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An Efficient Multigrid Algorithm for the Reactor Eigenvalue Problems

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Abstract

In this paper, a new multigrid method is developed to solve the reactor eigenvalue problems. The new algorithm can be used in any matrix equations concerned with the eigenvalue problem. The finite difference neutron diffusion problem is considered for demonstration of the performance of the new multigrid algorithm. The numerical results show that the new multigrid algorithm works well and requires much shorter (7~10 times) computing time compaired to the production code VENTURE.

I. Introduction

Solving the reactor eigenvalue problems involves two-level iterations: inner and outer iterations. In general, inner iteration takes much longer time than outer iteration. Therefore, an efficient inner iteration algorithm is required in reducing the computing time. SOR (or LSOR) and the Chebyshev acceleration are popularly used in solving the reactor eigenvalue problem in inner and outer iterations, respectively[1]. However, the convergence rate of those acceleration techniques is fairly slow when the problem size is large as in the fine-mesh finite difference calculations.

Among many numerical solvers, multigrid method is considered as one of the most efficient methods. Concerning the computational complexity, the multigrid method needs $O(N)$ operations in solving the elliptic boundary value problems with N number of unknowns, while most of the other methods have $O(N^{\alpha})$ with $\alpha > 1$. Meanwhile in application of the multigrid method[2] to reactor problems, most works so far were concerned with the fixed-source problem[3, 4]. In this paper, we investigate the multigrid method from the viewpoint of the eigenvalue problem, in particular, for dominance ratio estimation in outer iteration and compare its performance with that of a production code.

II. Methodology

The objective of the present work is to develop an efficient multigrid algorithm for solving the finite difference matrix equations for the multigroup neutron diffusion equations. As is usual, the within-group diffusion equation in the inner iteration is solved by using a multigrid method in the present approach.

It is essential for high performance of a multigrid algorithm that coarse-grid operators be good approximations and at the same time be obtained easily. There are two kinds of coarsening methods: the Galerkin method and the straightforward finite difference. Accounting for heterogeneity of the reactor core, the Galerkin coarse-gird approximation[5] is adopted here. Although the Galerkin coarse-grid approximation is a time-consuming process compaired to the finite difference approximation, the coarse-grid operators are set up only once and stored before the main calculation begins. Furthermore it works better in the case with rapidly varying coefficient as in the reactor core. In the Galerkin coarsening scheme, the coarse grid operator is obtained as:

$$A_g^{l+1} = R_l^{l+1} A_g^l P_{l+1}^l \,, \; l = 0, 1, 2, \ldots, \tag{1}$$

where A_g^l is the diffusion operator in l-th grid for g-th neutron group, R_l^{l+1} and P_{l+1}^l denote the restriction and prolongation operator, respectively. Note that $l=0$ indicates the fine-grid operator.

Taking into account numerical efficiency, the coarse-grid operator used in a multigrid method should be sparse as much as possible. However, at the same time, it should preserve properties of the original fine-grid operator as much as possible. To construct the coarse-grid operators, we used linear prolongation and corresponding 7-point restriction operators, which give 7-point coarse-grid operators for both 5-point and 7-point fine-grid operators. Note that 5-point conventional finite difference method is used in this work. In Fig. 1, graphical diagrams for the prolongation and restriction operators are shown. It is worthwhile to note that prolongation is linear interpolation and the 7-point restriction operator is its adjoint. The two operators result in a sparser matrix than all other linear interpolation operators.

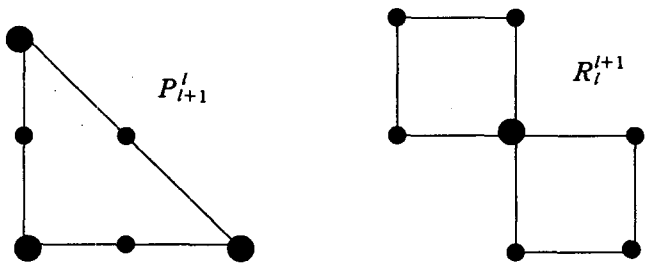

Fig. 1. Prolongation and restriction operators

Performance of a multigrid algorithm also depends on both multigrid cycles and smoothing schemes used. Numerical tests for typical multigrid cycles such as V-, W-, and saw-tooth cycles and smoothing schemes showed that the saw-tooth cycle combined with an ILU (Incomplete LU decomposition)[5, 6] is slightly better than others. In the saw-tooth cycle, each grid is visited only once in the course of post-smoothing as shown in Fig. 2.

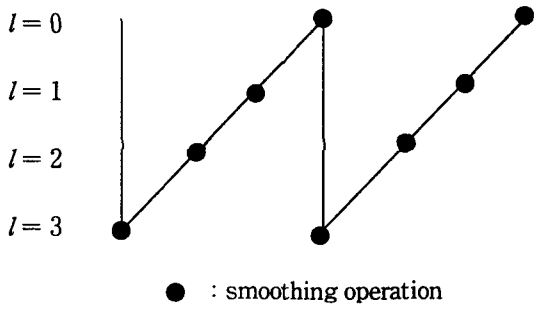

Fig. 2. Saw-tooth multigrid cycle in 4-grid case

No matter how fast the inner iteration converges, the overall computing time highly depends on the acceleration scheme in the outer iteration. As is well known, convergence of the outer iteration is governed by the dominance ratio, which is not known a priori, and the two-parameter Chebyshev method[1] is widely used as in the VENTURE code[7]. However, this method has a drawback in that the dominance ratio should be continuously updated during outer iterations. Meanwhile, a good initial dominance ratio can be estimated when using the multigrid algorithm. For instance, let us consider the two-group diffusion equations. Applying the Galerkin coarsening procedure to the fission operator F_g^l and scattering operator $\Sigma_{gg'}^l$, we can get the following reduced-order (coarsened) eigenvalue equations:

$$A_1^m \phi_1^m = \frac{1}{k_{eff}} (F_1^m \phi_1^m + F_2^m \phi_2^m) , \tag{2a}$$

$$A_2^m \phi_2^m = \Sigma_{21}^m \phi_1^m , \tag{2b}$$

where $m > 0$.

The dominance ratio estimated with Eq. (2) can be used as the initial dominance ratio for the original fine-grid equations. Besides the dominance ratio, the eigenvectors of Eq. (2) can serve as good initial guesses for the original fine-grid system. Through numerical experiments we note that the reduced-order eigenvalue problem $m = 2, 3, 4$, depending on the finest-grid system, provides fairly good dominance ratio and initial eigenvectors for the original fine-grid eigensystem. It is worthwhile to note that the

reduced-order eigenvalue calculation need not be performed completely. Consequently, this pre-calculation can be done cheaply.

III. Results and Discussion

Based on the above algorithm, a mesh-cornered finite difference code was programmed and compared with the VENTURE code. Note that VENTURE uses a mesh-centered finite difference scheme, giving a slightly different solution even in the case of the same mesh size. As a benchmark problem, the 2-D Zion 1 core[8] was analyzed on a SUN workstation (SUN Sparc Station 2). In the calculations, convergence criteria were as follows: 1.E-7 for the eigenvalue, and 1.E-5 for pointwise fission source (multigrid method) and fluxes (VENTURE). In the multigrid algorithm, the number of saw-tooth cycles per inner iteration was one. Table I compares performance of the multigrid algorithm and VENTURE.

Table I. Numerical results for the 2-D Zion 1 Core problem

Mesh System		72 x 72		128 x 128		144 x 144	
Multigrid Method	Dominance Ratio in Chebyshev Acceleration	conven-tional	new[b]	conven-tional	new	conven-tional	new
	Eigenvalue[a]	1.274835	1.274835	1.274873	1.274873	1.274887	1.274887
	No. of Outer Iterations	34	24	36	25	35	26
	Computing Time (sec)	23	20	76	60	94	78
VENTURE	Dominance Ratio in Chebyshev Acceleration	conventional		conventional		conventional	
	Eigenvalue	1.275070		1.274946		1.274916	
	No. of Outer Iterations	41		49		58	
	Computing Time (sec)	178		669		966	

[a] Reference eigenvalue[8] = 1.274890

[b] Pre-calculation with reduced-order eigensystem

As shown in Table I, the newly-developed multigrid algorithm is much faster than the VENTURE code. It should be noted that the reduction in the computing time

becomes larger (7 ~ 10 times) as the number of unknowns increases. Furthermore, the pre-calculation with reduced-order eigenvalue equations turns out to be effective for the reactor eigenvalue problem.

The correct dominance ratio of the Zion 1 core is ~0.9613. For this problem, the reduced-order eigensystem analysis gave 0.9595 ~ 0.9605 as the initial dominance ratio for the original fine-grid eigensystem. To find a good dominance ratio, the coarse-grid systems should provide good approximation to the orignial fine-grid system. Various numerical tests show that dominance ratio obtained with $l=2\sim3$ is good enough, i.e., coarse-grid operators with $l=2\sim3$ preserve the higher eigenmodes of the original operator.

Generally, performance of a multigrid method depends on the number of multigrid cycles per inner iteration. Numerical tests show that one saw-tooth cycle is best from the computational point of view. Performing several multigrid cycles in each inner iteration might reduce the number of outer iterations. However, reduction in the number of outer iterations is marginal, consequently the actual computing time slightly increases.

VI. Conclustions

An efficient multigrid algorithm for the reactor eigenvalue problems was developed and applied to solving the finite differece neutron diffusion equations. Unlike the conventional application of the multigrid method, new method is based on the coarse-grid eigenvalue equations. The 7-point Galerkin coarsening procesure is used and the dominance ratio for the Chebyshev acceleration is approximated by using the coarse-grid eigenvalue equations.

To show the peformance of the new multigrid algorithm, the Zion-1 reactor problem was solved by the new method. Numerical results confirmed that the newly developed algorithm was much faster than the VENTURE code.

In this paper, an ILU was used as the smoothing method. If more efficient smoothing scheme (e.g., combination of ILU and conjugate gradient method) is used, the peformance of the new multigrid method would be better.

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