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Development of a Consistent General Order Nodal Method for Solving the Three-Dimensional, Multigroup, Static Neutron Diffusion Equation

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ABSTRACT

A consistent general order nodal method for solving the 3-D neutron diffusion equation in (x-y-z) geometry has been derived by using a weighted integral technique and expanding the spatial variables by the Legendre orthogonal series function. The equation set derived can be converted into any order nodal schemes. It forms a compact system for general order of nodal moments. The method utilizes the analytic solutions of the transverse-integrated quasi-one dimensional equations and a consistent expansion for the spatial variables so that it renders the use of an approximation for the transverse leakages no necessary. Thus, we can expect extremely accurate solutions and the solution would converge exactly when the mesh width is decreased or the approximation order is increased since the equation set is consistent mathematically.

1. INTRODUCTION

Modern coarse-mesh nodal methods are a very efficient class of numerical methods that have proven to be superior in accuracy, computer storage requirement and computing time to finite difference and finite element methods for the solution of large, multidimensional, neutron diffusion problems[1]. However, in nodes the transverse currents at their surfaces not varying smoothly, substantial errors subsist and convergence difficulties may arise[2]. These problems arise mainly from the inconsistent quadratic fit approximations of the transverse leakage[3]. To overcome them, higher order scheme has been suggested[4]. A higher order nodal scheme for two-dimensional geometry showed greatly accurate results[2]. In the present work, a general order consistent nodal method without the transverse leakage approximation for 3-D multi-group diffusion equation is derived by using a weighted integral technique and expanding the spatial variables by the Legendre function. This work aims at deriving a mathematically consistent method so that the solution can be converged to the exact values in fine mesh limit and establishing the missing link between nodal methods and conventional finite difference methods.

2. GENERALIZED FLUX MOMENT FORMULATION

In a node of widths, Δx_i, Δy_j and Δz_k in x, y and z directions, respectively, the multigroup diffusion equation is written by the dimensionless variables,

$$- 4D_g/\Delta x_i^2 \ \partial^2\phi_g(u,v,w)/\partial u^2 - 4D_g/\Delta y_j^2 \ \partial^2\phi_g(u,v,w)/\partial v^2 - 4D_g/\Delta z_k^2 \ \partial^2\phi_g(u,v,w)/\partial w^2 + \Sigma_{tg}\phi_g(u,v,w) = s_g(u,v,w), \quad (1)$$

where

$$s_g(u,v,w) \equiv \sum_{g'=1}^{G}\{\chi_g/k_e \ \nu\Sigma_{fg'} + \Sigma_{sg'g}\}\phi_{g'}(u,v,w) + q_{0g}(u,v,w) \quad (2)$$

and $-1 \le u,\ v,\ w \le 1$.

We derive the transverse integrated quasi-one-dimensional nodal equation set by applying a weighted integration. Each resulting differential equation is for the general order moment of the flux in the transverse spatial direction. For example, for the v and w moments of the u-dependent flux, we multiply Eq. (1) by $P_{n_v}(v)$ and $P_{n_w}(w)$, the Legendre polynomials in v and w variables, respectively, and integrate over $-1 \le v,\ w \le 1$ in the node to arrive at the u-channel differential equation.

$$-4D/\Delta x^2 \ d^2\phi_{n_v n_w}(u)/du^2 + \Sigma_t\phi_{n_v n_w}(u) = A_{u n_v n_w}(u), \quad (3)$$

where

$$\phi_{n_v n_w}(u) = \int_{-1}^{1}\int_{-1}^{1} dv dw \ P_{n_v}(v)P_{n_w}(w)\phi(u,v,w), \quad (4)$$

$$A_{u n_v n_w}(u) \equiv S_{n_v n_w}(u) - L_v(u,n_v,n_w) - L_w(u,n_v,n_w), \quad (5)$$

$$L_v(u,n_v,n_w) \equiv 4/\Delta y^2 \ \{J_{v n_w}(u,1) + (-1)^{n_v+1}J_{v n_w}(u,-1) - \sum_{l=0}^{(n_v-1)/2}(2n_v-4l-1)J_{v(n_v-2l-1)n_w}(u)\}, \quad (6)$$

and

$$J_{v n_w}(u,1) = \int_{-1}^{1}dw \ P_{n_w}(w)J_v(u,1,w) = -D\int_{-1}^{1}dw \ P_{n_w}(w) \ \partial\phi(u,1,w)/\partial v, \quad (7)$$

and group index g is omitted.

After expanding the source term, $A_{u n_v n_w}(u)$ by the Legendre function in u and substituting into Eq. (3), solving this, we obtain the equation for the u-dependent, n_v and n_w moment fluxes, $\phi_{n_v n_w}(u)$,

$$\phi n_v n_w(u) = \{B_u n_v n_w 1 + \sum_{n_u=0} s_i n_u n_v n_w(u)\} \cosh(r_u u) + \{B_u n_v n_w 2 - \sum_{n_u=0} c_o n_u n_v n_w(u)\} \sinh(r_u u), \quad (8)$$

where

$$r_u \equiv (\Delta x/2) \sqrt{\Sigma_t/D}, \quad (9)$$

$$s_i n_u n_v n_w(u) \equiv \Delta x^2/(4r_u D)\ (2n_u+1)/2\ A_u n_u n_v n_w \int \sinh(r_u u) P n_u(u) du, \quad (10)$$

$$c_o n_u n_v n_w(u) \equiv \Delta x^2/(4r_u D)\ (2n_u+1)/2\ A_u n_u n_v n_w \int \cosh(r_u u) P n_u(u) du, \quad (11)$$

and $B_u n_v n_w 1$ and $B_u n_v n_w 2$ are the coefficients which are expressed by the net current moments on right (R) and left (L) surfaces, $J^R{}_u n_v n_w$ and $J_u{}^L n_v n_w$, and $s_i n_u n_v n_w(u)$ and $c_o n_u n_v n_w(u)$ at u=1 and -1. The equations for v- and w-channels can be derived similarly.

If we multiply Eq. (3) by $P n_u(u)$ and integrate over $-1 \le u \le 1$, we obtain the generalized moment balance equation in (u-v-w) coordinates,

$$L_u(n_u, n_v, n_w) + L_v(n_u, n_v, n_w) + L_w(n_u, n_v, n_w) + \Sigma_t \phi n_u n_v n_w = S n_u n_v n_w. \quad (12)$$

The solution for the flux moments given by Eq. (12) depends on the availability of the interface net current moments. When the continuity condition for the flux moments [Eq. (8)] at given interfaces are used, the spatial dependent flux moments are eliminated and a coupling equation relating the three net current moments at three consecutive interfaces is obtained. For example, for u-direction,

$$E_{u,i-1} J_u{}^L n_v n_{w,i-1} + E_{u,i} J_u{}^L n_v n_{w,i} + E_{u,i+1} J_u{}^L n_v n_{w,i+1} = Q_u n_v n_{w,i}, \quad (13)$$

where

$$E_{u,i-1} \equiv -1/\{r_u D \sinh(2r_u)\}_{i-1}, \quad (14)$$

$$E_{u,i-1} \equiv -1/\{r_u D \sinh(2r_u)\}_i, \quad (15)$$

$$E_{u,i} \equiv -E_{u,i-1} \cosh(2r_{u,i-1}) - E_{u,i+1} \cosh(2r_{u,i}), \quad (16)$$

and

$$\begin{aligned} Q_u n_v n_{w,i} \equiv\ & 1/\{2\cosh(r_{u,i-1})\} \sum_{n_u=0} \{s_i n_u n_v n_w(u=1) - s_i n_u n_v n_w(u=-1)\}_{i-1} \\ & + 1/\{2\cosh(r_{u,i})\} \sum_{n_u=0} \{s_i n_u n_v n_w(u=1) - s_i n_u n_v n_w(u=-1)\}_i \\ & + 1/\{2\sinh(r_{u,i-1})\} \sum_{n_u=0} \{c_o n_u n_v n_w(u=1) - c_o n_u n_v n_w(u=-1)\}_{i-1} \\ & - 1/\{2\sinh(r_{u,i})\} \sum_{n_u=0} \{c_o n_u n_v n_w(u=1) - c_o n_u n_v n_w(u=-1)\}_i. \end{aligned} \quad (17)$$

The process is repeated for all nodes and moments and for each spatial direction, giving special treatment only to boundary nodes by incorporating the boundary conditions. The resulting equation set for net current moments can be solved by the direct method.

3. NUMERICAL EXAMPLES

The acuracy and efficiency of the method have been studied for the three test cases[2,5]:

1. LMW light water reactor three-dimensional test problem.
2. LRA boiling water reactor three-dimensional benchmark problem.
3. IAEA pressurized water reactor three-dimensional benchmark problem.

The solutions for these problems are summarized in Table 1 through 4. Tables contain values for k_{eff} and the power distribution in fuel elements.

4. SUMMARY

The equation set derived constitutes a compact system for general order nodal moments. This system can be converted into any orders of nodal moments or the standard finite difference form. When the derived equation set is truncated at finite moment series for a desired order moment, they can be solved by the standard power method for eigenvalue iteration. The method uses a consistent polynomial expansion for the given spatial variables without any approximations, and therefore, we can expect extremely accurate solutions, and the solution would converge exactly when the mesh width is decreased or the approximation order is increased. The numerical examples tested with the leakage moments truncated at first order and the in-node moments at first or second order show very good accuracy in k_{eff} and local power densities compared with the references. The method should be a promising one for solving the multigroup neutron diffusion equation with highest accuracy and computational efficiency.

REFERENCES

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Table 1. Comparison of K-effective Value

The Problem	The Code	K-eff	Calculation Time (Min.)
LMW	Reference[1]	.99966	
	ILLICO[1-1]	.99962	
	The Solution[2] (1,1)	.99965	0.91*
LRA	Reference[3]	.99639	
		(.99644)[3-1]	
	ILLICO[3-2]	.99634	
	The Solution[4] (1,2)	.99633	3.47*
IAEA	Reference[5]	1.02903	
	IQSBOX[5-1]	1.02904	
	ILLICO[5-2]	1.02900	
	The Solution[6] (1,1)	1.02902	9.47*

1. The reference (QUANDRY): 10 x 10 x 20 (10) cm meshes in x-y-z.
1-1. The ILLICO: 10 x 10 x 10 cm meshes in x-y-z.
2. The calculation: 10 x 10 x 20 cm meshes in x-y-z.
3. The reference (QUANDRY): 7.5 x 7.5 x 25(15) cm meshes in x-y-z.
3-1. : 15 x 15 x x 25(15) cm meshes in x-y-z.
3-2. The ILLICO result: 15 x 15 x 30 cm meshes in x-y-z.
4. The calculation: 15 x 15 x 30 cm meshes in x-y-z.
5. The reference (VENTURE result exporated): 1 - 2/3 cm meshes.
5-1. The IQSBOX result: 10 x 10 x 20 (10) cm meshes in x-y-z.
5-2. The ILLICO result: 10 x 10 x 20 (10) cm meshes in x-y-z.
6. The calculation: 10 x 10 x 20 cm meshes in x-y-z.
* Convergence criteria of 1.0E-05, flat flux initial guess, and HP-715 computer used

Table 2. Nodal Power Densities for the 3-D LMW LWR Benchmark Problem

.7268 -.35	.7083 -.33	.6274 -.28	.4345 -.17
.9801 -.03	1.0833 .01	.9801 .01	.8597 .06
1.4401 .13	1.3959 .12	1.1228 .08	
1.6544 .18	1.5894 .17		
1.5542 .17		Reference Power % Error (1,1) -- $(P_{ref}-P_{cal})/P_{ref}$ x100	

Table 3. Nodal Power Densities for the 3-D LRA Benchmark Problem.

.9239	.8669	.8266	.8528	.9324	.9719	.8484	
.10	.13	.18	.23	.28	.32	.75	
1.4807	1.2806	1.1726	1.2211	1.4215	1.6796	1.6216	1.3319
-.02	-.14	.01	.07	.02	.24	.25	.76
1.6599	1.1506	.9667	1.0224	1.3381	2.0505	2.1607	
-.26	-.05	-.09	-.02	.03	.00	.09	
1.3844	.9397	.7826	.8434	1.1521	1.8515		
-.30	-.12	-.15	-.07	.07	-.03		
.7901	.6703	.6181	.6782	.8643			
-.17	-.33	-.18	-.12	-.16			
.5119	.4904	.4902	.5524				
-.31	-.27	-.62	-.19				
.4131	.4067	.4240					
-.37	-.33	-.30					
.4403	.3995						
-.36	-.48						
.6118		Reference Power					
-.57		% ERROR (1,2)	-- $(P_{ref}-P_{cal})/P_{ref}$ x100				

Table 4. Nodal Power Densities for the 3-D IAEA PWR Benchmark Problem

.7770	.7570	.7110			
-1.37	-1.26	-.95			
.9590	.9760	1.0000	.8660	.6110	
-.49	-.31	-.29	-.20	-.25	
.9530	1.0550	1.0890	.9230	.7000	.5970
-.04	.06	.04	.14	.11	.15
.6100	1.0720	1.1810	.9720	.4760	
-.22	.34	.25	.30	-.14	
1.1930	1.2910	1.3110	1.1780		
.42	.53	.41	.40		
1.4220	1.4320	1.3680			
.48	.56	.44			
1.2810	1.3970				
.59	.63				
.7290		Reference Power			
.17		% ERROR (1,1)	-- $(P_{ref}-P_{cal})/P_{ref}$ x100		