# NFL-ROO에 기준한 SMC의 안정도 증명 : Part 6

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# Stability Proof of NFL-ROO-based SMC : Part 6

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**[Abstract]** This paper presents the stability proof of a nonlinear feedback linearization-reduced order observer-based sliding mode controller (NFL-ROO-based SMC). The closed-loop stability is proved by a Lyapunov function candidate using an addition form of the sliding surface vector and the estimation error.

**Keywords** : nonlinear feedback linearization-reduced order observer-based sliding mode controller, Lyapunov function, stability proof

## 1. Introduction

In this paper, a nonlinear feedback linearization-reduced order observer-based sliding mode controller (NFL-ROO-based SMC) to design the simpler observer-based controller, and to solve the problem associated with the unmeasurable state variables in the conventional SMC is developed [1-18]. The proposed NFL-ROO-based SMC is obtained by the estimated state variable based on observer state in designing a sliding surface gain to ensure the stability by Lyapunov's method. The closed-loop stability is proved by a Lyapunov function candidate using an addition form of the sliding surface vector and the estimation error.

## 2. NFL-ROO-based SMC

The state equation for full-state feedback and the output equation based on nonlinear feedback linearization (NFL) can be expressed as [19]

$$z(t) = T(x(t)) \tag{1}$$

$$\dot{z}(t) = Az(t) + Bu(t) \tag{2}$$

$$y(t) = Cz(t) \tag{3}$$

The transformation matrix is introduced by

$$q(t) = Tz(t) \tag{4}$$

$$\begin{bmatrix} y(t) \\ q(t) \end{bmatrix} = \begin{bmatrix} C \\ T \end{bmatrix} z(t) \tag{5}$$

$$z(t) = \begin{bmatrix} C \\ T \end{bmatrix}^{-1} \begin{bmatrix} y(t) \\ q(t) \end{bmatrix} = \begin{bmatrix} P & M \end{bmatrix} \begin{bmatrix} y(t) \\ q(t) \end{bmatrix} = Py(t) + Mq(t) \tag{6}$$

The estimate $\hat{z}$ of $z$ is generated as

$$\hat{z}(t) = Py(t) + M\hat{q}(t) \tag{7}$$

A new realization for the system is expressed as

$$E\dot{z}(t) = EAz(t) + EBu(t) \tag{8}$$

Substituting for $z$ and $E$, we get

$$E\dot{z}(t) = \begin{bmatrix} C \\ T \end{bmatrix} \dot{z}(t) = \begin{bmatrix} \dot{y}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} C \\ T \end{bmatrix} Az(t) + \begin{bmatrix} C \\ T \end{bmatrix} Bu(t)$$

$$= \begin{bmatrix} C \\ T \end{bmatrix} A \begin{bmatrix} P & M \end{bmatrix} \begin{bmatrix} y(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} C \\ T \end{bmatrix} Bu(t) \tag{9}$$

Therefore, we get

$$\begin{bmatrix} \dot{y}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} CAP & CAM \\ TAP & TAM \end{bmatrix} \begin{bmatrix} y(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} CB \\ TB \end{bmatrix} u(t) \tag{10}$$

$$:= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \tag{11}$$

$$\dot{y}(t) = A_{11}y(t) + A_{12}q(t) + B_1u(t) \tag{12}$$

$$\dot{q}(t) = A_{22}q(t) + A_{21}y(t) + B_2u(t) \tag{13}$$

The estimator equation for $q$ is

$$\dot{\hat{q}}(t) = A_{22}\hat{q}(t) + A_{21}y(t) + B_2u(t) + L(y(t) - C\hat{z}(t)) \tag{14}$$

Let $CP = I$ and $CM = 0$

$$y(t) - C\hat{z} = y(t) - C(Py(t) + M\hat{q})$$

$$= y(t) - CPy(t) - CM\hat{q} = y(t) - y(t) - 0 = 0 \tag{15}$$

The reduced-order observer is represented by

$$\dot{\hat{q}}(t) = A_{22}\hat{q}(t) + A_{21}y(t) + B_2u(t)$$

$$+ L(\dot{y}(t) - A_{11}y(t) - B_1u(t) - A_{12}\hat{q}(t)) \tag{16}$$

Define $w$ and $\dot{w}$, we get

$$w(t) := \hat{q}(t) - Ly(t) \tag{17}$$

$$\dot{w}(t) := \dot{\hat{q}}(t) - L\dot{y}(t) \tag{18}$$

The final form of the reduced-order observer is

$$\dot{w}(t) = (A_{22} - LA_{12})w(t) + [(A_{22} - LA_{12})L + A_{21} - LA_{11}]y(t) + (B_2 - LB_1)u(t) \tag{19}$$

$$\hat{z}(t) = Mw(t) + (P + ML)y(t) \tag{20}$$

$$\dot{\hat{z}}(t) = M\dot{w}(t) + (P + ML)\dot{y}(t) \tag{21}$$

The equations (19) and (20) are simplified as [1]

$$\dot{w}(t) = Fw(t) + Gy(t) + Hu(t) \tag{22}$$

$$\hat{z}(t) = Mw(t) + Ny(t) \tag{23}$$

where,

$$F := (A_{22} - LA_{12}) \tag{24}$$

$$G := [(A_{22} - LA_{12})L + A_{21} - LA_{11}] \tag{25}$$

$$H := (B_2 - LB_1) \tag{26}$$

$$N := (P + ML) \tag{27}$$

The sliding surface vector and the differential sliding surface vector based on NFL are

$$\hat{z} = T^{-1}\hat{q}$$

$$\sigma(\hat{z}(t)) = G_{SS}^T\hat{z}(t) \tag{28}$$

$$\dot{\sigma}(\hat{z}(t)) = G_{SS}^T\dot{\hat{z}}(t) \tag{29}$$

The Lyapunov's function is

$$V(\hat{z}(t)) = \sigma^2(\hat{z}(t))/2 \tag{30}$$

The time derivative of $V(\hat{z}(t))$ is expressed as

$$\dot{V}(\hat{z}(t)) = \sigma(\hat{z}(t))\dot{\sigma}(\hat{z}(t)) \tag{31}$$

$$\begin{aligned}
&= G_{SS}^T\hat{z}(t)G_{SS}^T\dot{\hat{z}}(t) = G_{SS}^T\hat{z}(t)G_{SS}^T[M\dot{w}(t) + (P + ML)\dot{y}(t)] \\
&= G_{SS}^T\hat{z}(t)G_{SS}^T[M((A_{22} - LA_{12})w(t) \\
&\quad + ((A_{22} - LA_{12})L + A_{21} - LA_{11})y(t) \\
&\quad + (B_2 - LB_1)u_{ROO-SMC}(t)) + (P + ML) \\
&\quad (A_{11}y(t) + A_{12}q(t) + B_1u_{ROO-SMC}(t))] \\
&= G_{SS}^T\hat{z}(t)[G_{SS}^T M(A_{22} - LA_{12})w(t) \\
&\quad + (G_{SS}^T M(A_{22} - LA_{12})L + G_{SS}^T MA_{21} \\
&\quad - G_{SS}^T MLA_{11} + G_{SS}^T(P + ML)A_{11})y(t) \\
&\quad + (P + ML)A_{12}q(t) + (MB_2 + PB_1)u_{ROO-SMC}(t)] \le 0
\end{aligned} \tag{32}$$

The control inputs with switching function are

$$\begin{aligned}
u_{ROO-SMC}^{+}(t) \ge -[G_{SS}^T(MB_2 + PB_1)]^{-1}\,[G_{SS}^T(MA_{22} - MLA_{12})w(t) \\
+ G_{SS}^T(M(A_{22} - LA_{12})L + MA_{21} + PA_{11})y(t) \\
+ G_{SS}^T(P + ML)A_{12}q(t)] \quad for\ G_{SS}^T\hat{z}(t) > 0
\end{aligned} \tag{33}$$

$$\begin{aligned}
u_{ROO-SMC}^{-}(t) \le -[G_{SS}^T(MB_2 + PB_1)]^{-1}\,[G_{SS}^T(MA_{22} - MLA_{12})w(t) \\
+ G_{SS}^T(M(A_{22} - LA_{12})L + MA_{21} + PA_{11})y(t) \\
+ G_{SS}^T(P + ML)A_{12}q(t)] \quad for\ G_{SS}^T\hat{z}(t) < 0
\end{aligned} \tag{34}$$

The estimated control input with sign function is

$$\begin{aligned}
\hat{u}_{ROO-SMC}^{sign}(t) = -[G_{SS}^T(MB_2 + PB_1)]^{-1}\,[G_{SS}^T(MA_{22} - MLA_{12})w(t) \\
+ G_{SS}^T(M(A_{22} - LA_{12})L + MA_{21} + PA_{11})y(t) \\
+ G_{SS}^T(P + ML)A_{12}q(t)]\ sign(\sigma(\hat{z}(t)))
\end{aligned} \tag{35}$$

The equation (35) is simplified as follows

$$\hat{u}_{ROO-SMC}^{sign}(t) = -[RGK_1w(t) + RGK_2y(t) + RGK_3q(t)]\ sign(\sigma(\hat{z}(t))) \tag{36}$$

where,

$$RGK_1 := [G_{SS}^T(MB_2 + PB_1)]^{-1}\ G_{SS}^T M(A_{22} - LA_{12}) \tag{37}$$

$$RGK_2 := [G_{SS}^T(MB_2 + PB_1)]^{-1}\ G_{SS}^T(M(A_{22} - LA_{12})L + MA_{21} + PA_{11}) \tag{38}$$

$$RGK_3 := [G_{SS}^T(MB_2 + PB_1)]^{-1}\ G_{SS}^T(P + ML)A_{12} \tag{39}$$

***Theorem 1:*** Consider the full state equation based on NFL for the regulation problem and the reduced order observer problem based on NFL

$$\dot{z} = Az + B\hat{u}_{ROO-SMC}^{sign} \quad \text{and} \quad y = Cz$$

$$\dot{\hat{z}} = M\dot{w} + (P + ML)\dot{y}$$

$$\dot{w} = (A_{22} - LA_{12})w + [(A_{22} - LA_{12})L + A_{21} - LA_{11}]y + (B_2 - LB_1)\hat{u}_{ROO-SMC}^{sign}$$

$$\dot{y} = A_{11}y + A_{12}q + B_1\hat{u}_{ROO-SMC}^{sign}$$

$$\hat{z} = Mw + (P + ML)y$$

Suppose that $(A,C)$ is detectable and $(A - LC)$ is Hurwitz. The estimated sliding mode reduced order control law with sign function is guaranteed an *asymptotically stable* for the system (2)

$$\hat{u}_{ROO-SMC}^{sign}(t) = -[RGK_1w + RGK_2y + RGK_3q]sign(\sigma(\hat{z}))$$

where,

$$RGK_1 := [G_{SS}^T(MB_2 + PB_1)]^{-1}\ G_{SS}^T(MA_{22} - MLA_{12})$$

$$RGK_2 := [G_{SS}^T(MB_2 + PB_1)]^{-1}\ G_{SS}^T[M\ (A_{22} - LA_{12})L + MA_{21} + PA_{11}]$$

$$RGK_3 := [G_{SS}^T(MB_2 + PB_1)]^{-1}\ G_{SS}^T(P + ML)A_{12}$$

subject to $sign(\sigma(\hat{z})) = 1$ for $\sigma(\hat{z}) > 0$

$sign(\sigma(\hat{z})) = 0$ for $\sigma(\hat{z}) = 0$

$sign(\sigma(\hat{z})) = -1$ for $\sigma(\hat{z}) < 0$

***Proof.*** Let us define the estimation error and the differential estimation error

$$e = z - \hat{z}$$

$$\begin{aligned}
\dot{e} = \dot{q} - \dot{\hat{q}} &= A_{22}q + A_{21}y + B_2\hat{u}_{ROO-SMC}^{sign} - (A_{22}\hat{q} + A_{21}y + B_2\hat{u}_{ROO-SMC}^{sign} \\
&\quad + L(\dot{y} - A_{11}y - B_1\hat{u}_{ROO-SMC}^{sign} - A_{12}\hat{q})) \\
&= (A_{22} - LA_{12})(q - \hat{q}) = (A_{22} - LA_{12})e
\end{aligned}$$

Lyapunov's function candidate is chosen by

$$V=\frac{1}{2}\sigma^T\sigma+\frac{1}{2}e^Te$$

The derivative of a Lyapunov's function is

$$\dot{V}=\sigma^T\dot{\sigma}+e^T\dot{e}=\sigma^T\left(G_{SS}^T\dot{\hat{z}}\right)+e^T\left(A_{22}-LA_{12}\right)e$$

$$=\sigma^T\left(G_{SS}^T\left(M\dot{w}+\left(P+ML\right)\dot{y}\right)\right)+e^T\left(A_{22}-LA_{12}\right)e$$

$$=\sigma^T\left(G_{SS}^TM\left(A_{22}-LA_{12}\right)w+G_{SS}^T\left(\left(MA_{22}-MLA_{12}\right)L+\left(MA_{21}+PA_{11}\right)\right)y+G_{SS}^T\left(P+ML\right)A_{12}q-\left(G_{SS}^TMB_2+G_{SS}^TPB_1\right)\left(RGK_1w+RGK_2y+RGK_3q\right)sign\left(\sigma\left(\hat{z}\right)\right)\right)+e^T\left(A_{22}-LA_{12}\right)e$$

$$\dot{V}=\sigma^T\Big[G_{SS}^TM\left(A_{22}-LA_{12}\right)w+G_{SS}^T\left(\left(MA_{22}-MLA_{12}\right)L+\left(MA_{21}+PA_{11}\right)\right)y+G_{SS}^T\left(P+ML\right)A_{12}q$$
$$-\left(G_{SS}^TMB_2+G_{SS}^TPB_1\right)\left[G_{SS}^T\left(MB_2+PB_1\right)\right]^{-1}G_{SS}^T\left(MA_{22}-MLA_{12}\right)w\,sign\left(\sigma\left(\hat{z}\right)\right)$$
$$-\left(G_{SS}^TMB_2+G_{SS}^TPB_1\right)\left[G_{SS}^T\left(MB_2+PB_1\right)\right]^{-1}G_{SS}^T\left[M\left(A_{22}-LA_{12}\right)L+MA_{21}+PA_{11}\right]y(t)\,sign\left(\sigma\left(\hat{z}\right)\right)$$
$$-\left(G_{SS}^TMB_2+G_{SS}^TPB_1\right)\left[G_{SS}^T\left(MB_2+PB_1\right)\right]^{-1}G_{SS}^T\left(P+ML\right)A_{12}q\,sign\left(\sigma\left(\hat{z}\right)\right)\Big]+e^T\left(A_{22}-LA_{12}\right)e$$

Let $G_{SS}^T\left(MB_2+PB_1\right)\left[G_{SS}^T\left(MB_2+PB_1\right)\right]^{-1}=I$,

$$\dot{V}=\sigma^T\Big[G_{SS}^TM\left(A_{22}-LA_{12}\right)w+G_{SS}^T\left(\left(MA_{22}-MLA_{12}\right)L+\left(MA_{21}+PA_{11}\right)\right)y$$
$$+G_{SS}^T\left(P+ML\right)A_{12}q-G_{SS}^T\left(MA_{22}-MLA_{12}\right)w\,sign\left(\sigma\left(\hat{z}\right)\right)$$
$$-G_{SS}^T\left[M\left(A_{22}-LA_{12}\right)L+MA_{21}+PA_{11}\right]y\,sign\left(\sigma\left(\hat{z}\right)\right)$$
$$-G_{SS}^T\left(P+ML\right)A_{12}q\,sign\left(\sigma\left(\hat{z}\right)\right)\Big]+e^T\left(A_{22}-LA_{12}\right)e$$

$$=\sigma^TG_{SS}^TM\left(A_{22}-LA_{12}\right)w\left(1-sign\left(\sigma\left(\hat{z}\right)\right)\right)+\sigma^TG_{SS}^T\left(P+ML\right)A_{12}q\left(1-sign\left(\sigma\left(\hat{z}\right)\right)\right)$$
$$+\sigma^TG_{SS}^T\left(M\left(A_{22}-LA_{12}\right)L+MA_{21}+PA_{11}\right)y\left(1-sign\left(\sigma\left(\hat{z}\right)\right)\right)+e^T\left(A_{22}-LA_{12}\right)e$$

If $\left(A_{22}-LA_{12}\right)$ is stable, the estimation error is $e\to0$ as $t\to0$.

$$\dot{V}=\sigma^TG_{SS}^TM\left(A_{22}-LA_{12}\right)w\left(1-sign\left(\sigma\left(\hat{z}\right)\right)\right)+\sigma^TG_{SS}^T\left(P+ML\right)A_{12}q\left(1-sign\left(\sigma\left(\hat{z}\right)\right)\right)+\sigma^TG_{SS}^T\left(M\left(A_{22}-LA_{12}\right)L+MA_{21}+PA_{11}\right)y\left(1-sign\left(\sigma\left(\hat{z}\right)\right)\right)\le0$$

subject to, if $\sigma>0$, $\dot{V}=0$

if $\sigma=0$, $\dot{V}=0$

if $\sigma<0$, $\dot{V}=2kG_{SS}^TM\left(A_{22}-LA_{12}\right)w+2kG_{SS}^T\left(P+ML\right)A_{12}q+2kG_{SS}^T\left(M\left(A_{22}-LA_{12}\right)L+MA_{21}+PA_{11}\right)y<0$

where, $k$ is positive constant. The above condition is satisfied on negative definite, and is *asymptotically stable*. This completes the proof of this theorem. □

## 3. Conclusion

A nonlinear feedback linearization-reduced order observer-based sliding mode controller (NFL-ROO-based SMC) has been proposed and a stabilty proof of a closed-loop system has been done.

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