Relative Performance of Group CUSUM Charts

SUNGWOON CHOI and SANGHOON LEE
Department of Industrial Engineering
Kyung Won University, Seongnam City, Kyungki Do, Korea

Abstract

Performance of the group cumulative sum(CUSUM) control scheme using multiple univariate CUSUM charts is more sensitive to the change of quality control(QC) characteristics than the control chart schemes based on the Hotelling statistic. We examine three group charts for multivariate normal data sets simulated with various correlation structures and shift directions in the mean vector. These group schemes apply the original measurement vectors, the scaled residual vectors from the regression of each variable on all others and the principal component vectors respectively to calculating the CUSUM statistics. They are also compared to the multivariate QC charts based on the Hotelling statistic by estimating average run lengths, coefficients of variation of run length and ranks in signaling order. On the basis of simulation results, we suggest a control chart scheme appropriate for specific quality control environment.

INTRODUCTION

Performance of the directionally-invariant control scheme for a multivariate normal processes depends on the mean vector μ and covariance matrix Σ only through the magnitude of noncentrality parameter

$$\eta_c = \sqrt{(\mu-\mu_0)'\Sigma^{-1}(\mu-\mu_0)}. \qquad (1)$$

While the control charts based on the T^2 or T statistic are generally directionally invariant like the Shewhart and CCU charts, it is not true for the group CUSUM charts.

Performance of the group schemes vary in the mean shift direction and the covariance structure for a constant value of noncentrality parameter. Performance of QC charts is usually evaluated by the ARL that is the average number of successive observations without an out-of-control signal. Monte Carlo simulation method is often used to compare the ARL performance of several control charts for monitoring multivariate processes (Crosier[1]; Hawkins[4]; Lowry, et al.[7]). Another performance measure is the coefficient of variation of run length (CVRL) (Yashchin, [10]). It is defined a function of ARL and standard deviation of the run length (SDRL), CVRL = SDRL/ARL and expresses a relative variation of the run length as a measure of dispersion of the run length. This study also suggests "ranks in signaling order" (RISO) as a measure to compare performance of different control charts. It is considered as a relative measure for the run length for out-of-control signal, while the ARL is an absolute measure.

Doganaksoy, et al.[2] used three correlation structures for identification of out-of-control quality characteristics in a multivariate manufacturing environment. These structures correspond respectively with the cases that all the variables are positively correlated, only some variables are correlated and the variables are correlated with mixed sign. This study uses six different types of correlation structure. The six structures are categorized into two classes: the positive type, in which all pairs of variables have an equal positive correlation, and the negative type, in which variables i and j for $i \neq j$ have a negative correlation if $i+j$ is odd and a positive correlation $i+j$ is even.

To model the out-of-control mean process, three types of shift direction are considered:

Equal Shift, in which all components of μ are equal;
Symmetric Shift, which differs from Equal Shift in that the first half components of μ have different signs to the second half;
Only Shift, in which only a single component of μ is nonzero.

GROUP CUSUM CHARTS

A group CUSUM chart was suggested by extending the univariate CUSUM procedure to the multivariate normal process in Woodall and

Ncube[9]. In this approach, p two-sided CUSUM charts are operated simultaneously to detect a shift in the mean vector of p-variate normal distribution and the out-of-control signal is given if any of the p univariate charts exceeds its control limit. The ith univariate CUSUM is operated for a given reference value $k_i > 0$ by forming the cumulative sums

$$U_{n,i} = \max(0, U_{n-1,i} + x_{n,i} - k_i)$$

and

$$L_{n,i} = \min(0, U_{n-1,i} + x_{n,i} + k_i)$$

for n = 1,2,⋯. Under the assumption of the zero in-control mean vector and the normalized covariance matrix, the reference values k_i are given equivalently for all the variables. This group chart, so called MCX, signals an out-of-control condition when

$$MCX = \max[\max(U_{n,i}, -L_{n,i})] > h \quad (2)$$

for a given threshold h. Hawkins[5] discusses how to generalize these CUSUMs to others that are optimal for detecting shifts in arbitrary but specific directions.

Departures from the in-control condition in multivariate processes may be expected to affect only a minority of the variables. With this idea, Hawkins[4] proposed a measure for multivariate normal processes to test for shifts in mean. Realization X_n can be transformed to the regression-adjusted vector

$$Z_n = [\text{ diagonal}(\Sigma^{-1})]^{-1/2} \Sigma^{-1}(X_n - \mu_0)$$

that is the vector of scaled residuals from the regression of each variable on all the others. In case of a shift affecting a single variable, the control scheme using the vector Z_n has the advantages of better separation of location from scale and immediate association to a signal with a particular variable. The group CUSUM chart that uses the regression-adjusted vector Z_n for (2) is referred as to MCZ. The approach of using Z_n is equivalent to the proposal of Healy[6] when the "bad" mean vector differs from the in-control mean vector in only one component. Hawkins[4] also suggested a measure using Z_n to test for shifts in variability and a control chart using the Euclidean norm of the "resultant" vectors of the cumulative sums in the MCZ procedure.

Multivariate normal variables can be transformed to independent principal components by the spectral decomposition of the covariance matrix. The linear transformation for the successive measurements provides the possibility of separate control of the individual variables of a multivariate normal process (Woodall and Ncube[9]; Jackson[3]; Pignatiello and Runger[8]). For $X_n \sim N(\mu_0, \Sigma)$, the normalized principal component vector is

$$W_n = \Lambda^{-1/2} \Psi' (X_n - \mu_0)$$

where Ψ is the matrix of eigenvectors and the Λ diagonal matrix of eigenvalues associated with Ψ . Each component of W_n has the independent standard normal distribution, that is, $W_n \sim N(0, I)$. Thus, we directly applies the principal component vector W_n to the group CUSUM chart of (2) and denotes it as MCW.

COMPARATIVE PERFORMANCE OF QC CHART SCHEMES

The three group CUSUM charts are compared in their performance of detecting a mean shift from in-control to the two multivariate QC schemes, the Shewhart (the acronym SHW in this section) and CCU charts for p = 4 with three measure described in the second section . As in the analysis of the previous section, all the schemes are designed to give an out-of-control signal when the test statistic is greater than the threshold h of in-control ARL = 300, which are obtained by the simulation. The results in this section are obtained by operating simultaneously the four charts for an identical multivariate normal process at each run for the two extreme situations of mean shift direction respectively: SHW, CCU, MCX, MCW for Equal Shift and SHW, CCU, MCX, MCZ for Only Shift.

Table 1 and Table 2 contain the results of three performance measures for the Equal and Only Shift directions respectively. The CVRL value represents a relative variation of the run length as a ratio of the average and sample standard deviation of independent run lengths, and the RISO value is the percentage of runs giving the earliest signal among the charts when simultaneously operating them for an identical out-of-control process. Better performance of the chart scheme is associated with smaller values of ARL and CVRL and lager values of RISO. As illustrated in these tables, the simulation results confirm that the multivariate QC charts based on T^2, SHW and CCU are directionally invariant. As long as the noncentrality parameter of the shift in mean is constant, they perform equivalently in ARL for both the mean shift directions and both the different correlation types for the same number of variables. The ARL performance of CCU is best among these charts for small changes in process means. It is not true for large shifts, however. When shifting a large distance, MCX and MCW have shorter ARLs than CCU for P-5

Table 1. Performance Comparsion of QC charts for multivariate normal processes of Equal shift

		N-5					P-5				
	QC	η_c=0.2	η_c=0.4	η_c=0.8	η_c=1.6	η_c=3.2	η_c=0.2	η_c=0.4	η_c=0.8	η_c=1.6	η_c=3.2
ARL	SHW	281.8	231.4	123.3	26.8	2.7	279.1	230.2	122.3	26.6	2.7
	CCU	145.7	54.7	18.4	7.6	3.6	143.7	55.2	18.4	7.6	3.6
	MCX	239.5	146.2	52.8	14.8	5.9	145.4	54.4	16.1	6.1	2.9
	MCW	197.4	88.7	24.7	8.2	3.6	190.3	71.8	17.9	6.1	2.8
CVRL	SHW	1.00	1.01	1.01	0.99	0.79	0.99	0.99	1.00	1.00	0.80
	CCU	0.91	0.73	0.47	0.29	0.19	0.91	0.75	0.47	0.28	0.19
	MCX	0.98	0.94	0.84	0.48	0.27	0.94	0.86	0.59	0.35	0.23
	MCW	0.97	0.91	0.66	0.37	0.24	0.96	0.87	0.59	0.35	0.24
RISO	SHW	0.29	0.19	0.14	0.26	0.85	0.28	0.17	0.12	0.21	0.77
	CCU	0.36	0.57	0.70	0.65	0.27	0.29	0.37	0.24	0.09	0.09
	MCX	0.25	0.16	0.09	0.06	0.02	0.40	0.52	0.67	0.66	0.41
	MCW	0.27	0.26	0.31	0.45	0.27	0.25	0.27	0.47	0.68	0.49

Table 2. Performance Comparsion of QC charts for multivariate normal processes of Only shift

		N-5					P-5				
	QC	η_c=0.2	η_c=0.4	η_c=0.8	η_c=1.6	η_c=3.2	η_c=0.2	η_c=0.4	η_c=0.8	η_c=1.6	η_c=3.2
ARL	SHW	279.1	230.9	121.2	26.6	2.8	281.4	229.5	121.1	26.8	2.8
	CCU	144.6	55.0	18.4	7.6	3.6	145.3	54.9	18.5	7.6	3.6
	MCX	223.2	108.7	27.4	8.3	3.5	220.4	108.6	27.8	8.3	3.5
	MCZ	186	69.7	17.7	6.1	2.8	183.8	68.5	17.7	6.1	2.8
CVRL	SHW	1.00	1.01	0.99	0.99	0.79	1.00	1.00	0.99	1.00	0.79
	CCU	0.91	0.74	0.47	0.28	0.19	0.91	0.73	0.46	0.28	0.19
	MCX	0.97	0.92	0.70	0.41	0.26	0.98	0.93	0.70	0.41	0.26
	MCZ	0.98	0.86	0.59	0.35	0.24	0.95	0.85	0.59	0.34	0.24
RISO	SHW	0.29	0.19	0.13	0.22	0.77	0.29	0.18	0.13	0.22	0.77
	CCU	0.34	0.49	0.39	0.15	0.10	0.34	0.49	0.39	0.15	0.10
	MCX	0.25	0.21	0.20	0.23	0.19	0.25	0.21	0.21	0.23	0.20
	MCZ	0.30	0.36	0.64	0.82	0.51	0.30	0.37	0.65	0.82	0.51

and Equal Shift, in which the ARL performance of MCX is better than of that of CCU even for a small shift for p = 4. For the situation of shifting only in a particular variable, MCZ shows the best ARL performance for relatively larger values of noncentrality parameter. As shown in the estimated results of CVRL, the run lengths for signaling out-of-control are distributed from ARL with smaller scale as the shift is larger and CCU has the smallest values for our cases of measurement and out-of-control characteristics. For the process of higher variable dimension, CCU improves in its CVRL performance, but the performance of SHW becomes worse. The CVRL values of the group charts more or less change according to the quality control environment considered and the values of the directionally-invariant approaches depend only on the magnitude of noncentrality parameter for the equal variable dimension.

SUMMARY AND CONCLUSIONS

This study has presented extensive comparisons between three group CUSUM charts and two multivariate QC charts for multivariate normal processes using simulated data with various correlation structures and mean shift directions. The performance of the group CUSUM charts appears to be more or less affected by the correlation structure of the process, but MCW and MCZ work without significant variation when the process mean vector changes only in one variable. The CCU chart scheme is generally more effective in detecting a small departure from the in-control mean vector than the group CUSUM charts and has the smallest relative variation of the run length. When the process is positively correlated and experiences substantial shifts in the means of all variables, MCX and MCW have better ARL performance than CCU and the ARL performance of MCZ is superior for the shift of relatively larger amounts only in the mean of one variable. Although the Shewhart chart has an advantage in computation over the other complicate schemes as MCX does, it appears to be ineffective in detecting a small mean shift for multivariate processes. The Shewhart chart offers a good detection for a considerably large shift in the mean level, however, it is likely to fail in giving a signal on the out-of-condition that the process changes without extreme outliers. For the out-of-control situation only in one variable, MCX and MCZ operates with a high degree of correctness in detecting the right out-of-control variable, thereby being appropriate for the QC procedure to concern the interpretation in terms of original variables rather than monitoring the signal.

Overall, the multivariate QC chart CCU appears to provide good protection for the shifts in unanticipated directions in the mean level of multivariate processes. For some special QC environments, however, the group CUSUM charts offer better protection for the out-of-control situation. If one is interested in detecting a shift only in one specified direction, MCZ is the optimal proposal. When the process is positively correlated, it is susceptible to simultaneous shifts in the means of all variables. For this situation, MCX is quite effective for the protection, and the signal from MCX can correspond to an immediately identified variable. Another group chart MCW appears to be competitive to the other superior schemes for shifting relatively larger distances in the mean vector and is suggested for controlling the means of multivariate processes if the principal components are interpretable.

REFERENCES

[1] Crosier, R.B. (1988), "Multivariate Generalizations of Cumulative Sum Quality Control Schemes," *Technometrics*, 30, 291-303.

[2] Doganaksoy, N., Faltin, F.W. and Tucker, W.T. (1991), "Identification of Out-of-Control Quality Characteristics in a Multivariate Manufacturing Environment," *Communications in Statistics-Theory and Methods*, 20, 2775-2790.

[3] Jackson, J.E. (1985), "Multivariate Quality Control," *Communications in Statistics-Theory and Methods*, 14, 2657-2688.

[4] Hawkins, D.M. (1991), "Multivariate Quality Control Based on Regression-Adjusted Variables," *Technometrics*, 33, 61-75.

[5] Hawkins, D.M. (1993), "The Regression Adjustment for Variables in Multivariate Quality Control," *Journal of Quality Technology*, 25, 170-180.

[6] Healy, J.D. (1987), "A Note on Multivariate CUSUM Procedures," *Technometrics*, 29, 409-412.

[7] Lowry, C.A., Woodall, W.H., Champ, C.W. and Rigdon, S.E. (1992), " Multivariate Exponentially Weighted Moving Average Control Chart," *Technometrics*, 34, 46-53.

[8] Pignatiello, J.J., Jr. and Runger, G.C. (1990), "Comparisons of Multivariate CUSUM Charts," *Journal of Quality Technology*, 22, 173-186.

[9] Woodall, W.H. and Ncube, M.M. (1985), "Multivariate CUSUM Quality Control Procedures," *Technometrics*, 27, 285-292.

[10] Yashchin, E. (1992), "Analysis of CUSUM and Other Markov-type Control Schemes by Using Empirical Distributions," *Technometrics*, 34, 54-63.