퍼지확률변수에 대한 Kolmogorov의 대수의 강법칙

Kolmogorov's Strong Law of Large Numbers for Fuzzy Random Variables

Yun Kyong Kim
Department of Mathematics, Dongshin University
Naju, Chonnam, Korea, 520-714

ABSTRACT

In this paper, we generalize Kolmogorov's strong law of large numbers to the the case of independent fuzzy random variables.

1. Introduction

Strong Laws of large numbers for independent fuzzy random variables have been stuied by several people. A SLLN for i.i.d. fuzzy random variables was obtained by Kruse [12], and SLLN for independent fuzzy random variables by Miyakoshi and Shimbo [13]. Also, Klement, Puri and Ralescu [11] proved some limit theorem containing a SLLN, and Inoue [7] obtained a SLLN for independent tight fuzzy random sets, Hong and Kim [6] studied Marcinkiewicz -type law of large numbers for fuzzy random variables under additional assumption. Recently, Kim [8] obtained SLLN for levelwise independent fuzzy random variables and Kim [9] generalized Chung's law of large numbers to the case of fuzzy random variables.

The purpose of this paper is to obtain Kolmogorov's strong law of large numbers for sums of independent fuzzy random variables. Section 2 is devoted to describe some basic concepts of fuzzy numbers. The main result is given in section 3.

2. Preliminaries

In this section, we describe some basic concepts of fuzzy random variables. First, we introduce the concept of fuzzy numbers. Let R denote the real line. A fuzzy number is a fuzzy set $\tilde{u} : R \to [0,1]$ with the following properties;

(1) $\tilde{u}$ is normal, i.e., there exists $x \in R$ such that $\tilde{u}(x) = 1$.

(2) $\tilde{u}$ is upper semicontinuous.

(3) $supp\ \tilde{u} = cl\ \{ x \in R : \tilde{u}(x) > 0 \}$ is compact.

(4) $\tilde{u}$ is a convex fuzzy set, i.e., $\tilde{u}(\lambda x + (1-\lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y))$ for $x, y \in R$ and $\lambda \in [0,1]$.

We denote the family of all fuzzy numbers by $F(R)$. For a fuzzy set $\tilde{u}$, the α-level set of $\tilde{u}$ is defined by

$$L_\alpha \tilde{u} = \begin{cases} \{x : \tilde{u}(x) \geq \alpha\} & \text{if } 0 < \alpha \leq 1 \\ supp\ \tilde{u} & \text{if } \alpha = 0 \end{cases}$$

Then it follows that $\tilde{u}$ is a fuzzy number if and only if $L_1 \tilde{u} \neq \emptyset$ and $L_\alpha \tilde{u}$ is a closed bounded interval for each $\alpha \in [0,1]$. From this characterization of fuzzy numbers, a fuzzy number $\tilde{u}$ is completely determined by the end points of the intervals $L_\alpha \tilde{u} = [u_\alpha^-, u_\alpha^+]$.

Theorem 2.1. **([5])** For $\tilde{u} \in F(R)$, denote $u^-(\alpha) = u_\alpha^-$ and $u^+(\alpha) = u_\alpha^+$ by considering as functions of $\alpha \in [0,1]$. Then the followings hold.

(1) u^- is a bounded increasing function on $[0,1]$.

(2) u^+ is a bounded decreasing function on $[0,1]$.

(3) $u^-(1) \leq u^+(1)$.

(4) u^- and u^+ are left continuous on $(0,1]$ and right continuous at 0.

(5) If v^- and v^+ satisfy above (1)-(4), then there exists a unique $\tilde{v} \in F(R)$ such that $L_\alpha \tilde{v} = [v^-(\alpha), v^+(\alpha)]$.

The above theorem implies that we can identify a fuzzy number $\tilde{u}$ with the parametrized representation $\{(u_\alpha^-, u_\alpha^+) \mid 0 \leq \alpha \leq 1\}$. Suppose now that $\tilde{u}, \tilde{v} \in F(R)$ are fuzzy numbers whose representations are $\{(u_\alpha^-, u_\alpha^+) \mid 0 \leq \alpha \leq 1\}$ and $\{(v_\alpha^-, v_\alpha^+) \mid 0 \leq \alpha \leq 1\}$, respectively. If we define

$$(\tilde{u}+\tilde{v})(z) = \sup_{x+y=z} \min(\tilde{u}(x), \tilde{v}(y))$$

$$(\lambda\tilde{u})(z) = \begin{cases} \tilde{u}(z/\lambda), & \lambda \neq 0 \\ \tilde{0}, & \lambda = 0 \end{cases}$$

where $\tilde{0} = I_{\{0\}}$ is the indicator function of $\{0\}$, then

$$\tilde{u} + \tilde{v} = \{(u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+) \mid 0 \leq \alpha \leq 1\}$$

$$\lambda\tilde{u} = \begin{cases} \{(\lambda u_\alpha^-, \lambda u_\alpha^+) \mid 0 \leq \alpha \leq 1\} & \text{if } \lambda \geq 0 \\ \{(\lambda u_\alpha^+, \lambda u_\alpha^-) \mid 0 \leq \alpha \leq 1\} & \text{if } \lambda < 0 \end{cases}$$

Now we define a metric d on $F(R)$ by

$$d(\tilde{u}, \tilde{v}) = \sup_{0 \leq \alpha \leq 1} d_H(L_\alpha\tilde{u}, L_\alpha\tilde{v}) \tag{2.1}$$

where d_H is the Hausdorff mertic defined as

$$d_H(L_\alpha\tilde{u}, L_\alpha\tilde{v}) = \max(|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|).$$

Also, the norm $\|\tilde{u}\|$ of fuzzy number $\tilde{u}$ will be defined as

$$\|\tilde{u}\| = d(\tilde{u}, \tilde{0}) = \max(|u_0^-|, |u_0^+|).$$

3. Main Result

In this section, we will generalize Kolmogorov's strong law of large numbers to the case of fuzzy random variables by using the metric d defined as in (2.1). This will be accomplished by similar arguments as in Kim [9]. First we review the definition of fuzzy random variables. Let (Ω, A, P) denote a complete probability space. For a fuzzy number valued function $\tilde{X} : \Omega \to F(R)$ and a subset B of R, $\tilde{X}^{-1}(B)$ denotes the fuzzy subset of Ω defined by

$$\tilde{X}^{-1}(B)(\omega) = \sup_{\omega \in B} \tilde{X}(\omega)(x)$$

for every $\omega \in \Omega$. The function $\tilde{X}: \Omega \to F(R)$ is called a fuzzy random variables if for every closed subset B of R, the fuzzy set $\tilde{X}^{-1}(B)$ is measurable when considered as a function from Ω to $[0,1]$. If we denote $\tilde{X}(\omega)=\{(X_\alpha^-(\omega), X_\alpha^+(\omega))|0\le\alpha\le 1\}$, then it is well-known that $\tilde{X}$ is a fuzzy random variable if and only if for each $\alpha\in[0,1]$, X_α^- and X_α^+ are random variables in the usual sense. A fuzzy random variables $\tilde{X} = \{(X_\alpha^-, X_\alpha^+)|0 \le \alpha \le 1\}$ is called integrable if for each $\alpha\in[0,1]$, X_α^- and X_α^+ are integrable, equivalently, $\int \| \tilde{X} \| dP < \infty$. In this case, the expectation of $\tilde{X}$ is defined by

$$E\tilde{X} = \int \tilde{X}\, dP = \{(\int X_\alpha^-\, dP, \int X_\alpha^+\, dP) \mid 0 \le \alpha \le 1\}.$$

For detailes, see Kim and Ghil [10].

Definition 3.1. Let $\tilde{X}, \tilde{Y}$ be two fuzzy random variables whose representations are $\{(X_\alpha^-, X_\alpha^+) \mid 0 \le \alpha \le 1\}$ and $\{(Y_\alpha^-, Y_\alpha^+) \mid 0 \le \alpha \le 1\}$, respectively.

(1) $\tilde{X}$ and $\tilde{Y}$ are called independent if the σ-fields $\sigma(\{X_\alpha^-, X_\alpha^+ \mid 0 \le \alpha \le 1\})$ and $\sigma(\{Y_\alpha^-, Y_\alpha^+ \mid 0 \le \alpha \le 1\})$ are independent.

(2) $\tilde{X}$ and $\tilde{Y}$ are called level-wise identically distributed if for each $\alpha\in[0,1]$, (X_α^-, X_α^+) and (Y_α^-, Y_α^+) are identically distributed random vectors.

Now we obtain Kolmogorov's strong law of large numbers for independent fuzzy random variables. The following lemma is useful in proving the main result.

Lemma 3.2. For each $\tilde{u}\in F(R)$ and each $\varepsilon > 0$, there exist a partition of $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_r$ of $[0,1]$ such that

(3.1) $$\max\ (u_{\alpha_k}^- - u_{\alpha_{k-1}^+}^-,\ u_{\alpha_{k-1}^+}^+ - u_{\alpha_k}^+) < \varepsilon,\ k=1,2,\ \ldots\ ,r.$$

Proof. See Lemma 3.2 of Kim [9].

Theorem 3.3. Let $\{\tilde{X}_n\}$ be a sequence of independent and levelwise identically distributed fuzzy random variables. There exists $\tilde{b} \in F(R)$ such that

$$d\left(\frac{1}{n}\sum_{i=1}^{n}\tilde{X}_i, \tilde{b}\right) \to 0 \quad a.s. \tag{3.2}$$

if and only if $E\|\tilde{X}_1\| < \infty$. Furthermore, if (3.2) holds, then $\tilde{b} = E\tilde{X}_1$.

Proof. (Sufficiency) Let $\tilde{X}_n = \{(X^-_{n\alpha}, X^+_{n\alpha}) | 0 \le \alpha \le 1\}$ and $\varepsilon > 0$ be given. Then by lemma 3.2, there exists a partition $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_r = 1$ of $[0,1]$ such that

$$\max\left(EX^-_{1\alpha_k} - EX^-_{1\alpha^+_{k-1}},\ EX^+_{1\alpha^+_{k-1}} - EX^+_{1\alpha_k}\right) < \varepsilon, \quad k = 1, 2, \ldots, r. \tag{3.3}$$

Now if $0 < \alpha \le 1$, then $\alpha_{k-1} < \alpha \le \alpha_k$ for some k. Thus, using (3.3), we obtain

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}X^-_{i\alpha} - EX^-_{1\alpha} &\le \frac{1}{n}\sum_{i=1}^{n}X^-_{i\alpha_k} - EX^-_{1\alpha^+_{k-1}} \\ &\le \frac{1}{n}\sum_{i=1}^{n}X^-_{i\alpha_k} - EX^-_{1\alpha_k} + \varepsilon\end{aligned}$$

and

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}X^-_{i\alpha} - EX^-_{1\alpha} &\ge \frac{1}{n}\sum_{i=1}^{n}X^-_{i\alpha^+_{k-1}} - EX^-_{1\alpha_k} \\ &\ge \frac{1}{n}\sum_{i=1}^{n}X^-_{i\alpha^+_{k-1}} - EX^-_{1\alpha^+_{k-1}} - \varepsilon\end{aligned}$$

Hence,

$$\begin{aligned}&\sup_{0\le\alpha\le1}\left|\frac{1}{n}\sum_{i=1}^{n}X^-_{i\alpha} - EX^-_{1\alpha}\right| \\ &\le \max_{1\le k\le r}\left(\left|\frac{1}{n}\sum_{i=1}^{n}X^-_{i\alpha_k} - EX^-_{1\alpha_k}\right| \vee \left|\frac{1}{n}\sum_{i=1}^{n}X^-_{i\alpha^+_{k-1}} - EX^-_{1\alpha^+_{k-1}}\right|\right) + \varepsilon\end{aligned}$$

By Kolmogorov's strong law of large numbers,

$$\overline{\lim_{n\to\infty}}\ \sup_{0\le\alpha\le1}\left|\frac{1}{n}\sum_{i=1}^{n}X^-_{i\alpha} - EX^-_{i\alpha}\right| \le \varepsilon \quad a.s.$$

Similarly, it can be proved that

$$\overline{\lim_{n\to\infty}} \sup_{0\le\alpha\le1} \left| \frac{1}{n} \sum_{i=1}^{n} X_{i\alpha}^{+} - EX_{1\alpha}^{+} \right| \le \varepsilon \quad a.s.$$

Therefore, we obtain

$$\overline{\lim_{n\to\infty}} \, d\left(\frac{1}{n} \sum_{i=1}^{n} \tilde{X}_i, \; E\tilde{X}_1\right) \le \varepsilon \quad a.s.$$

Since ε is arbitrary, this gives the sufficiency.

(Necessity) If (3.2) holds, then for any $\alpha \in [0,1]$,

$$\frac{1}{n} \sum_{i=1}^{n} X_{i\alpha}^{-} \to b_{\alpha}^{-} \quad a.s. \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^{n} X_{i\alpha}^{+} \to b_{\alpha}^{+} \quad a.s.$$

By the converse of Kolmogorov's strong law of large numbers,

$$E \mid X_{1\alpha}^{-} \mid < \infty \quad , \quad E \mid X_{1\alpha}^{+} \mid < \infty \quad \text{for each } \alpha \in [0,1]$$

and

$$b_{\alpha}^{-} = EX_{1\alpha}^{-} \, , \quad b_{\alpha}^{+} = EX_{1\alpha}^{+} \quad \text{for each } \alpha \in [0,1]$$

which implies $E\| \tilde{X}_1 \| < \infty$ and $\tilde{b} = E\tilde{X}_1$. Q.E.D.

Example. Let $\tilde{u} \in F(R)$ be fixed and let $\{Y_n\}$ be i.i.d. with $E\,|Y_1| < \infty$ in the usual sense. Define $\tilde{X}_n(\omega) = \tilde{u}\,(x - Y_n(\omega))$ *i.e.*, $\tilde{X}_n(\omega)$ is the translation of $\tilde{u}$ by $Y_n(\omega)$ in the real axis. Then,

$$X_{n\alpha}^{-}(\omega) = u_{\alpha}^{-} + Y_n(\omega) \quad \text{and} \quad X_{n\alpha}^{+}(\omega) = u_{\alpha}^{+} + Y_n(\omega).$$

Hence the above theorem implies that

$$\frac{1}{n} \sum_{i=1}^{n} \tilde{X}_i \xrightarrow{d} E\tilde{X}_1 \quad a.s.$$

where $(E\tilde{X}_1)(x) = \tilde{u}(x - EY_1)$.

REFERENCES

[1]. Z. Artstein and R. A. Vitale, A strong law of large numbers for random compact sets, Ann. Probab. 3 (1975), 879-882.

[2]. W. Congxin and M. Ming, Embedding problem of fuzzy number space : part II, Fuzzy Sets and Systems 45 (1992), 189-202.

[3]. P. Z. Daffer and R. L. Taylor, Laws of large numbers for D[0,1], Ann. Prob. 7 (1979), 85-95.

[4]. P. Z. Daffer and R. L. Taylor, Tightness and strong laws of large numbers in Banach spaces, Bull. Inst. Math. Academia Sinica 10 (1982), 253-263.

[5]. R. Goetschel and W. Voxman, Elementary fuzzy calculus, Fuzzy Sets and Systems 18 (1982), 31-43.

[6]. D. H. Hong and H. J. Kim, Marcinkiewicz-type law of large numbers for fuzzy random variables, Fuzzy Sets and Systems 64 (1994), 387-393.

[7]. H. Inoue, A strong law of large numbers for fuzzy random sets, Fuzzy Sets and Systems 41 (1991), 285-291.

[8]. Y. K. Kim, Strong law of large numbers for level-wise independent fuzzy random variables, Comm. Korean Math. Soc. 12 (1997), 769-778.

[9]. Y. K. Kim, Chung's law of large numbers for fuzzy random variables, Korean J. Comput. and Applied Math. preprint.

[10]. Y. K. Kim and B. M. Ghil, Integrals of fuzzy number valued functions, Fuzzy Sets and Systems 86 (1997), 213-222.

[11]. E. P. Klement, M. L. Puri and D. A. Ralescu, Limit theorems for fuzzy random variables, Proc. Roy. Soc. London ser. A 407 (1986), 171-182.

[12]. R. Kruse, The strong law of large numbers for fuzzy random variables, Inform. Sci. 28 (1982), 233-241.

[13]. M. Miyakoshi and M. Shimbo, A strong law of large numbers for fuzzy random variables, Fuzzy Sets and Systems 12 (1984), 133-142.

[14]. M. L. Puri and D. A. Ralescu, Fuzzy random variables, J. Math. Anal. Appl. 114 (1986), 402-422.

[15]. R. L. Taylor and H. Inoue. A strong law of large numgers for random sets in Banach spaces, Bull. Inst. Math. Academia Sinica 13 (1985), 403-409.

[16]. T. Uemura, A law of large numbers for random variables, Fuzzy Sets and Systems 59 (1993), 181-188.