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Robust stability for discrete-time systems with delayed perturbations

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Abstract— This paper presents a sufficient condition for robust stability of discrete-time systems with delayed nonlinear perturbations. Using state evolution method, the bound on the norms of nonlinear perturbation which guarantees the exponential stability of the systems, is found. The numerical example is given to illustrate the results.

Nomenclature

R^n	n – real vector space of dimension n.
R^{nxm}	real matrix space of dimension n x m.
A^T	transposse of matrix A.
$\lambda_i(A)$	ith eigenvalue of A.
$\lambda_{max}(A)$	maximum eigenvalue of A.
$\rho(A)$	spectral radius of $A(= \max_i \lvert\lambda_i(A)\rvert)$.
$\lVert A\rVert$	matrix norm of $A(= [\lambda_{max}(A^TA)]^{1/2})$.
$\forall\ i$	for all i.

I. Introduction

The problem of robust stability of linear discrete-time systems subjected to perturbations has been studied by many researchers for decades [1]-[13]. Some of them pay attention to the systems with nonlinear perturbations [6]-[7] and some others concentrate on the systems with delayed linear perturbations [10]-[13]. However, in the real world, we frequently encounter the systems which have delayed nonlinear perturbations. Therefore, in this paper, we consider the problem of robust stability for linear discrete-time systems subject to the delayed nonlinear perturbations for the first time. The perturbation is a nonlinear function of time and delayed state, and its norm is assumed to be bounded. Then, the problem becomes to derive a sufficient condition on the perturbation which guarantees the robust stability of the system. In the literature, various techniques were introduced to deal with such problem. Using characteristic equation in complex z-domain, Mori[10], Lee[11], Su and Shyr[12], and Trinh and Aldeen[13] derived the sufficient conditions for robust stability of discrete-time systems with delayed linear perturbation. And, the state evolution method were used to handle nonlinear perturbations by Yaz and Niu[6] and Su[7]. In this paper, a new sufficient condition for robust stability is derived utilizing comparison lemma. The new sufficient condition gives the allowable bound on the delayed nonlinear perturbation so that the systems remain stable in the presence of the perturbation. Furthermore, the exponential decay bound of the state trajectory of the system is found.

The outline of the paper is as follows. In Section II, we state the problem to be considered. In Section III, the main results of this paper are presented. A simple example is given to illustrate the new result in Section IV. Brief conclusions are given in Section V.

II. Problem Formulation

Let us consider the discrete-time dynamical system described by the difference state equation

$$\left.\begin{array}{l} x(k+1) = Ax(k) + F(x(k-h), k) \\ x(k) = \xi(k), \quad \text{for } k \in [-h, 0] \end{array}\right\} \qquad (1)$$

where $x(k) \in \mathrm{R}^n$ is a state vector, $A \in \mathrm{R}^{nxn}$ is a stable matrix, $F(\cdot,\cdot) : \mathrm{R}^n\mathrm{x}\mathrm{R} \rightarrow \mathrm{R}^n$ is a unknown nonlinear vector function which represents the perturbation acting on the system (1), $h \geq 1$ is an integer denoting time delay, and $\xi(k)$ is a vector-valued initial condition function.

We assume that the perturbation term $F(\cdot,\cdot)$ is bounded on its norm with respect to the delayed state $x(k-h)$ in the form of

$$\lVert F(x(k-h), k)\rVert \leq \beta\lVert x(k-h)\rVert \qquad (2)$$

where β is a nonnegative constant.

Now, the problem is to determinine the allowable bound β, which gurantee the robust stability of the system (1).

III. Main Results

In this section, we present a sufficient condition to gurantee the stability for the system (1), which gives the bound on the perturbations. In order to derive the condition, we need a definition and lemmas given below.

Definition 1: The system (1) is said to be exponentially stable if there exist real numbers $\eta > 0$ and $0 \le \zeta < 1$ such that

$$\|x(k)\| \le \eta\zeta^k \quad \forall\, k > 0. \tag{3}$$

Lemma 1:[14] For the system matrix A given in (1), there exist positive constants $\mu(\mu \ge 1)$ and $\tau(0 < \tau < 1)$ such that

$$\|A^k\| < \mu\tau^k. \tag{4}$$

Lemma 2(Comparison lemma): Let the scalar function $g(k,i,w) : N\mathrm{x}N\mathrm{x}R \to R$ be nondecreasing with respect to scalar function $w(k)$. For a positive integer h and arbitrary scalar function $p(k)$, assume that

$$w(k) \le p(k) + \sum_{i=0}^{k-1} g(k,i,w(i-h)) \quad \text{for } k \ge 1 \tag{5}$$

with initial condition function $w(k) = \psi(k)$ defined on $[-h, 0]$. And let $z(k)$ satisfy the following difference equation

$$z(k) = p(k) + \sum_{i=0}^{k-1} g(k,i,z(i-h)) \quad \text{for } k \ge 1 \tag{6}$$

with initial condition function $z(k) = \varphi(k)$ defined on $[-h, 0]$. Then, $\psi(k) \le \varphi(k)$ implies

$$w(k) \le z(k), \quad \forall\, k \ge 0. \tag{7}$$

The proof is in Appendix.

Now, using Lemma 1 and 2, we claim the following theorem.

Theorem 1: For given μ and τ in Lemma 1, the perturbed system (1) is exponentially stable in large if the following inequality holds

$$\beta < \frac{1-\tau}{\mu}. \tag{8}$$

Furthermore, the bound of state trajectory is given by

$$\|x(k)\| \le \sup_{\rho\in[-h,0]} \|\xi(\rho)\|\mu\gamma^k, \quad k \ge 0 \tag{9}$$

where γ is a positive constant which satisfy the following relation

$$\left.\begin{array}{l} \mu\beta\gamma^{-h} = \gamma - \tau \\ \tau < \gamma < 1. \end{array}\right\} \tag{10}$$

Proof: We can rewrite (1) as

$$x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-i-1} F(x(i-h), i). \tag{11}$$

By taking the norm on both sides of (11), we obtain

$$\|x(k)\| \le \|A^k\|\|x(0)\| + \sum_{i=0}^{k-1} \|A^{k-i-1}\|\|F(x(i-h), i)\|. \tag{12}$$

Using inequalities (2) and (4), (12) becomes

$$\|x(k)\| \le x_0\mu\tau^k + \sum_{i=0}^{k-1} \mu\tau^{k-i-1}\beta\|x(i-h)\| \tag{13}$$

where $x_0 = \sup_{\rho\in[-h,0]} \|\xi(\rho)\|$.
In order to utilize Lemma 2, we define scalar function

$$v(k) = x_0\mu\gamma^k \quad \text{for } k \ge -h, \tag{14}$$

and we consider the following equation whose right hand side is same form as the one in (13)

$$v(k) = x_0\mu\tau^k + \sum_{i=0}^{k-1} \mu\tau^{k-i-1}\beta v(i-h), \quad k \ge -h. \tag{15}$$

Then, it is easy to show the equality of (15) by substituting (10) and (14) into (15). By comparing (13) with (15), Lemma 2 can be utilized to obtain the following

$$\|x(k)\| \le v(k) = x_0\mu\gamma^k \quad \forall\ k \ge 0. \tag{16}$$

Hence, from (16), it is obvious that the system (1) is exponentially stable.

Remark 1: The allowable bound of β in (8) is dependent on the parameters μ and τ. Therefore, in order to have less conservative bound of β, we have to choose the smaller μ and τ. The simple choice is $\mu = 1$ and $\tau = \|A\|$. Then the allowable bound for stability is as follows

$$\beta < 1 - \|A\|. \tag{17}$$

Remark 2: If the system matrix A is diagonalizable, there exists a nonsingular matrix T such that

$$A^k = T\Lambda^k T^{-1} \tag{18}$$

where Λ is the diagonal matrix having the eigenvalues of A on the diagonal. By taking norm of (18), we have

$$\|A^k\| \le \|T\|\|\Lambda^k\|\|T^{-1}\| = \|T\|\|T^{-1}\|\rho^k(A) \tag{19}$$

Then, we can choose μ and τ in (4) as

$$\mu = \|T\|\|T^{-1}\|, \tag{20}$$

$$\tau = \rho(A). \tag{21}$$

Therefore, the bound of β becomes

$$\beta < \frac{1-\rho(A)}{\|T\|\|T^{-1}\|}. \tag{22}$$

Remark 3: Consider the properties of the nominal system($x(k+1) = Ax(k)$). Then, there exists a positive constant α such that A has all its eigenvalues insides a disk of radius $(1+\alpha)^{-1/2}$. This stability margin was used in deriving quantitative measure of robustness of the nominal system in the presence of various perturbations by Yaz [4], Yaz and Niu [6], and Niu *et al.* [8]. Similarly, to utilize the parameter α, if the following inequality is used

$$\rho(A) \le (1+\alpha)^{-1/2}, \tag{23}$$

then the allowable bound is

$$\beta < \frac{1-(1+\alpha)^{-1/2}}{\|T\|\|T^{-1}\|}, \tag{24}$$

which is same to those presented by Yaz and Niu [6] for the nondelayed perturbations. And, obviously, the bound of (22) is less conservative than that of (24).

Remark 4: The stability analysis developed in this paper can be also applied to the systems[11-13]

$$x(k+1) = (A+\Delta A)x(k) + (B+\Delta B)x(k-h) \tag{25}$$

where A and B are system matrices of nominal time-delay system, and ΔA and ΔB are time-varying or time-invariant perturbation matrices.

IV. Numerical Example

To illustrate the application of the presented method, we give the following nemerical example.

Consider the discrete system with delayed nonlinear perturbation

$$x(k+1) = Ax(k) + F(x(k-h), k) \tag{26}$$

with initial condition function $\xi(k) = [2\cos(k) \quad -\cos(k)]^T$
where

$$A = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.15 \end{bmatrix}, \; F(\cdot,\cdot) = \begin{bmatrix} 0.5\sin(x_2(k-3)) \\ 0.5\sin(x_1(k-3)) \end{bmatrix}. \tag{27}$$

The system matrix have eigenvalues $\lambda_1 = 0.2781$ and $\lambda_2 = 0.0719$. Therefore, let's define the parameter τ as the spectral radius of A. That is, $\tau = 0.2781$.

Selecting the similarity transformation matrix T

$$T = \begin{bmatrix} -0.7882 & 0.6154 \\ -0.6154 & -0.7882 \end{bmatrix}$$

yields $\mu = 1$ from (21). Therefore, from (8), the allowable bound for robust stability is

$$\beta < \frac{1-\tau}{\mu} = 0.7219.$$

Since, from (2), the magnitude of the perturbation (27) is bouded with $\beta = 0.7071$, the system (26) is robustly stable in the presence of the perturbation. To confirm this, the simulation of this example is depicted in Fig.1. It is shown from Fig. 1 that the system is indeed exponentially stable.

V. Conclusion

In this work, using the state evolution method associated with norm inequality, the allowable bounds on delayed nonlinear perturbation for robust stability of uncertain discrete-time systems have been obtained. These bounds provide quantitive measures for control engineers to design closed-loop discrete-time system with robustness property. Furthermore, we can give a rate for exponential decay of the state trajectory of the system. And also, the procedure developed in this paper can be easily applied to the system with delayed linear time-varying perturbations. Finally, numerical example is given to illustrate effectiveness of the presented results.

Appendix
Proof of Lemma 2

Let the difference between $w(k)$ and $z(k)$ be

$$\delta(k) \triangleq w(k) - z(k). \tag{A.1}$$

Then, from initial condition functions of $w(k)$ and $z(k)$, it follows that for $k \in [1, 1+h]$

$$\delta(k) \le \sum_{i=0}^{k-1}[g(k,i,w(i-h)) - g(k,i,z(i-h))] \le 0. \tag{A.2}$$

That is,

$$w(k) \le z(k), \quad k \in [1, 1+h]. \tag{A.3}$$

Using the above inequality (A.3) of the interval $[1, 1+h]$, we can obtain that $\delta(k) \le 0$ for the interval $k \in [2+h, 2+2h]$. Utilizing the inequality of the previous interval, the procedure can be easily extended to the every $h+1$ dulation until $k \to \infty$. Therefore, the inequality (7) is obviously satisfied. This completes the proof and also, the similar proof is given in [15].

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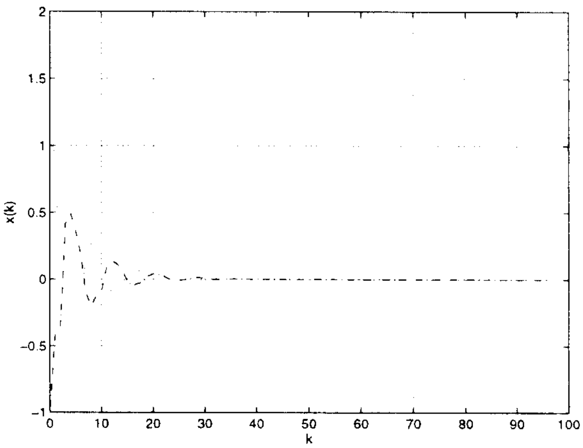

Figure 1: Simulation of example for $\beta = 0.707$