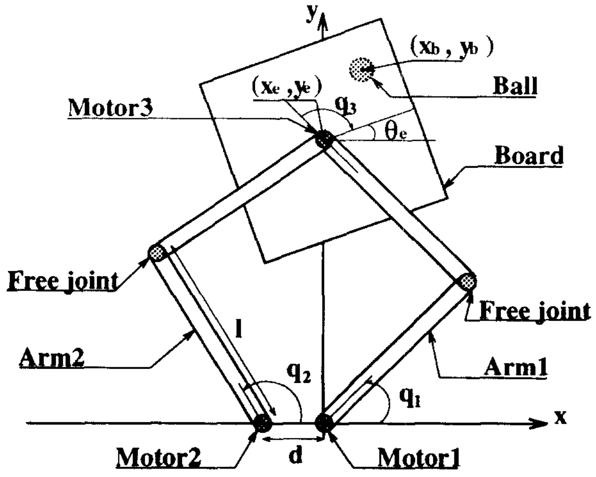

Figure 1: Model of The System

Figure 1 shows a two-link pantograph type manipulator which is placed parallel to the floor. A board, rotated by Motor3, is attached to the end of the link 2. The joints between link 1 and link 2 of both arms are free. Angular velocity of link 1 of both arms and the board can be given by Motor1,2 and Motor3 independently as input. Motor3 is fixed to the end of Arm1.

We consider a rigid ball put between the board and the floor. The ball does not move itself and its motion is dependent only of the motion of the board. Suppose the board and the floor is parallel and the board and the ball does not slide. The board is wide enough that the ball does not get out of it.

4.2 The state equation of the system

Define each variables and constants as follows.

(x_b, y_b)	:	relative position of the ball from the center of the
(x_e, y_e)	:	coordinates of the board
q_1	:	angle of link1 of Arm1
q_2	:	angle of link1 of Arm2
θ_e	:	rotating angle of the board
(x_g, y_g)	:	goal point
d	:	distance between two arms

The origin of coordinates is the root of Arm1.
Take the state of the system χ as below,

$$\chi = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \end{bmatrix} = \begin{bmatrix} x_b \\ y_b \\ x_e - x_g \\ y_e - y_g \\ \theta_e \end{bmatrix} \quad (10)$$

and the input $\mathbf{u}$ as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \dot{x}_e \\ \dot{y}_e \\ \dot{\theta}_e \end{bmatrix} \quad (11)$$

Note that real input is an angular velocity of the arm but here we give a velocity of end point as input. This makes the equation simple, so we use this input first and then transform to the real input at the end.

The rolling constraint of the ball and the board can be written as

$$\dot{x}_b = -\frac{1}{2}y_b\dot{\theta}_e - \frac{1}{2}\dot{x}_e \quad (12)$$

$$\dot{y}_b = \frac{1}{2}x_b\dot{\theta}_e - \frac{1}{2}\dot{y}_e \quad (13)$$

Hence, the state equation of the system is

$$\frac{d}{dt}\begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} -\frac{1}{2}\chi_2 \\ \frac{1}{2}\chi_1 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_3 \quad (14)$$

Now our interest is to stabilize (14) at the equilibrium $\chi = 0$.

4.3 Designing controller

4.3.1 Transformation to time-state control form

System(14) is a 5 dimension 3 input drift-less nonlinear system.

This system is a nonholonomic system and it is shown that there does not exist a static continuous state feedback that asymptotically stabilizes the equilibrium even the system is controllable [1]. Thus we apply the method of transformation into the time-state control form to the system as written in Section3.
Take new states $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4)^T$ as

$$\boldsymbol{\xi} \triangleq \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{bmatrix} \quad (15)$$

and new time axis τ as

$$\tau \triangleq \chi_5 \quad (16)$$

Also apply input transformation of

$$\mu_1 \triangleq \frac{u_1}{u_3} \quad (17)$$

$$\mu_2 \triangleq \frac{u_2}{u_3} \quad (18)$$

$$\mu_3 \triangleq u_3 \quad (19)$$

then the system is transformed into time-state control form.

$$\frac{d\boldsymbol{\xi}}{d\tau} = \begin{bmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad (20)$$

$$\frac{d\tau}{dt} = \mu_3 \quad (21)$$

Note that the system (20) is a controllable linear system of $\boldsymbol{\xi}$ and that $\chi \to 0$ when $\boldsymbol{\xi} \to 0$ and $\tau \to 0$, so our interest now is stabilizing both $\boldsymbol{\xi}$ and τ at 0.

4.3.2 Designing controller

Using LQ optimal control, derive a state feedback $\mu = \mathbf{F}\boldsymbol{\xi}$ for each τ increasing and τ decreasing. With this controller and the strategy of Section3 all states can be asymptotically stabilized at the equilibrium.

4.3.3 Input and state transformation

Originally, the input of the system is the velocity of the manipulator, so we have to transform the input.

Transformation from (μ_1, μ_2, μ_3) to $(\dot{q}_1, \dot{q}_2, \dot{q}_3)$ is derived as below.
Clearly, $(\dot{x}_e, \dot{y}_e)$ is

$$\begin{bmatrix} \dot{x}_e \\ \dot{y}_e \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \mu_3. \quad (22)$$

and let ν_1 and ν_2 be

$$\begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad (23)$$

then $(\dot{q}_1, \dot{q}_2)$ can be denoted as

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \dot{x}_e \\ \dot{y}_e \end{bmatrix} = \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \mu_3 \quad (24)$$

where $\mathbf{J}$ is the manipulator's Jacobian. Also, θ_e is

$$\theta_e = q_2 + q_3, \quad (25)$$

thus $\dot{q}_3$ is denoted as

$$\dot{q}_3 = \dot{\theta}_e - \dot{q}_2 = \mu_3 - \nu_2\mu_3. \tag{26}$$

Finally, we get

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} \nu_1 \\ \nu_2 \\ 1 - \nu_2 \end{bmatrix} \mu_3. \tag{27}$$

In the next Section, the simulation results using this input are shown.

5 Simulation Result

The simulation result is shown in Figure 2.

The goal point of the ball and the board is (0,0.35)[m] and $d = 0.05$[m], $l = 0.238$[m]. The velocity of the new time axis μ_3 is 2[rad/sec].

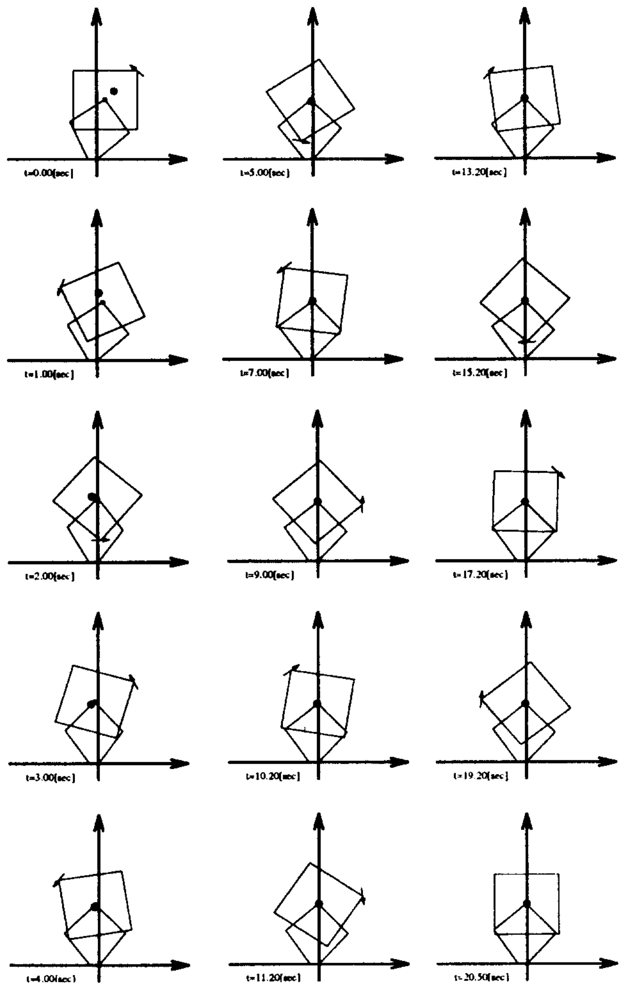

Figure 2: Simulation result

6 Conclusion

We proposed a method to control a system with a class of nonholonomic constraint. The example we showed has a rolling constraint and the constraint was a nonholonomic constraint. We could transform this system into the time-state control form and it appeared to be a controllable linear system. We used the ordinary linear control theory to stabilize the system.

The effectiveness of the method was shown form the simulation results.

References

[1] R.W.Brokett: Asymptotic Stability and Feed back Stabilization.Differential Geometric Control Theory Progress in Mathematics, Vol.27, pp.181-191(1983).

[2] Nakamura: Lecture. Nonholonomic Systems, (1st-5th). The Journal of Robotics Society of Japan, Vol.11 No.4-7, Vol.12 No.2(1993-1994)(In Japanese)

[3] Isidori: Nonlinear Control Systems, second edition, Springer-Verlag, (1989).

[4] Sampei, Tamura, Kobayashi: A Control Strategy for a Class of Nonlinear Systems which can not be Stabilized with Continuous State Feedback, Proceedings of The 36th Japan Joint Automatic Control Conference, pp29-30(1993)(In Japanese)

[5] Sampei, Itoh: Path Planning and Path Tracking Control of Mobile Robot Using Nonlinear System Theory --Parking Control Using Forward and Backward Movement--, The Institute of Systems, Control and Information Engineers, 6-1,pp37-47(1993)(In Japanese)

[6] Sampei, Ishikawa, Kiyota, Koga: Time-State Control Form and its Application to Non-Holonomic Systems, Proceedings of The 12th Annual Conference of Robotics Society of Japan, No2, pp437-438 (1994)(In Japanese).