An Optimum Maintenance Policy: A Bayesian Approach to Periodic Incomplete Preventive Maintenance with Minimal Repair at Failure

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ABSTRACT

In this paper we consider a Bayesian theoretic approach to periodic incomplete preventive maintenance with minimal repair at failure. We assume that the system failure rate is increasing as the frequency of PM increases and that the system is replaced at the K-th PM under this maintenance strategy. The optimal policies which minimize the expected cost rates are discussed. We seek the optimal periodic PM interval x and replacement time K under a Weibull failure intensity. Assuming suitable prior distribution for the Weibull parameters, we derive the posterior distribution incorporating failure data and obtain the updated optimal replacement strategies.

1. Introduction

After the work by Barlow and Hunter(1960), many researchers have proposed various maintenance polices which include age replacement, block replacement, periodic replacement with minimal repair at failure etc.. Though age replacement and condition based maintenance policies are studied by many researchers, time based maintenance where replacements are done at specified time intervals with minimal repair at failure is the most popular maintenance policy in heavy industries or steel companies. These industries generally have periodic preventive maintenance (PM) schedule two or three times a month and have a major overhaul one or two times a year. A simple periodic replacement policy assumes that the system becomes new after each PM. In reality, however, the improvement depends on the age of the system as well as the cost and the frequency of PM. Hence, the system has a different failure distribution after each PM and generally the failure rate increases with the frequency of PM.

Thus, in this paper, we assume that (i) PM are done at kx (k=1, 2, ..., K), where x is a PM interval and K is the number of PM's until the complete replacement, (ii) the system is replaced at K-th PM for an infinite time span, i.e. Kx is the replacement interval, (iii) the system undergoes minimal repair at failure, hence the failure rate remains unchanged by any repair, (iv) but the system failure rate, say $\lambda_k(t)$, in the k-th period of PM is increased by k increasing as such $\lambda_k(t) < \lambda_{k+1}(t)$ for any t > 0, which means that the system is undergoing incomplete PM, (v) the times for PM, minimal repair and replacement are negligible or reflected at costs. Under these assumption, Nakagawa(1986) proposed a expected cost rate function and proposed the optimal PM interval and replacement time with deterministic failure rate parameters.

We propose a Bayesian approach to periodic incomplete PM and replacement policy for determining the optimal PM interval and replacement time interval with minimal repair at failure. Minimizing the total cost is used as an object function for infinite time span. The underlying assumption behind such strategies is that an in-service failure of the system is more costly than a planned replacement. We assume that the failure behavior of system can be described by a Weibull hazard function. One of the rationales adopting the Bayesian approach is based on the fact that the failure parameters vary over time as the system is stablizing or deteriorating.

2. Cost Function and Prior Distribution of Failure Parameters

To make the usual assumption of aging, we model the failure process by assuming that the number of system failure during k-th PM interval which is the time interval between (k-1)-th PM and k-th PM is a non-homogeneous Poisson process with intensity function,

$$\lambda_k(t) = w_k \beta t^{\beta-1} \quad (1)$$

where β is unknown coefficient, and w_k is represented by αr^{k-1} with unknown α and known r coefficients so that $w_k < w_{k+1}$ (k=1, 2, ..., K) to ensure $\lambda_k(t) < \lambda_{k+1}(t)$. Thus, when coefficients α and β are given, the mean failure rate function during the k-th PM interval which is the conditional expectation of $N_k(x)$, the number of failures during k-th PM interval is

$$R_k(x) = E[N_k(x)|\alpha,\beta] = \int_0^x \lambda_k(t)dt$$

$$= \int_0^x \alpha r^{k-1}\beta t^{\beta-1}dt = \alpha r^{k-1}x^{\beta}. \quad (2)$$

Therefore, the probability of n_k failures during k-th PM in an interval of length x is given as

$$\Pr\{N_k(x) = n_k\} = \frac{\{R_k(x)\}^{n_k}}{n_k!}\exp\{-R_k(x)\}. \quad (3)$$

From these, the distribution for T_{k1}, the time to first failure during k-th PM interval, could be written as

$$\Pr\{T_{k1} \geq x|\alpha,\beta\} = \Pr\{N_k(x) = 0\}$$

$$= \exp\left(-\alpha r^{k-1}x^{\beta}\right) \quad (4)$$

and the probability density function of T_{k1} is

$$f_{k1}(t|\alpha,\beta) = w_k\beta t^{\beta-1}\exp\left(w_k t^{\beta}\right)$$

$$= \alpha r^{k-1}\beta t^{\beta-1}\exp\left(-\alpha r^{k-1}t^{\beta}\right). \quad (5)$$

The underlying model for times to first failures during the k-th PM is Weibull density function which is used extensively in reliability and maintenance analysis.

With the failure intensity rate $\lambda_k(t)$ given at equation (1), Nakagawa(1986) proposed the following expected cost rate function:

$$C(x, K) = \frac{c_1\sum_{k=1}^{K}\int_0^x \lambda_k(t)dt + (K-1)c_2 + c_3}{Kx} \quad (6)$$

where c_1 is the cost of minimal repair, c_2 is the cost of PM, and c_3 is the cost of replacement with $c_3 \geq c_2$. When we assume that unknown coefficients α and β are given, which is the common approach in the literature of Bayesian analysis for developing an optimal PM interval and replacement strategy, the expected cost per unit time is

$$E[C(x, K)|\alpha, \beta]$$

$$= E\left[\frac{c_1\sum_{k=1}^{K}E\{N_k(x)|\alpha,\beta\} + (K-1)c_2 + c_3}{Kx}\right]$$

$$= \frac{c_1\alpha x^{\beta}(r^K - 1)/(r-1) + (K-1)c_2 + c_3}{Kx}. \quad (7)$$

Since our analysis involves expressing our uncertainty about the unknown coefficients α and β via prior distributions and coefficient r was assumed as known constant in the previous expression of w_k, we select, as an appropriate prior for α, the gamma distribution Gamma(a, b) given by

$$g(\alpha) = \frac{b^a}{\Gamma(a)}\alpha^{a-1}e^{-b\alpha}, \alpha > 0 \quad (8)$$

where a, b > 0 are the specified parameters. For the prior distribution of the shape parameter β, we use a discrete distribution as in Mazzuchi(1996) by using a discretization of the beta density on (β_L, β_U) since this allows for great flexibility in representing prior uncertainty. The beta density is given by

$$h(\beta) = \frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}\frac{(\beta-\beta_L)^{c-1}(\beta_U-\beta)^{d-1}}{(\beta_U-\beta_L)^{c+d-1}}$$

$$\text{for } 0 \leq \beta_L \leq \beta \leq \beta_U \quad (9)$$

where β_L, β_U, c, d > 0 are specified parameters. The definition of distribution for β is

$$P_l = \Pr\{\beta=\beta_l\} = \int_{\beta_l-\delta/2}^{\beta_l+\delta/2}h(\beta)d\beta \quad (10)$$

where $\beta_l = \beta_L + \delta(2l-1)/2$ and $\delta = (\beta_U - \beta_L)/m$ for l = 1, 2, ..., m.

We assume that the quantities α and β are independent initially and thus the joint prior distribution is the product of the distributions of α and β. Because β controls the rate at which the system(or component) ages and for items which experience aging as assumed at failure rate density, β must greater than 1. Thus, β_L may be assumed as 1 and other parameters β_U, c, d may be guessed by an expert or evaluated by effective methods. Once the prior distribution for β has been established, the parameters of prior of α could be obtained by eliciting information about the time to first failure during k-th PM and equating these with appropriate expression from the predictive distribution

$$f_{k1}(t) = \sum_{l=1}^{m}\int_0^{\infty}f_{k1}(t|\alpha,\beta_l)g(\alpha)d\alpha \cdot P_l$$

$$= \sum_{l=1}^{m}\frac{ab^a\beta_l r^{k-1}t^{\beta_l-1}}{(b+r^{k-1}t^{\beta_l})^{a+1}}\cdot P_l. \quad (11)$$

3. The Optimal Periodic PM Interval and Replacement Time

The optimal PM interval and replacement strategy for the periodic PM with minimal repair at failure is obtained as the value of replacement number K and interval time x which minimizes the expected cost

$$E[C(x, K)] = E_{\alpha,\beta}E[C(x, K)|\alpha, \beta]$$

$$= \sum_{l=1}^{m}\frac{c_1\frac{a}{b}\frac{r^K-1}{r-1}x^{\beta_l} + (K-1)c_2 + c_3}{Kx}\cdot P_l$$

$$= \sum_{l=1}^{m}\frac{c_1 a(r^K-1)x^{\beta_l}}{bKx(r-1)}\cdot P_l + \frac{(K-1)c_2 + c_3}{Kx}. \quad (12)$$

To find an x^* which minimizes E[C(x, K)], we differentiate right-hand side of the equation (12) with respect to x and set it equal to 0, then we should solve the following equation (13) with numerical techniques when K is given, which is elicited at the following equations (18) or (19),

$$\frac{dE[C(x, K)]}{dx} = \sum_{l=1}^{m} \frac{c_1(\beta_l - 1)\frac{a}{b}\frac{r^K - 1}{r-1} x^{\beta_l} - (K-1)c_2 - c_3}{Kx^2} \cdot P_l = 0. \quad (13)$$

Since the second derivative of E[C(x, N)] should be greater than 0 for the existence and uniqueness of x satisfying above equation (13),

$$\frac{d^2E[C(x, K)]}{dx^2} = \sum_{l=1}^{m} c_1(\beta_l - 1)(\beta_l - 2)\frac{a}{Kb}\frac{r^K - 1}{r-1} x^{\beta_l - 3} \cdot P_l + \frac{2(K-1)c_2 + 2c_3}{Kx^3} > 0 \quad (14)$$

the lower limit β_L of β_l should be greater than 2 or less than 1, which is in order to be sufficient for the convexity of E[C(x, K)]. From the equation (13), we have

$$\sum_{l=1}^{m} (\beta_l - 1)\frac{a}{b}\frac{r^K - 1}{r-1} x^{\beta_l} \cdot P_l = \{(K-1)c_2 - c_3\}/c_1 . \quad (15)$$

The left hand side of previous equation is also increasing to infinity, since

$$\left\{\sum_{l=1}^{m} (\beta_l - 1)\frac{a}{b}\frac{r^K - 1}{r-1} x^{\beta_l} \cdot P_l\right\}' = \sum_{l=1}^{m} (\beta_l - 1)\beta_l\frac{a}{b}\frac{r^K - 1}{r-1} x^{\beta_l - 1} \cdot P_l > 0, \quad (16)$$

$$\sum_{l=1}^{m} (\beta_l - 1)\frac{a}{b}\frac{rK - 1}{r-1} x^{\beta_l} \cdot P_l \geq \sum_{l=1}^{m} (\beta_l - 1)\frac{a}{b}\frac{r^K - 1}{r-1} t^{\beta_l} \cdot P_l \quad (17)$$

for t < x. Thus if $\beta > 1$, for the satisfaction of inequality (17) and assumption of an aging system, then there exists a finite and unique x^* which satisfies previous equality (13) or (15) for any integer K. Next, to find an K^* which minimizes E[C(x, K)], we form the inequalities

$$E[C(x, K+1)] \geq E[C(x, K)] \text{ and}$$
$$E[C(x, K)] < E[C(x, K-1)], \quad (18)$$

which imply

$$L(x, K) \geq (c_3 - c_2)/c_1 \text{ and}$$
$$L(x, K-1) < (c_3 - c_2)/c_1 \quad (19)$$

where

$$L(x, K) = K\sum_{l=1}^{m} \frac{a}{b} r^K x^{\beta_l} \cdot P_l - \sum_{l=1}^{m} \frac{a}{b}\frac{r^K - 1}{r-1} x^{\beta_l} \cdot P_l , \quad (K = 1, 2, \ldots,). \quad (20)$$

and when K is 0, then L(x, K) = 0. From the assumption that $\lambda_k(t) < \lambda_{k+1}(t)$ for any t>0, that is, under the condition r>1, we have

$$L(x, K) - L(x, K-1) = K(r-1)r^{K-1}\frac{a}{b}\sum_{l=1}^{m} x^{\beta_l} \cdot P_l > 0. \quad (21)$$

This means that L(x, K) is increasing in K and tends to infinity as K goes to infinity. Hence there exists a finite and unique K^* which satisfies above two inequalities (20) and (21) for any x>0.To solve out optimal x^* and K^*, we must use the numerical method. For the optimal x^*, dissimilar with the case of the Nakagawa(1986), equation (13) or (15) can not have a closed form.

To develop the concept of an adaptive replacement strategy, we consider a system which is minimally repaired upon at i-th failure $t_{k,i}$, i = 1, 2, ...,n_k where n_k is the number of failures during the k-th PM interval, k = 1, 2, ..., K and receives periodic PMs at kx, k = 1, 2, ..., K-1 and is replaced by new system at the end of the replacement cycle Kx. If there are n_k failures during the k-th PM interval and system failures are observed at times $0<t_{11}<t_{12}<\ldots<t_{1n_1}<x<x+t_{21}<x+t_{22}<\ldots<x+t_{2n_2}<2x<\ldots<(K-1)x<(K-1)x+t_{K1}<(K-1)x+t_{K2}<\ldots<(K-1)x+t_{Kn_K}<Kx$, then we may write the likelihood as

$$\pounds(\alpha, \beta) = \prod_{k=1}^{K}\left[\left\{\prod_{i=1}^{n_k} \alpha r^{k-1}\beta t_{ki}^{\beta-1}\right\} \cdot \exp\left\{-\alpha r^{k-1} x^{\beta}\right\}\right]. \quad (22)$$

The posterior distribution of α and β, given $t^{(K,n_K)} = (t_{11}, t_{12}, \ldots, t_{1n_1}, t_{21}, t_{22}, \ldots, t_{2n_2}, \ldots, t_{K1}, t_{K2}, \ldots, t_{Kn_K})$, is obtained via Bayes theorem as

$$f(\alpha, \beta_l | t^{(K,n_K)}) \propto \prod_{k=1}^{K}\left[\left\{\prod_{i=1}^{n_k} \alpha r^{k-1}\beta_l t_{ki}^{\beta_l - 1}\right\} \cdot \exp\left\{-\alpha r^{k-1} x^{\beta_l}\right\}\right] \times \alpha^{a-1}\exp\{-b\alpha\} \cdot P_l \quad (23)$$

and by rearranging terms we could obtain the posterior joint distribution of α and β as

$$f(\alpha,\beta_l|t^{(K,n_K)}) = \frac{\beta_l^{\sum_{k=1}^{K} n_k}\left(\prod_{k=1}^{K}\prod_{j=1}^{n_k} t_{kj}\right)^{\beta_l-1} \cdot P_l}{\sum_{i=1}^{m}\left[\frac{\beta_i^{\sum_{k=1}^{K} n_k}\left(\prod_{k=1}^{K}\prod_{j=1}^{n_k} t_{kj}\right)^{\beta_i-1} \cdot P_i}{\left(b+\sum_{k=1}^{K} r^{k-1}x^{\beta_i}\right)^{a+\sum_{k=1}^{K} n_k}}\right]} \times \frac{\alpha^{a-1+\sum_{k=1}^{K} n_k}\exp\left[-\alpha\left(b+\sum_{k=1}^{K} r^{k-1}x^{\beta_l}\right)\right]}{\Gamma\left(a+\sum_{k=1}^{K} n_k\right)} \quad (24)$$

which can be written as the product of $f(\alpha|\beta_l,t^{(K,n_K)})$ and $P\{\beta=\beta_l|t^{(K,n_K)})$ where $f(\alpha|\beta_l,t^{(K,n_K)}) \sim \text{Gamma}(a^*, b^*)$ with $a^* = a + \sum_{k=1}^{K} n_k$, $b^* = b + \sum_{k=1}^{K} r^{k-1}x^{\beta_l}$, and the posterior distribution of β can be obtained as

$$P\{\beta=\beta_l|t^{(K,n_K)}) = P_l^* = \frac{\frac{\beta_l^{\sum_{k=1}^{K} n_k}\left(\prod_{k=1}^{K}\prod_{j=1}^{n_k} t_{kj}\right)^{\beta_l-1} \cdot P_l}{\left(b+\sum_{k=1}^{K} r^{k-1}x^{\beta_l}\right)^{a+\sum_{k=1}^{K} n_k}}}{\sum_{i=1}^{m}\left[\frac{\beta_i^{\sum_{k=1}^{K} n_k}\left(\prod_{k=1}^{K}\prod_{j=1}^{n_k} t_{kj}\right)^{\beta_i-1} \cdot P_i}{\left(b+\sum_{k=1}^{K} r^{k-1}x^{\beta_l}\right)^{a+\sum_{k=1}^{K} n_k}}\right]}. \quad (25)$$

Thus the posterior distribution of α and β are no longer independent. And revision of the expected cost function $E\left[C(x,K)|t^{(K,n_K)}\right]$ for the optimal PM interval and replacement strategy with the posterior distribution and the calculation of its optimal x^* and K^* are achieved in a straight manner by replacing the prior quantities a, b and P_l by the posterior quantities a^*, b^* and P_l^* respectively, in equations (12), (15) and (19).

4. Discussion

This paper uses an incomplete periodic PM policies to introduce the concept that the optimal PM interval and replacement and uncertainty analysis can be possible easily by adopting a Baysian theoretic view. The selection of Weibull failure density model and the priors was mainly for illustrative purpose. However, these are central in the theory of repairable systems and preventive maintenance reliability theory. In the context, though we assumed that the increasing factor r of scaling parameter w_k of failure density function is known or given and should be greater than 1 for the existence and uniqueness of K^*, this value could be given by expert or methods of the classical parameter estimation. Further, many alternative selection of the type of failure rate or the use of prior distributions are possible.

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