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Nonlinear Control of Underactuated Mechanical Systems Via Feedback Linearization and Energy Based Lyapunov Function

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Abstracts In this paper a nonlinear control strategy via feedback linearization and energy based Lyapunov function for underactuated mechanical systems is investigated. Underactuated mechanical system is a system of which the number of actuators is less than the number of degrees of freedom. Developed algorithm is applied to a crane system of grab operation. Positioning of the trolley as well as swing-up of the pendulum to the up-right position including maintaining the sway angle at some desired degree are demonstrated. Simulations are provided.

Keywords Control Theory, Feedback Linearization, Underactuated System, Robotics, Lyapunov Function, Crane System.

1. Introduction

Underactuated mechanical system is a system of which the number of actuators is less than the number of degrees of freedom. This class of systems includes crane systems, gymnastic robots such as the Acrobot, the classical cart-pole system, mobile robot systems in which a manipulator arm is attached to a mobile platform, a space platform, and in some undersea vehicles. Further if we include the flexibility of the joint in the mathematical model, then the flexibility itself remains as underactuated part of the system dynamics. In the case that the number of actuators is same with the degrees of freedom, many different control laws are easily available.

This research was motivated from the automated grab operation when unloading ships with bulk materials. When the grab reaches the target (bunker or ship hold), the sway motion generally needs to be suppressed as in the case of container crane. However when removing materials remaining at the corner in the bunker, the grab needs necessarily to swing to reach there. Further the method in the paper can swing up the pendulum to the upright position if it is applied a traditional cart-pendulum system.

The approach in the paper is based on control of energy level of the sway motion. When the energy is poorly controllable from the input, the input is instead used to position the trolley.

In this paper, we propose an autonomous nonlinear state feedback control law to regulate the trolley position as well as the swinging energy of the pendulum. Namely nonlinear feedback linearization and energy based Lyapunov function for stability analysis. The resulting closed-loop system will possess a stable periodic orbit.

This paper is organized as follows. We illustrate the equations of motion for the general underactuated mechanical systems and construct the partial feedback linearization control law following the work of (Spong, 1996) in Section 2. We apply the nonlinear controller to a crane system of pendulum and trolley motion in Section 3, as an example of underactuated mechanical system. The proposed nonlinear controller is to regulate the passive joints as well as the active joints and stability of the controller is also investigated. In Section 4, simulations are provided.

2. Underactuated Mechanical Systems

Consider an n - degrees of freedom open loop mechanism with joint variables $q^1,\dots,q^n$. It is assumed that each joint has a single degree of freedom and only $m<n$ joints are active, i.e. the only m joints have ability to move neighboring link and $l=n-m$ joints have no actuation. Each joint which is capable of actuation is called an active joint. The remaining $l=n-m$ joints with no actuation are called passive joints. A general underactuated mechanical system is shown in Fig. 1.

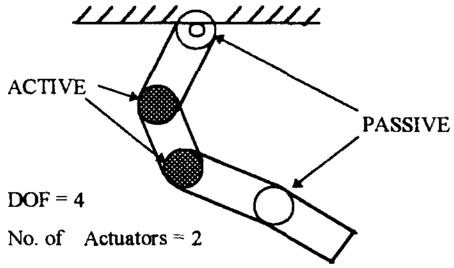

Fig. 1. Schematics of an Underactuated Mechanical System.

After obtaining the equations of motion through the Lagrange method, we can rearrange equations so that the coordinates for passive joints are grouped in $q_1 \in R^l$ and the coordinate for active joints are grouped in $q_2 \in R^m$. Hence a general underactuated mechanical system is represented as

$$M_{11}\ddot{q}_1 + M_{12}\ddot{q}_2 + C_1(q,\dot{q}) + G_1(q) = 0 \quad (1)$$

$$M_{21}\ddot{q}_1 + M_{22}\ddot{q}_2 + C_2(q,\dot{q}) + G_2(q) = f \quad (2)$$

where the vector functions $C_1(q,\dot{q}) \in R^l$ and $C_2(q,\dot{q}) \in R^m$ contain Coriolis and centrifugal terms, the vector functions $G_1(q) \in R^l$ and $G_2(q) \in R^m$ contain gravitational terms, and $f \in R^m$ represents the input generalized force produced by the m actuators at the active joints. Hence like a completely controllable robot, the dynamic equations of underactuated system are also written as

$$M(q)\ddot{q} + C(q,\dot{q}) + G(q) = Bf \quad (3)$$

where

$$q = [q_1, q_2]^T,\ M(q) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},\ B = \begin{bmatrix} O_{l\times m} \\ I_{m\times m} \end{bmatrix}$$

Note that M is a symmetric, positive definite inertia matrix. For notational simplicity we will henceforth not write the explicit dependence on q in M, C and G. It is emphasized again that the dynamic of underactuated systems is represented as standard dynamics of n link robots except that there is no control input to the first l equations.

Now consider the equation (1). The term M_{11} is an invertible $l \times l$ matrix as a consequence of the uniform positive definiteness of the robot inertia matrix M in (3). Therefore we may solve for $\ddot{q}_1$ in equation (1) as

$$\ddot{q}_1 = -M_{11}^{-1}(M_{12}\ddot{q}_2 + C_1 + G_1) \quad (4)$$

and substitute the resulting expression (4) into (2) to obtain

$$\overline{M}_{22}\ddot{q}_2 + \overline{C}_2 + \overline{G}_2 = f \quad (5)$$

where the terms $\overline{M}_{22}$, $\overline{C}_2$ and $\overline{G}_2$ are given by

$$\overline{M}_{22} = M_{22} - M_{21}M_{11}^{-1}M_{12},$$

$$\overline{C}_2 = C_2 - M_{21}M_{11}^{-1}C_1,$$

$$\overline{G}_2 = G_2 - M_{21}M_{11}^{-1}G_1.$$

Note that the $m \times m$ matrix $\overline{M}_{22}$ is itself symmetric and positive definite as shown in (Gu et al., 1993).

A partial feedback linearizing controller can therefore be defined for equation (5) according to

$$f = \overline{M}_{22}u + \overline{C}_2 + \overline{G}_2 \quad (6)$$

where $u \in R^m$ is an additional control input yet to be defined. The complete system up to this point may be written as

$$M_{11}\ddot{q}_1 + C_1 + G_1 = -M_{12}u \quad (7)$$

$$\ddot{q}_2 = u \quad (8)$$

Since the input-output relation from u to q_2 in equation (8) is linear, the active part of equation (2) has been completely linearized. The complete system therefore has m-vector relative degree $(2,\ldots,2)^T$ (Isidori, 1989) with respect to the output q_2. The nonlinear equation (7) now represents the internal dynamics whose stability has to be investigated.

We first apply a feedback control u as following

$$u = -k_1 q_2 - k_2\dot{q}_2 + k_3\overline{u} \quad (9)$$

where k_1, k_2 and k_3 are constant and $\overline{u}$ is a design variable. Note that the linear subsystem (8) is asymptotically stable for $\overline{u} = 0$. The remaining design problem then is how to choose the additional term $\overline{u}$. Detailed procedure including stability will be addressed in the next section through the example of crane.

In the case of full, or exact, feedback linearization the control design problem becomes complete once the system is linearized. However for partial, or input-output, linearization with asymptotically stable zero dynamics only local stabilization is achieved. Global or semi-global stability requires further investigation such as peaking, etc. For the underactuated systems, the collocated linearization approach above may results in non-minimum systems having unstable zero dynamics. Then the second stage control, i.e. the design of the outer loop terms u in (8), more to point, the choice of $\overline{u}$ in (9), becomes nontrivial. In the next section by using explicit expressions for (1) and (2) a partial feedback linearization is investigated for the design of $\overline{u}$ that combines the high gain and energy based Lyapunov function.

3. Nonlinear Control of a Crane System

Consider a trolley and pendulum system shown in Fig. 2.

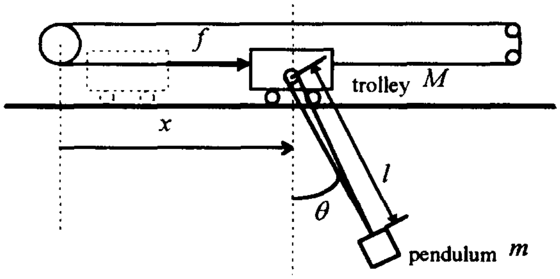

Fig. 2. Trolley and Pendulum System.

Using the Lagrange mechanics, the equations of motion with the assumptions of massless rod and point mass are obtained.

$$ml^2\ddot{\theta} + ml\cos\theta\ddot{x} + mgl\sin\theta = 0 \quad (10)$$

$$ml\cos\theta\ddot{\theta} + (M+m)\ddot{x} - ml\sin\theta\dot{\theta}^2 = f \quad (11)$$

where f is the external force applied to the trolley and g is the gravitational acceleration. Note that equations (10) and (11) are in the form of the underactuated mechanical system (1) and (2), respectively. Specifically the terms in equations (1) and (2) correspond to

$$M_{11} = ml^2,\ M_{22} = M + m,$$

$$M_{12} = M_{21} = ml\cos\theta,$$
$$C_1 = 0,\ C_2 = -ml\sin\theta\dot\theta^2,$$
$$G_1 = -mgl\sin\theta,\ G_2 = 0$$

We now apply the partial feedback linearization control. Solve the equation (10) for $\ddot\theta$ and substitute it into the equation (11). Then we obtain

$$(M + m\sin^2\theta)\ddot x - m\sin\theta(g\cos\theta + l\dot\theta^2) = f \qquad (12)$$

If we take the input to the trolley f as

$$f = (M + m\sin^2\theta)u - m\sin\theta(l\dot\theta^2 + g\cos\theta) \qquad (13)$$

where u is the additional control input to be decided as was introduced in equation (6). Then the resulting system can be written as

$$l\ddot\theta + g\sin\theta = -\cos\theta u \qquad (14)$$
$$\ddot x = u \qquad (15)$$

Let u have the form as

$$u = -k_1 x_e - k_2\dot x_e + k_3\bar u \qquad (16)$$

where $\bar u$ is a new design variable and $x_e = x - x_d$, where x_d is a desired trolley position which is constant. With the outer loop term given by (16), the closed loop system becomes

$$l\ddot\theta + g\sin\theta = k_1\cos\theta x_e + k_2\cos\theta\dot x_e - k_3\cos\theta\bar u \qquad (17)$$
$$\ddot x_e + k_2\dot x_e + k_1 x_e = k_3\bar u \qquad (18)$$

In order to assure the stability, consider the following function.

$$V(\theta,\dot\theta,x,\dot x) = \frac{1}{2}m(l\dot\theta)^2 + mgl(1-\cos\theta) + \frac{1}{2}mlk_1k_2x_e^2 + \frac{1}{2}mlk_2\dot x_e^2 \qquad (19)$$

And let V_d denote the desired energy level of the system to be regulated, which assumes constant value. For instance if we want to swing up the pendulum to the up-right position when a target trolley position is reached, $V_d = 2mgl$ will be chosen. Define $V_e = V - V_d$. Then differentiating (19) with respect to t and by utilizing (17) and (18) yields

$$\dot V = \dot V_e = ml\cos\theta\dot\theta(k_1x_e + k_2\dot x_e)^2 + mlk_3(k_2\dot x_e - \cos\theta\dot\theta)\bar u - mlk_2{}^2\dot x_e$$
$$= \text{Term 1} + \text{Term 2}\ \bar u - \text{Term 3} \qquad (20)$$

Hence by choosing

$$\bar u = -(k_2\dot x_e - \cos\theta\dot\theta)V_e + \begin{cases} -\dfrac{\text{Term 1}}{\text{Term 2}}, & \text{if } |\text{Term 2}| \ge \varepsilon \\ -\dfrac{\text{Term 1}}{\varepsilon}, & \text{if } 0 < \text{Term 2} < \varepsilon \\ \dfrac{\text{Term 1}}{\varepsilon}, & \text{if } -\varepsilon < \text{Term 2} < 0 \end{cases} \qquad (21)$$

(20) becomes

$$\dot V_e \le -mlk_3(k_2\dot x_e - \cos\theta\dot\theta)^2 V_e - mlk_2{}^2\dot x_e{}^2 + |\text{Term 1}| \qquad (22)$$

Therefore $V_e \to 0$ as $t \to \infty$.

4. Simulations

The development in Section 3 has been computer simulated. The following are the data used in the simulations. $m = 1\,kg$, $l = 1\,m$, $x_d = 3m$, $\theta(0) = 0.1\,radian$, $\dot\theta(0) = 0$, $\omega_c = 0.25$, $k_1 = \omega_c{}^2$, $k_2 = 2\omega_c$, $k_3 = 10$, $\varepsilon = 0.03$. When the ε in equation (21), which bounds the magnitude of Term 2, gets smaller, the control input becomes sharper and therefore trolley movement gets more abrupt.

5. Conclusion

In this paper we have investigated a nonlinear control for the underactuated mechanical system. The partial feedback linearization control together with energy based Lyapunov function have been applied to a crane system for grab operation. The developed theory is not complete, however the procedure shown in the paper may demonstrate a good example for developing control strategy based on combination of the high gain and energy based Lyapunov function. Further analyses for asymptotic stability and convergence are planned.

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