Proceedings of the Korean Nuclear Society Spring Meeting
Cheju, Korea, May 1996

Fast Solution of Linear Systems by Wavelet Transform

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Abstract

We develop in this study a wavelet transform method to apply to the flux reconstruction problem in reactor analysis. When we reconstruct pinwise heterogeneous flux by iterative methods, a difficulty arises due to the near singularity of the matrix as the mesh size becomes finer. Here we suggest a wavelet transform to lower the spectral radius of the near singular matrix and thus to converge by a standard iterative scheme. We find that the spectral radius becomes smaller than one after the wavelet transform is performed on sample problems.

1. Introduction

Wavelets are lately attracting a considerable amount of attention and becoming a useful tool in many different fields of science and engineering. We note that the wavelet basis has the following three outstanding characteristics [1] :

(1) The operators and functions are represented in an *orthonormal* basis.
(2) The basis functions have *vanishing moments* leading to the sparsity of representations.
(3) The algorithms are *recursive* due to the multiresolution properties of the basis.

From the above properties, several people have tried to solve differential equations using the orthonormal wavelet function[2], and we also developed a wavelet method to solve the neutron diffusion equation.[3] These wavelets have the property of localization in time and in frequency, and can take advantage of the sparsity after wavelet transform so as to obtain computationally fast algorithms.

The wavelet functions $\psi_{n,k}(x)$ are generated by dilation and translation operation such as

$$\psi_{n,k}(x) = 2^{n/2}\psi(2^n x - k), \tag{1}$$

for some $\psi \in L^2(\mathbf{R})$ and $(n,k) \in \mathbf{Z}^2$. Here $\mathbf{Z}$ and $\mathbf{R}$ respectively denote the set of integers and real numbers. $L^2(\mathbf{R})$ denotes the space of measurable, square-integrable functions.

We can generate these wavelet functions from the scaling functions $\phi_{n,k}(x)$ which have the same form

$$\phi_{n,k}(x) = 2^{n/2}\phi(2^n x - k), \tag{2}$$

for some $\phi \in L^2(\mathbf{R})$.

Suppose we define

$$V_n = closure < \phi_{n,k} : \ k \in \mathbf{Z} >, \tag{3}$$

$$W_n = closure < \psi_{n,k} : \ k \in \mathbf{Z} >, \tag{4}$$

then the scaling functions and wavelet functions have the following subspace relations :

$$\phi_{n,k} \in V_n, \tag{5}$$

$$\psi_{n,k} \in W_n, \tag{6}$$

$$\cdots V_{-1} \subset V_0 \subset V_1 \subset V_2 \cdots \tag{7}$$

$$V_n = V_{n-1} \oplus W_{n-1}, \tag{8}$$

$$\overline{\bigcup_n V_n} = L^2(\mathbf{R}), \tag{9}$$

$$\bigoplus_n W_n = L^2(\mathbf{R}), \tag{10}$$

where $\oplus$ stands for orthogonal sum.

From Eq. (8),

$$V_n = V_{n-m} \oplus W_{n-m} \oplus \cdots \oplus W_{n-2} \oplus W_{n-1}.$$

Thus, a wavelet decomposition at scale n becomes

$$f_n(x) = f_{n-m}(x) + \sum_{j=n-m}^{n-1} g_j(x), \quad f_j \in V_j,\ g_j \in W_j, \tag{11}$$

or

$$\begin{aligned} f_n(x) &= \sum_k a_{n,k}\phi_{n,k}(x) \\ &= \sum_k a_{n-m,k}\ \phi_{n-m,k}(x) + \sum_{j=n-m}^{n-1}\sum_k b_{n-m,k}\ \psi_{j,k}(x). \end{aligned} \tag{12}$$

In Eq. (12), the first expression represents the function f at a single "fine" scale n, while the second expression gives a multiscale representation of the function f at the coarser scales $\{n-m, \cdots, n-2, n-1\}$.

Because of Eqs. (7) and (9), there exist sufficiently large values of n such that $\|f - f_n\|$ is arbitrarily small for any $f \in L^2$. A fundamental result in wavelet theory on approximation of smooth functions is as follows[4] :

$$\|f - f_n\| \le C\|f^{(N)}\|2^{-Nn}, \tag{13}$$

where the first N moments of the wavelet $\psi(x)$ chosen are zero :

$$\int \psi(x)x^p dx = 0, \quad \text{for } p = 0, 1, 2, \cdots, N-1. \tag{14}$$

Eq.(13) states that smooth functions can be approximated with error $O(h^N)$ by combinations at every scale $h = 2^{-n}$.

Since $\phi \in V_0$, $\psi \in W_0$, and $V_1 = V_0 \oplus W_0$, we have the following two-scale relations :

$$\phi(x) = \sum_k h_k \phi(2x-k), \tag{15}$$

$$\psi(x) = \sum_k g_k \phi(2x-k). \tag{16}$$

If the two-scale relations are restricted by finite sums, the scaling and wavelet functions have compact supports. In addition, if ψ is required to generate orthonormal basis, the following relation holds :

$$g_k = (-1)^k h_{1-k}. \tag{17}$$

Among several wavelet bases, we have chosen the orthonormal compactly supported wavelets that were constructed by Daubechies.[1]

2. Fast Wavelet Transform (FWT)

Now, let us describe the algorithms of FWT and inverse fast wavelet transform. Assume that a finite sequence $s_k^0, k = 1, 2, \cdots, K$ is given.

i) Fast Wavelet Transform (Decomposition)

$$\begin{aligned} s_k^j &= \sum_{n=0}^{2N-1} h_n s_{n+2k-1}^{j-1}, \\ d_k^j &= \sum_{n=0}^{2N-1} g_n s_{n+2k-1}^{j-1}. \end{aligned} \tag{18}$$

ii) Inverse Fast Wavelet Transform (Reconstruction)

$$
\begin{aligned}
s_{2n}^{j-1} &= \sum_{k=1}^{N} h_{2k-1} s_{n-k+1}^{j} + \sum_{k=1}^{N} g_{2k-1} d_{n-k+1}^{j}, \\
s_{2n-1}^{j-1} &= \sum_{k=1}^{N} h_{2k-2} s_{n-k+1}^{j} + \sum_{k=1}^{N} g_{2k-2} d_{n-k+1}^{j},
\end{aligned}
\tag{19}
$$

where N is the Daubechies order.

For a two–dimensional case, fast wavelet transform (FWT) can be easily obtained by transforming a row vector and then a column vector, or transforming a column vector and then a row vector. So we can calculate the entries of the matrices $\alpha^j, \beta^j, \gamma^j$ with $j = 1, 2, \cdots, n$, that are the coefficients of two–dimensional wavelet transform. In Figure 1, we show the mapping of the entries in two–dimensional wavelet transform. Now, given a set of coefficients $t_{i,l}^0$ with $i, l = 1, 2, \cdots, K$, repeated application of the formula for the one–dimensional case produces

$$
\begin{aligned}
\alpha_{i,l}^{j} &= \sum_{k,m=0}^{2N-1} g_k g_m t_{k+2i-1,m+2l-1}^{j-1}, \quad \beta_{i,l}^{j} = \sum_{k,m=0}^{2N-1} g_k h_m t_{k+2i-1,m+2l-1}^{j-1}, \\
\gamma_{i,l}^{j} &= \sum_{k,m=0}^{2N-1} h_k g_m t_{k+2i-1,m+2l-1}^{j-1}, \quad t_{i,l}^{j} = \sum_{k,m=0}^{2N-1} h_k h_m t_{k+2i-1,m+2l-1}^{j-1},
\end{aligned}
\tag{20}
$$

with $i, l = 1, 2, \cdots, 2^{n-j}$, $j = 1, 2, \cdots, n$. Clearly, we formulate two–dimensional version of the wavelet transform in Eqs. (18) and (19), and provide an order N^2 scheme for the evaluation of the elements of all matrices $\alpha^j, \beta^j, \gamma^j$ with $j = 1, 2, \cdots, n$.

3. Fast Solution of Linear Systems in Reactor Analysis

Like fast Fourier transform (FFT), fast wavelet transform (FWT) is a fast and linear operation that operates on a data vector whose length is an integer power of two, transforming it into a numerically different vector of the same length. Also like FFT, FWT is invertible and in fact orthogonal. For FFT, the new domain has basis functions which are the familiar sines and cosines. In the wavelet domain, the basis functions are somewhat more complicated and have "wavelets". Unlike sines and cosines, individual wavelet functions are quite localized in space; simultaneously, quite localized in frequency or in characteristic scale. As we shall see below, this particular kind of dual localization of wavelets renders a large class of functions and operators sparse, or sparse to some high accuracy, when transformed into the wavelet domain.

One of the most interesting and promising wavelet applications is linear algebra. The basic idea is to think of an integral operator (that is, a large matrix) as a digital image. Suppose that the operator compresses well under a two–dimensional wavelet transform, i.e., that a large fraction of its wavelet coefficients are so small as to be negligible. Then any linear system involving the operator becomes a sparse system in the wavelet basis.

Our interest is in finding its properties of the wavelet transformed matrix. In other words, we want to solve

$$
\mathbf{A}\mathbf{x} = \mathbf{b}. \tag{21}
$$

We first wavelet–transform matrix $\mathbf{A}$ and vector $\mathbf{b}$ such that

$$
\tilde{\mathbf{A}} \equiv \mathbf{W}\mathbf{A}\mathbf{W}^{\mathbf{T}}, \quad \tilde{\mathbf{b}} \equiv \mathbf{W}\mathbf{b}, \tag{22}
$$

where $\mathbf{W}$ represents the one–dimensional wavelet transform. Then Eq. (24) can be transformed into

$$
\tilde{\mathbf{A}}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}, \tag{23}
$$

and finally get the answer by the inverse wavelet transform

$$
\mathbf{x} = \mathbf{W}^{\mathbf{T}}\tilde{\mathbf{x}}. \tag{24}
$$

We recall that the condition number of a matrix is defined as the ratio of the largest and the smallest singular values. If a matrix has a null space (the actual null space or a null space for a given accuracy), then by the condition number we ascertain the ratio of the largest singular value to the smallest singular value above the threshold of accuracy. The condition number controls the rate of convergence of a number of iterative algorithms for solving linear systems ; for example the number of iterations of the conjugate gradient method is $O(\sqrt{\kappa})$, where κ is the condition number of the matrix. After applying a particular preconditioner, the condition number κ_p of the operator is uniformly bounded with respect to the size of the matrix. In the following examples, the standard form D_w of the periodized second derivative D of size $M \times M$, where $M = 2^m$, is preconditioned by the diagonal matrix P ,

$$D_w^p = PD_wP, \tag{25}$$

where $P_{ij} = \delta_{il}2^j$, $1 \leq j \leq m$, and where j is chosen depending on i, l so that $N - N/2^{j-1} + 1 \leq i, l \leq N - N/2^j$, and $P_{MM} = 2^m$. That is,

$$P = \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2^{m-1} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 2^m & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 2^m \end{pmatrix}.$$

For example, the matrix D is given as follows:

$$D = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}.$$

In Table I we compare the original condition number κ of D, κ_w of D_w and κ_p of D_w^p. Such a preconditioning technique is useful to treat singular matrix.

From the above FWT, the linear equation which is nearly singular can be solved by a standard iterative method. In this paper, we take as an example the flux reconstruction problem in reactor analysis without preconditioning.

Reconstruction of pinwise flux and power has been of much interest in reactor analysis. There are many methods such as form function method and global/local iteration method[5,6]. We can divide them into two kinds of methods for reconstruction of pinwise heterogeneous flux from nodal values. The one finds the homogeneous distribution and then multiplying form function, the other constitutes solving a linear matrix equation using boundary flux or current.[7] We try to develop the latter method using fast wavelet transform (FWT). Here our interest is how to solve the linear equation overcoming singularity occurring when we perform fine–mesh calculation by a finite difference scheme.

FWT converts the matrix equation to a fast form, that is less singular, to be iterated. We may also select other methods, such as singular value decomposition (SVD), multiplying the transpose matrix which is similar to conjugate gradient (CG), or a direct inverse scheme. But the results show that the FWT method is fastest among the above methods.

In the reconstruction problem, we have nine nodal values : four corner point fluxes, four surface average fluxes and one nodal average flux. We also have multiplication factor k_{eff}. From the nine nodal values, we can find surface flux or current distribution which can be used as boundary conditions. AFEN[8,9], which is analytic function expansion nodal method, is used to find the surface boundary condition. Then the diffusion equation may be solved in the usual way. But it happens often times that the system is not convergent by standard iteration methods, because the spectral radius of the iteration matrix is greater one. However, if the system is transformed by wavelets, the spectral radius may be reduced below one and any iterative schemes are able to solve the system.

The two–dimensional two–group diffusion equation,

$$\begin{aligned} -\nabla D_1 \cdot \nabla \phi_1 + (\Sigma_{12} + \Sigma_{a1} - \frac{1}{k_{eff}} \nu \Sigma_{f1}) \phi_1 - \frac{1}{k_{eff}} \nu \Sigma_{f2} \phi_2 &= 0, \\ -\nabla D_2 \cdot \nabla \phi_2 + \Sigma_{a2} \phi_2 - \Sigma_{12} \phi_1 &= 0, \end{aligned} \tag{26}$$

can be transformed, when we reconstruct pinwise heterogeneous flux, to a matrix equation as follows:

$$\mathbf{Ax} = \mathbf{b}, \tag{27}$$

if we discretize Eq. (26) by a standard finite difference box scheme.

Eq. (27) can be transformed as in Eq. (23), so as to achieve convergence. We solved the above equation by Gauss-Seidel and successive overrelaxation (SOR) methods. We tested our scheme on sample problems.

First we take a simple one-dimensional reconstruction problem of which solution is known to us as cosine shape. The boundary condition we used is current boundary condition (Neumann boundary condition). Figure 2 shows the configuration of the problem and the results are provided in Table II. We compare the computing time and spectral radius of the Jacobi matrix. From the results we find that FWT converts the property of the matrix to converge by standard iterative schemes.

As a second test, we take a two-dimensional eigenvalue problem, that is, a two-dimensional two-group diffusion equation. We also use Neumann boundary condition. The configuration is shown in Figure 3 and the cross section data we used are given in Table III. We also compare the computing time and spectral radius of the Jacobi matrix in Table IV. We also observe that FWT can decrease the spectral radius of the Jacobi matrix below one.

From the results we also find the FWT method is faster than the other methods in solving near-singular problems.

4. Conclusions

The fast wavelet transform (FWT) has high potential for many applications. For a linear equation system, we developed a new solver with FWT. One of the properties of FWT is reducing the spectral radius of the Jacobi matrix so as to obtain convergence of the iteration. This property was used in this study to the flux reconstruction problem in reactor analysis. Usually, when we reconstruct the flux, the iteration matrix becomes nearly singular, so we cannot use standard iteration schemes. In this problem, FWT did reduce the spectral radius of the iteration matrix. We conclude that it is worthwhile investigating further FWT for its fuller utilization in nuclear engineering problems.

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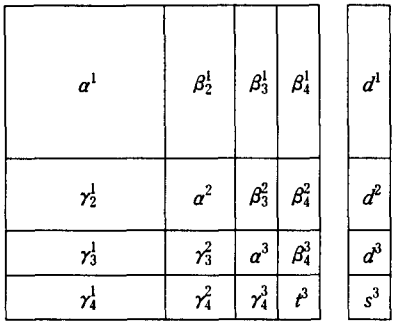
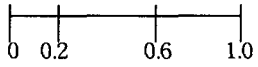


Fig.1 Representation of the decomposed matrix

Table I

Condition Number of the Matrix of Periodized Second Derivative

matrix size	Daubechies' order	condition number		
		χ	χ_u	χ_p
32×32	3	1.04087E+02	1.14666E+02	8.08654E+00
	6		1.04916E+02	5.20035E+00
64×64	3	4.15350E+02	4.58204E+02	9.09907E+00
	6		4.18739E+02	5.26105E+00
128×128	3	1.66041E+03	1.83078E+03	1.00210E+01
	6		1.67577E+03	5.28970E+00



* Boundary condition : $\phi(0) = \phi(1) = 0$
* To reconstruct flux in [0.2,0.6] with current boundary condition

Fig. 2 Configuration of sample problem 1

Table II

Comparison of Computing Times and Spectral Radii

matrix size	Daubechies' order	computing time (sec)				spectral radius of Jacobi matrix	
		FWT		CG	SVD	FWT	original
		GS	SOR				
32×32	N=3	0.91	0.38	0.36	0.52	0.97988	1.00008
	N=6	0.93	0.47			0.96072	
64×64	N=3	6.70	1.80	2.34	3.17	0.99017	1.00020
	N=6	6.71	1.79			0.98073	

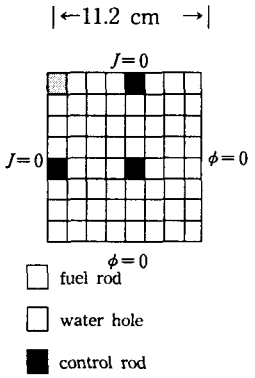


Fig. 3 Configuration of sample problem 2

Table III

Cross Sections for Sample Problem

type	group	D (cm)	Σ_a (cm^{-1})	$\nu\Sigma_f$ (cm^{-1})	Σ_{12} (cm^{-1})
fuel	1	1.500	0.0130	0.0065	0.020
	2	0.400	0.1800	0.2400	
water	1	1.700	0.001	0.000	0.035
	2	0.350	0.050	0.000	
control rod	1	1.1133	0.0836661	0.000	0.037529
	2	0.18401	0.96726	0.000	

Table IV

Comparison of Computing Times and Spectral Radii

Daubechies order	computing time (sec)				spectral radius of Jacobi matrix	
	FWT		CG	SVD	FWT	original
	GS	SOR				
N=3	13.95	7.32	30.53	25.94	0.97872	1.02256
N=6	12.74	11.36			0.97249	