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A Spectral-Galerkin Nodal Method for Solving the Two-Dimensional Multigroup Diffusion Equations

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Abstract- A novel nodal method is developed for the two-dimensional multi-group diffusion equations based on the Spectral-Galerkin approach. In this study, the nodal diffusion equations with Robin boundary condition are reformulated in a weak (variational) form, which is then approximated spatially by choosing appropriate basis functions. For the nodal coupling relations between the neighbouring nodes, the continuity conditions of partial currents are utilized. The resulting discrete systems with sparse structured matrices are solved by the Preconditioned Conjugate Gradient Method (PCG) and sweeping technique. The method is validated on two test problems.

I. Introduction

Over the past several decades, owing to the characteristics of both accuracy and efficiency that advanced nodal methods own, they have been extensively and successfully applied in the design of nuclear reactors to estimate the power distribution and effective multiplication factor instead of the finite difference method (FDM) and finite element method (FEM). Of a large number of various coarse mesh methods, the Nodal Expansion Method (NEM)[1], the Analytical Nodal Method (ANM)[2] and the Nodal Green's Function Method (NGFM)[3] are most well-known and representative. In general, these modern nodal methods are characterized by three distinct features:

1. The unknowns chosen are the node-averaged group fluxes and surface-averaged partial or net currents.
2. A transverse integration procedure (TIP) is used for deriving auxiliary one-dimensional diffusion equations which are needed to relate the surface currents and nodal fluxes.
3. The spatial dependence of the transverse leakage term which appears in these one-dimensional equations is approximated by a parabolic fit.

Clearly, these principles can endow advanced nodal methods with high efficiency. However, due to employing TIP, some disadvantages are brought about. Firstly, TIP affects the accuracy of these nodal methods to a greater extent. So far, even though a few better approximation methods of the transverse leakage term have been developed, yet they are not still accurate enough to predict the power distribution for some problems with the large flux gradients near nodal interfaces of strong material discontinuity. Meanwhile, the more accurately TIP is handled, if not impossible, the more slowly these advanced nodal methods run. Secondly, they cannot offer the detailed nodal flux distributions which lead to the process of pin-power reconstructing required. Around 1990s, Noh and Cho put forward a new nodal method arising from Analytic Basis Function Expansion Method (AFEN)[4], in which TIP is not used. As a result, it overcomes the above-mentioned shortcomings. In this paper, a novel nodal method without TIP is as well posed based on the Spectral-Galerkin

method. As compared with AFEN, this method is self-consistent, that is, does not require additional assumptions.

II. Derivation of Nodal Equations

For a homogeneous node (k,l) of volume $\Omega_{k,l}$, the two-dimensional multigroup diffusion equations are in the form of:

$$-\nabla\cdot D_g\nabla\Phi_g(x,y)+\Sigma_{Rg}\Phi_g(x,y)=Q_g(x,y),\quad (x,y)\in\Omega_{k,l},\quad 1\le g\le G \tag{1}$$

where the source term $Q_g(x,y)$ contains the fission and scattering contributions:

$$Q_g(x,y)=\frac{\chi_g}{k_{eff}}\sum_{g'=1}^{G}\nu\Sigma_{fg'}\Phi_{g'}(x,y)+\sum_{g'\neq g}^{G}\Sigma_{sgg'}\Phi_{g'}(x,y).$$

The notations in Eqs. (1)-(2) are standard. The G coupled partial differential equations (1) are subject to the following Robin boundary condition at the nodal surfaces which may be cast into:

$$D_g\frac{\partial\Phi_g(x,y)}{\partial\mathbf{n}}+\frac{\Phi_g(x,y)}{2}=2J_g^{in}(x,y),(x,y)\in\Gamma=\Gamma_l\cup\Gamma_r\cup\Gamma_b\cup\Gamma_t \tag{2}$$

where $\partial/\partial\mathbf{n}$ denotes the outward normal derivative and Γ, Γ_l, Γ_r, Γ_b and Γ_t denote the total boundary, left surface, right surface, bottom surface and top surface of node (k,l), respectively.

Before the Spectral-Galerkin Method is applied to solve Eqs. (1)-(3), it is necessary to map $\Omega_{k,l}=[-a_x,+a_x]\times[-a_y,+a_y]$ where a_x and a_y stand for the nodal half pitches of x-direction and y-direction (see Fig. 1) to the 2D reference node $\Delta=[-1,+1]\times[-1,+1]$ and Eqs. (1)-(3) in reduced coordinates (u,v) become:

$$-\left(\frac{D_{gx}}{a_x}\frac{\partial^2\phi_g}{\partial u^2}+\frac{D_{gy}}{a_y}\frac{\partial^2\phi_g}{\partial v^2}\right)+\Sigma_{Rg}\phi_g=Q_g(u,v),\quad (u,v)\in\Delta \tag{3}$$

and

$$D_{gd}\frac{\partial\phi_g(u,v)}{\partial\mathbf{n}}+\frac{\phi_g(u,v)}{2}=2J_g^{in}(u,v),(u,v)\in\Gamma=\Gamma_l\cup\Gamma_r\cup\Gamma_b\cup\Gamma_t \tag{4}$$

where the coefficients D_{gx} and D_{gy} are nondimensional diffusion coefficients depending on the horizontal and vertical dimensions of the node:

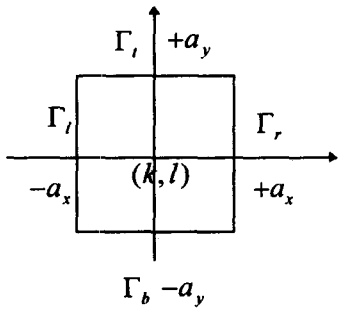

Fig.1 Coordinate system of node (k,l)

$$D_{gx}=\frac{D_g}{a_x},\ D_{gy}=\frac{D_g}{a_y}\ \text{and}\ D_{gd}=\begin{cases}D_{gx},(u,v)\in\Gamma_l\cup\Gamma_r\\ D_{gy},(u,v)\in\Gamma_b\cup\Gamma_t\end{cases}$$

By using the Galerkin variational principle, Eqs. (3)-(4) are rewritten as

$$B(\phi_g, \phi_g^*) = f(\phi_g^*), \quad \forall \phi_g^* \in H^1(\Delta) \tag{5}$$

and

$$\begin{aligned} B(\phi, \phi^*) = & \int_\Delta \left(\frac{D_{gx}}{a_x}\frac{\partial \phi_g}{\partial u}\frac{\partial \phi_g^*}{\partial u} + \frac{D_{gy}}{a_y}\frac{\partial \phi_g}{\partial v}\frac{\partial \phi_g^*}{\partial v}\right) dV \\ & + \int_\Delta \Sigma_R \phi_g \phi_g^* dV + \int_\Gamma \frac{1}{2a_d}\phi_g \phi_g^* dS, \forall \phi_g, \phi_g^* \in H^1(\Delta) \end{aligned} \tag{6}$$

$$f(\phi_g^*) = \int_\Delta Q_g \phi_g^* dV + \int_\Gamma \frac{2}{a_d} J_g^{in} \phi_g^* dS, \quad \forall \phi_g^* \in H^1(\Delta) \tag{7}$$

where $H^1(\Delta)$ is a Hilbert space and in Eqs. (5)-(6)

$$a_d = \begin{cases} a_x, (u,v) \in \Gamma_l \cup \Gamma_r \\ a_y, (u,v) \in \Gamma_b \cup \Gamma_t \end{cases}.$$

We denote by $P_i(u)$ the i th degree Legendre polynomial, and we set

$$V_N^2 = \text{span}\ \{P_i(u)P_j(u): i, j = 0,1,\ldots,N\}.$$

Then the standard Spectral-Galerkin approximation to Eq. (5) is to find the weak solution $\phi_{gN} \in V_N^2$ such that

$$B(\phi_{gN}, \phi_g^*) = f(\phi_g^*), \quad \forall \phi_g^* \in H^1(\Delta). \tag{8}$$

Setting

$$\phi_g \approx \phi_{gN} = \sum_{i,j=0}^{N} \phi_{gij} P_i(u)P_j(v) \text{ and } \phi_g^* \approx \phi_{gN}^* = \sum_{m,n=0}^{N} \phi_{gmn}^* P_m(u)P_n(v), \tag{9}$$

if the above formula is substituted into Eq.(8), we obtain

$$\sum_{i,j=0}^{N} \phi_{gij} B[P_i(u)P_j(v), P_m(u)P_n(v)] = f[P_m(u)P_n(v)], \quad m,n = 0,\ldots,N. \tag{10}$$

Let

$$\Phi = (\phi_{g00}, \phi_{g01}, \cdots, \phi_{gNN})^T$$

$$\mathbf{f} = (f[P_0(u)P_0(v)], f[P_0(u)P_1(v)], \cdots, f[P_N(u)P_N(v)])^T$$

$$\mathbf{K} = (B[P_i(u)P_j(v), P_m(u)P_n(v)])_{i,j=1,\ldots,N;m,n=1,\ldots,N}$$

Eq. (10) can be recast in matrix form

$$\mathbf{K}\Phi = \mathbf{f} \tag{11}$$

Closely examining Eqs. (5)-(11), we find that discrete equation system (11) of node (k,l) could be solved if $\mathbf{f}$ would be represented analytically since matrix $\mathbf{K}$ depends only on the nodal material property and dimensions, not on the nodal flux distributions. For the calculation of the source term contained in $\mathbf{f}$, fission source iteration method (FSI) could be employed to deal with it. However, for the treatment of the boundary incoming current also embodied in $\mathbf{f}$, we encounter a difficulty which arises from the fact that the nodal surface in-currents is as usual to be determined. Fortunately, the continuity conditions of nodal partial

currents can make up for this inadequacy, that is, the in-currents of node (k,l) are equal to those of its neighboring node at the interface formed by these two nodes.

The iterative procedure may be a conventional one that involves two levels of iterative schemes: inner iteration and outer iteration. Outer iteration begins with the initial k_{eff} and initial flux estimates to form the right side of the linear system (11). At each inner iteration, equation system (11) could be solve sequentially by PCG. $\mathbf{K}$ is obviously symmetric, positive definite and sparse because of the orthogonal property of Legendre polynomials. In this method, the one-parameter Chebyshev iterative method is applied to accelerate the outer iteration.

III. Numerical Tests and Conclusions

The new nodal method based on the Legendre-Galerkin method described in Section II has been implemented in a compute code SGNM-2D (Spectral-Galerkin Nodal Method). In order to examine its accuracy, two PWR problems are chosen to be computed.

One is the small PWR problem which stems from Ref. 5. In this problem, the core configuration is a little modified so that the baffle between the core and reflector is deleted and assemblies are homogeneous (see Fig. 2). The cross sections data of different assembly type are displayed in Table I. Reference solutions are computed by NGFM with (6 × 6) nodes per assembly. They along with those produced by SGNM-2D with different orders of nodal flux expansion are summarized in Table II. It is obvious that the power errors and the errors of effective multiplication factors are very small and satisfactory, what is more, these errors are gradually reduced with the increase of degree N of nodal flux expansion.

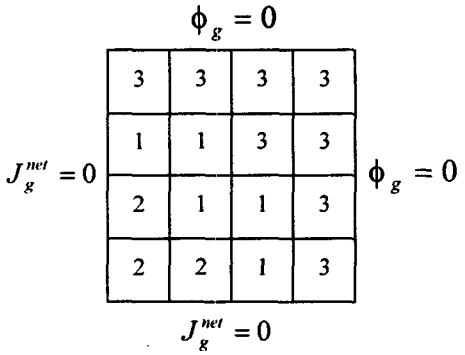

Fig. 2 The quadrant core configuration for Problem 1

Table I

Assembly Cross Sections for Problem 1

Assembly type	Group	D	Σ_R	Σ_{21}	$\nu\Sigma_f$
1	1	1.5130	0.03323	0.02113	0.006012
	2	3.9500	0.16480	0.0	0.218600
2	1	1.5130	0.03045	0.02113	0.004625
	2	3.9510	0.14140	0.0	0.164500
3	1	1.7000	0.03600	0.03500	0.0
	2	3.5000	0.05000	0.0	0.0

The other is the well-known 2D-IAEA benchmark problem with four control rods inserted in the core. The results calculated by SGNM-2D with degree 3 of nodal flux expansion are shown in Fig. 3 and Fig. 4. In the figures, the nodal average fluxes are

compared with those of the VENTURE reference calculation and NEM-MSS calculation in the active core, which are the same as in Ref. 4. We find that although the accuracy of SGNM-2D is not so high as AFEN, it is better than MSS-NEM, especially for the rodded assemblies and those positions close to the core periphery.

From the above numerical experiments, this nodal method is accurate enough to predict the nodal flux distribution (or power distribution) as well as the effective multiplication factor as compared with traditional nodal methods in which TIP is commonly used. However, during the process of calculating 2D-IAEA benchmark problem, we meet with a problem that when degree N of nodal flux expansion is given a larger value, the flux accuracy becomes poor while for a small PWR problem this phenomenon does not take place. In this case, Cond($\mathbf{K}$) in Eq. (11) would be larger. So, the round-off error is not significantly reduced. Hence, we should find an excellent iterative method to solve Eq. (11) and efficiently cut down the number of outer iterations for solving large reactor cores. In the meanwhile, CPU time is naturally saved.

Table II
Comparison of Results for Problem 1

Codes	N	k_{eff}	$\frac{\Delta k_{eff}}{k_{eff}} \times 100$	$\frac{\Delta P}{P}\|_{ave} \times 100$	$\frac{\Delta P}{P}\|_{max} \times 100$
NGFM	----	0.932228	----	----	----
SGNM-2D	3	0.932224	0.0	0.65	1.55
	4	0.931715	-0.055	0.15	0.45
	5	0.931802	-0.046	0.17	0.27

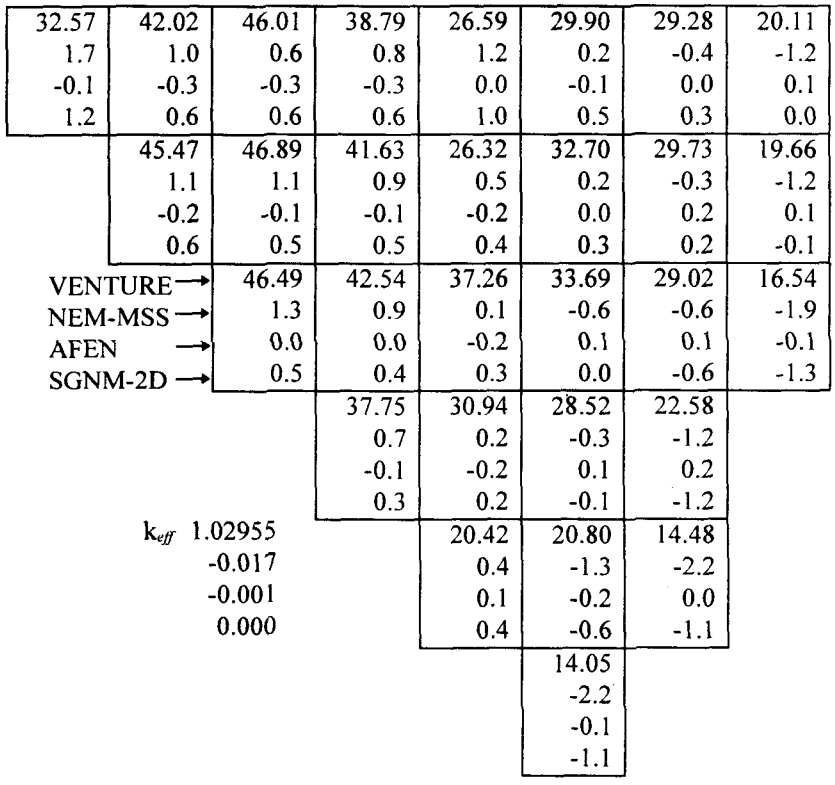

Fig. 3 Results of 2D-IAEA benchmark problem: fast flux

5.535	9.734	10.80	8.992	4.519	6.919	6.898	5.520
1.2	1.1	0.7	0.8	0.6	0.2	-0.7	-1.3
0.0	-0.3	-0.3	-0.3	0.1	-0.2	0.0	0.2
1.0	0.8	0.6	0.7	0.8	0.5	0.1	0.0
	10.66	10.99	9.763	7.935	7.668	7.012	5.385
	1.1	1.1	0.9	0.5	0.2	-0.6	-1.7
	-0.2	-0.1	-0.1	-0.2	0.0	0.2	0.2
	0.6	0.5	0.5	0.5	0.3	0.0	-0.4
VENTURE →		10.91	9.985	8.746	7.972	7.025	5.082
NEM-MSS →		1.3	0.9	0.2	0.0	-0.9	-2.0
AFEN →		0.0	0.0	-0.2	0.1	0.0	-0.4
SGNM-2D →		0.5	0.3	0.3	0.0	-0.6	-1.4
			8.854	7.176	6.719	6.263	
			0.7	0.3	-0.7	-1.4	
			0.0	-0.2	0.1	0.5	
			0.3	0.3	-0.4	-1.2	
				3.488	5.088	4.435	
				-0.2	-1.8	-2.1	
				0.2	-0.3	0.6	
				0.2	-1.0	-1.2	
					4.342		
					-1.9		
					-0.6		
					-1.3		

Fig. 4 Results of 2D-IAEA benchmark problem: thermal flux

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