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Differential Game of Approach with an Inertial Evader and Two Noninertial Pursuers

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Abstract. This paper is concerned with a coplanar pursuit-evasion game of one inertial evader and two identical noninertial pursuers. The terminal time is fixed and the payoff is the distance between the evader and the nearest pursuer when the game is terminated. The value functions and the strategies is constructed for all the game surface. To get a value function, we use the generalization of the Bellman-Isaacs fundamental equation.

Key Words. Pursuit-evasion game, value function, programmed maximin function, dispersal surface, fundamental equation.

1. Introduction

Diffrential games which treat many pursuers and evaders is not only an interesting problem but also an important subject which can be applied in many situations. It has been studied extensively from the past. In the present paper, we consider one evader which can be accelerated to any directions with constant acceleration and two pursuers which have the same constraints each other, that is, each pursuers can move any directions with constant velocity. Evader wants to maximize the distance between itself and pursuers, but pursuers hope to approach evader to catch evader within the terminal time. The game is played on the plane and terminal time is fixed.

In the diffrential game under consideration, the value function coincide with the programmed maximin function, which is a continuous, piecewise-smooth function. But in this game, there is a dispersal surface. So, we can not use the Bellman-Iasscs fundamental equation, because on this singular surface, the value function is nondifferentiable (Ref. 1). To prove the fact that this programmed maximum function is the value function of the game under consideration, we use a generalization of the Bellman-Isaacs fundamental equation (Ref. 2)

After proving that the programmed maximin function is the value function of the game, we construct the value function and strategies for all the game surface which is subdivided into four domain. In Section 2, the game is formulated and in section 3, game surface is subdivided and the programmed maximin function is defined. In section 4, all the values of the game and strategies is constructed. All game is considered in the first quadrant. The other quadrants have the same attitude.

2. Problem Formulation

The motion of the evader $E(x)$ is described by the equations

$$\dot{x}_1 = x_3, \quad \dot{x}_3 = v_1, \quad \dot{x}_2 = x_4, \quad \dot{x}_4 = v_2. \quad (1a)$$

And, pursuers $P_i(z^i)$ is described by the equations

$$\dot{z}_1^1 = u_1^1, \quad \dot{z}_2^1 = u_2^1, \quad \dot{z}_1^2 = u_1^2, \quad \dot{z}_2^2 = u_2^2. \quad (1b)$$

The control vectors of the evader and the pursuers are constrainted as

$$[(v_1)^2+(v_2)^2]^{1/2} \le \nu, \ [(u_1^i)^2+(u_2^i)^2]^{1/2} \le \mu, \ i=1, 2. \quad (1c)$$

Game is played in the time interval $[t_0, \theta]$. The terminal time θ at which the game ends is fixed. The payoff functional σ is the distance between the evader and the pursuer closest to it at the time instant θ; i.e.,

$$\sigma = \min_i [(z_1^i(\theta)-x_1(\theta))^2 + (z_2^i(\theta)-x_2(\theta))^2]^{1/2}. \quad (2)$$

As can be seen at (1a)~(1b), this model has eight states. It is exactly what we want to treat in this game, but too complex. So we introduce the change of variables,

$$y_j = x_j + (\theta - t)x_{j+2}, \quad j = 1, 2,$$

with reference to the region of attainability of the inertial evader. The state equations (1a), (1b) and (1c) are changed as,

$$\dot{y}_1 = (\theta - t)v_1, \quad \dot{y}_2 = (\theta - t)v_2, \quad (3a)$$

$$y_j(t_0) = x_j(t_0) + (\theta - t)x_{j+2}(t_0) \quad (3b)$$

$$\sigma = \min_i [(y_1^i(\theta) - z_1(\theta))^2 + (y_2^i(\theta) - z_2(\theta))^2]^{1/2} \quad (3c)$$

The y state is the center of the attainability region at which the evader can reach in terminal time θ. The changed payoff (3c) is the same value to the original payoff (2).

Let us introduce a moving coordinate system (q_1, q_2). We direct the axis q_1 from the position of the first pursuer $P_1(z^1)$ to the position of the second pursuer $P_2(z^2)$. And we direct the ordinate axis q_2 through the midpoint of the segment $[P_1, P_2]$ perpendicular to it (See Fig 1). The dynamics of the states of relative coordinates is described by the following differential equations.

$$\dot{x} = (\theta - t)v_1 - (u_1^1 + u_1^2)/2 + y(u_2^2 - u_2^1)/2z \quad (4a)$$

$$\dot{y} = (\theta - t)v_2 - (u_2^1 + u_2^2)/2 - x(u_2^2 - u_2^1)/2z \quad (4b)$$

$$\dot{z} = (u_1^2 - u_1^1)/2 \quad (4c)$$

Constraints on the control of the players coincide with (1e)

$$[(v_1)^2+(v_2)^2]^{1/2} \le \nu, \ [(u_1^i)^2+(u_2^i)^2]^{1/2} \le \mu, \ i=1, 2. \quad (4d)$$

The payoff functional is

$$\sigma = [(z(\theta) - |x(\theta)|)^2 + y^2(\theta)]^{1/2} \quad (4e)$$

In the system (4), the vector $v = (v_1, v_2)$ is absolute velocity of the evader E, and the vectors $u^i = (u_1^i, u_2^i)$ are the proportional to the velocity of the pursuers P_i with the factor $\theta - t$. In this system, the point (x, y) is the position of the evader and $(0, \pm z)$ is the position of each pursuers. Clearly $(\dot{x}, \dot{y})$ is the relative velocity of the evader E and $\dot{z}$ is one half of the relative velocity of the pursuer in the moving coordinate system (q_1, q_2). Game is played using system (4a)~(4e). Let us define $\xi(t, x(t), y(t))$ as the position of the evader and $\zeta_i(t, z(t))$ as the positions of each pursuers at time t.

At time t_0, which is start time of the game, let us freeze the axis (q_1, q_2). We will call this coordinate as a fixed coordinate system (η_1, η_2). All trajectories and game values will be displayed in this coordinate system. The domain of attainability $G_e(t_0, \theta)$ of the evader from the position $\xi(t_0, x(t_0), y(t_0))$ at the instant θ is a circle of radius $r(t_0)=\nu(\theta-t_0)^2/2$, with center at the point $\xi(t_0, x(t_0), y(t_0))$. The domain of attainability $G_p^i(t_0, \theta)$ of the pursuers from the position $\zeta_i(t_0, z(t_0))$ is a circle of radius $R(t_0)=\mu(\theta-t_0)$, with center at the point $\zeta_i(t_0, z(t_0))$. All positions and the domains of attainability are displayed at Fig 1.

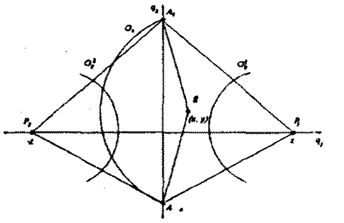

Fig. 1. Moving coordianate system.

3. Partitioning the Game Domain

At the instant t_0, the evader E is located at the position $\xi(t_0, x(t_0), y(t_0))$ and the attainability domain $G_e(t_0, \theta)$ of the evader intersect the axis q_2 at the point $A_1(0, d_1)$ and $A_2(0, d_2)$ if attainability domain is sufficiently large. Then following two cases are possible:

(i) the quadrangle $P_1A_1P_2A_2$ does not exist, or $\xi(t_0, x(t_0), y(t_0)) \notin int(P_1A_1P_2A_2)$

(ii) $\xi(t_0, x(t_0), y(t_0)) \in int(P_1A_1P_2A_2)$.

Case (i) is occured when the following condition is satisfied.(Fig.2) $\cos\theta_1 \geq \cos\theta_2$, i.e.,

$$|x|/\frac{\nu}{2}(\theta-t)^2 \geq (z-|x|)/[(z-|x|)^2+y^2]^{1/2} \quad (5)$$

The situation correspond to case (ii) is described by the opposite inequility.

By the inequility (5) we can get a three dimensional domain, D^{11}. In this domain the game is restricted to one pursuer-one evader game because the availability of the second pursuer is not beneficial. Only nearest pursuer at initial time and evader play a role in a game. In case (ii), by chaging the inequility in equation (5), a three dimensional D^{21} is constructed. D^{11} and D^{21} is saparated by S_1. In case (i), the value function, ρ^{21}, is defined as following,

$$\rho^{21}=\rho^{11}, \qquad \xi(t_0, x(t_0), y(t_0)) \in D^{11}.$$

3.1 One to One Pursuit Evasion Game

When evader's position, $\xi(t, x(t), y(t))$, is in D^{11}, we can subdivide this region into DR^{11} (regular domain) and DS^{11} (singular domain). The regular domain, DR^{11} is constructed if the following equations

$$(t-t_0)^2-2(t-t_0)(\theta-t_0-\mu/\nu)-2d/\nu=0 \quad (6a)$$

$$d=[(z-|x|)^2+y^2]^{1/2} \quad (6b)$$

do not have roots in the segment $[t_0, \theta]$ (see Appendix A). DS^{11} is made by the other condition, i.e., the equations (6a), (6b) have roots in $[t_0, \theta]$. Clearly the equation (6a) always have a one real root which is larger than t_0. So, the root must have smaller value than θ if evader is located inside the domain DS^{11}. The surface S_0 is a boundary between the subdomain DR^{11} and DS^{11}. If evader is located at the surface S_0, not only evader has the same position with pursuer but also has the same velocity at the terminal time θ. If evader is located in DR^{11}, pursuer nerver catch evader for any terminal time.

Let γ^{11} is the programmed maximin function of the one to one game.

$$\gamma^{11}=\max(\hat{\gamma}, 0) \quad (7a)$$

$$\hat{\gamma}=[(z(t_0)-x(t_0))^2+y^2(t_0)]^{1/2}-\frac{\nu(\theta-t_0)^2}{2}-\mu(\theta-t_0) \quad (7b)$$

3.2 Two to One Pursuit Evasion Game

When evader is positioned in D^{21}(case (ii)), its optimal strategy is diffrent from the case (i). D^{21} is also subdevided into the regular domain, DR^{21}, and the singular domain, DS^{21}. If the equations

$$(t-t_0)^2-2(t-t_0)(\theta-t_0-\mu\sin\beta_0/\nu\sin\alpha_0)-2|y(t_0)|/\nu\sin\alpha_0=0 \quad (8a)$$

$$\sin\alpha_0=[V^2-x^2(t_0)]^{1/2}/V \quad (8b)$$

$$\sin\beta_0=\frac{y(t_0)+x(t_0)\tan\alpha_0}{[(y(t_0)+x(t_0)\tan\alpha_0)^2+z^2(t_0)]^{1/2}} \quad (8c)$$

$$V=\nu(\theta-t_0)^2/2 \quad (8d)$$

do not have roots in the segment $[t_0, \theta]$, then the evader is in DR^{21}, i.e., pursuer can not catch evader permanently (see Appendix A). The other case is DS^{21}. If the quadratic equations (8) have a root in $[t_0, \theta]$ then two pursuers will catch the evader within the final time instantaneously. The surface S_2 is the boundry of the regular domain, DR^{21} and singular domain, DS^{21}. If the evader's initial position is located at surface S_2, pursuers catch evader in terminal time θ and they have the same speed with evader. At D^{21}, we denote γ^{21} is the programmed maximin function of this game. γ^{21} is defined as follows:

$$\gamma^{21}=\max\{\gamma_1, \gamma_2, 0\} \quad (9a)$$

where γ_k, $k=1, 2$, are smooth functions, defined via the formulas

$$\gamma_k=[z^2(t_0)+a_k^2]^{1/2}-\mu(\theta-t_0) \quad (9b)$$

$$a_k=y(t_0)\pm[(\nu(\theta-t_0)^2/2)^2-x^2(t_0)]^{1/2} \quad (9c)$$

All domain is shown in Fig 3.

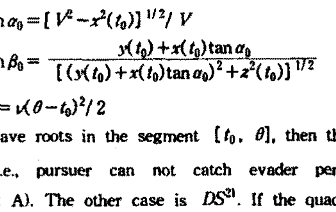

Fig. 2. The positions of the evader and pursuers.

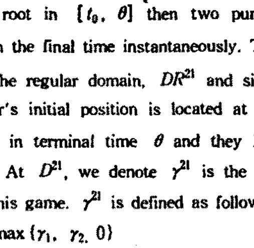

Fig. 3. Structure of coordinate space on the plane (x, y) for $\mu=3$, $\nu=1$, $\theta=3$, $z=5$.

4. Value function and strategy

The programmed maximin function in the domain D^{11} is the value function of the game. In the domain DS^{11}, value function has a zero value, trivially. As can be seen in section 3, the programmed maxmin function γ^{21} is the value function in the domain D^{21}. Finally the value function in the region DS^{21} has zero value, also.

At the surface S_1, value function can be defined as γ^{11} or γ^{21}. They have the same value in S^1. The value function of the all the game surface is defined as

$$\rho^{21}=\begin{cases}\gamma^{11}, & (t,\xi)\in D^{11},\\ \gamma^{21}, & (t,\xi)\in D^{21},\\ \gamma^{11} \text{ or } \gamma^{21}, & (t,\xi)\in S_1.\end{cases} \quad (10)$$

The strategy is diffrent according to the position of the evader. If the evader is located at one-to-one game region (D^{11}), the strategy of the evader and the nearest pursuer are defined as

$$v_1=\nu\cos\delta_0,\ v_2=\nu\sin\delta_0, \quad (11a)$$

$$u_1^1=\mu\sin\delta_0, \qquad u_2^1=\mu\sin\delta_0. \quad (11b)$$

$$\sin\delta_0=(x(t_0)-z(t_0))\,/\,[\,(x(t_0)-z(t_0))^2+y^2(t_0)]^{1/2} \quad (11c)$$

$$\cos\delta_0=y(t_0)\,/\,[\,(x(t_0)-z(t_0))^2+y^2(t_0)]^{1/2} \quad (11d)$$

and the other pursuer's moving is not concern. When evader is located at two-to-one game region (D^{21}), the strategy of the evader and each pursuers are defined as

$$v_1=\nu\cos\alpha_0,\ \ v_2=\nu\sin\alpha_0, \quad (12a)$$

$$u_1^1=-u_1^2=\mu\sin\beta_0, \quad u_2^1=u_2^2=\mu\sin\beta_0 \quad (12b)$$

$$\sin\alpha_0=[\,V^2-x^2(t_0)]^{1/2}/\,V \quad (12c)$$

$$\sin\beta_0=\frac{y(t_0)+x(t_0)\tan\alpha_0}{[\,(y(t_0)+x(t_0)\tan\alpha_0)^2+z^2(t_0)]^{1/2}} \quad (12d)$$

$$V=\nu(\theta-t_0)^2/\,2 \quad (12e)$$

The surface S_1 which divides D^{11} and D^{21} is the barrier between two diffrent strategy. The one-to-one strategy is simple. At D^{11}, it is a optimal strategy that evader and pursuer move direct line passing each other. When evader is in DS^{21}, it is optimal for evader to be catched by two pursuers instantaneously and in DR^{21}, it is optimal that evader is positioned at a same distance with each pursuers when it is catched. So evader move toward the point A_1 (or A_2 if it is located below the q_2 axis. see Fig 1), and each pursuers also move toward the point A_1 (or A_2).)

The value of the game in the all game domain is described at Fig 4. The other three quadrants has exactly the mirror image of the first quadrant.

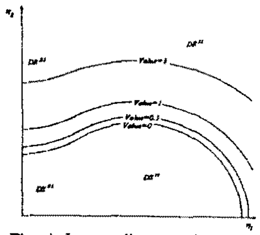

Fig. 4. Isocost lines of the game.

5. Conclusions

In this paper, the game of an inertial evader and two noninertial pursuers is solved. The value function of the game is found by using the programmed maximin function. The strategy and value function are constructed for all the game surface. The game surface is divided into an one-to-one game region and a two-to-one game region. Each region is again subdivided into a regular domain where it is impossible for pursuers to catch evader and a singular domain where pursuers can catch evader. In each region the game is played with diffrent strategies and game values.

The singular surface is also described. When evader is located between two pursuers, evader's trajectory toward each sides makes dispersal surface.

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Appendix A

(I) Finding the Capture Region in the One to One Game

In one to one game, the evader and the pursuer nearest to it move always on the line connecting each other. For the pursuer to catch the evader, it must be satisfied that they are located at the same position within the terminal time. Equation (A1) is denote this condition (see Fig 2. to get a more graphical appreciation).

the distance between the evader and the pursuer
+ the total moved distance of the evader to time t
= the total moved distance of the pursuer to time t. (A1)

$$d+\int_{t_0}^{t}(\theta-\tau)\nu d\tau=\int_{t_0}^{t}\mu d\tau \quad (A2)$$

where

$$d=[\,(z(t_0)-x(t_0))^2+y^2(t_0)]^{1/2}. \quad (A3)$$

Expanding above equation (A1),

$$\frac{\nu}{2}(t-t_0)^2-\nu(t-t_0)(\theta-t_0-\frac{\mu}{\nu})-d=0 \quad (A4)$$

(A3), (A4) make equations (6a) and (6b).

(II) Finding the Capture Region in the Two to One Game

In this case, situation is somewhat diffrent. The evader moves toward the extremal point A_1 (or A_2) and each pursuers also move toward that point(see Fig. 2). At this situation, we consider the projection of the evader and the pursuer to the η_2 axis. If two projections of the evader and the pursuer coincide, pursuer can catch evader within the terminal time. Let us denote these as equations.

the projected distance fo the evader and the pursuer
+ the projected total distance of the evader to time t
= the projected total distance of the pursuer to time t. (A5)

$$|y(t_0)|+\int_{t_0}^{t}(\theta-\tau)\nu d\tau\sin\alpha_0=\int_{t_0}^{t}\mu d\tau\cos\beta_0. \quad (A6)$$

where

$$\sin\alpha_0=[\,V^2-x^2(t_0)]^{1/2}/\,V. \quad (A7)$$

$$\sin\beta_0=\frac{y(t_0)+x(t_0)\tan\alpha_0}{[\,(y(t_0)+x(t_0)\tan\alpha_0)^2+z^2(t_0)]^{1/2}}. \quad (A8)$$

$$V=\nu(\theta-t_0)^2/\,2. \quad (A9)$$

Expanding above equation (A5),

$$\frac{\nu}{2}(t-t_0)^2\sin\alpha_0-\nu(t-t_0)(\theta-t_0-\frac{\mu\sin\beta_0}{\nu\sin\alpha_0})\sin\alpha_0-|y(t_0)|=0. \quad (A10)$$

(A7)~(A10) make the equations (8a)~(8d).