미분적으로 평활한 불확정 비선형 시스템의 강인 안정화

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Robust Stabilization of Differentially Flat Uncertain Nonlinear Systems

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Abstract

This paper describes a robust stabilization of single input nonlinear systems with parametric uncertainty. We first investigate differential flatness of the nominal nonlinear systems. If a single input system is differentially flat, it possesses a flat output. And we define coordinate transformation functions via successively differentiating the flat output, and we also consider the robust fictitious controls at every differentiation of the flat output. In the new coordinates the nonlinear system is transformed into the Brunovsky normal form with matched uncertainty. With a robust control based on the Lyapunov method, the robust stabilization is achieved.

1. Introduction

The feedback linearization has been a successful technique for the control of nonlinear systems during the past decades. However, when one considers the parametric uncertainty or the unmodelled system dynamics, it is difficult to apply feedback linearization method. The principal difficulty with feedback linearization is that the system dynamics must be known exactly for the nonlinearities to be successfully canceled. Uncertainty in the system will result in residual nonlinear terms which will make the stabilization of the system more difficult. In this paper we consider robust stabilization of nonlinear systems with parametric uncertainty whose structural conditions are relaxed. To this end, we first investigate the differential flatness of the nominal nonlinear system. Differentially flat systems are underdetermined systems of (nonlinear) ordinary differential equations whose solution curves are in smooth one-one correspondence with arbitrary curves in a space whose dimension equals the number of equations by which the system is underdetermined [3, 4, 6]. The components of the map from the system space to the smaller dimensional space are referred to as the flat outputs. With the flat output function we can define coordinate transformation functions which make the original nonlinear system into the controllable linear system with uncertainty. Since the coordinate transformation functions are derived from successively differentiating the flat outputs, we also consider the robust stabilizing fictitious controls at every design stage. With a robust control based on the Lyapunov method, the robust control is designed.

2. Preliminaries

In this section, we shall introduce the notions of *Pfaffian systems*, *Cartan prolongations* and *absolute equivalence* and provide a definition of differential flatness in terms of absolute equivalence. We assume that all manifolds and mappings are smooth (C^∞) unless explicitly stated otherwise.

Definition 1 *A* Pfaffian system *I on a manifold M is a submodule of the module of differential one-forms $\Omega^1(M)$ over the commutative ring of smooth functions $C^\infty(M)$. A set of one-forms $\omega^1, \ldots, \omega^n$, generates a Pfaffian system $I = \{\omega^1, \ldots, \omega^n\} = \{\sum f_k \omega^k \mid f_k \in C^\infty(M)\}$.*

We restrict our attention to finitely generated Pfaffian systems on finite dimensional manifolds. Since we are only interested in changes of coordinates that preserve time, we shall be dealing with manifold M equipped with a notion of time given by a map $\pi : M \to \mathbb{R}$ which is a submersion [5]. The time coordinate on M is $\pi^* t = t \circ \pi$, which we shall often write as t for notational ease. And we treat dt as an *independence condition*, i.e., a one-form that is not allowed to vanish on any of the solution curves. For $p \in M$, the *codimension at* p of the system is $\dim M - \dim I(p)$. A system is *trivial* if $I = \{0\}$.

Definition 2 *A Pfaffian system with independence condition (I, dt) is called a* control system *if $\{I, dt\}$ is integrable.*

Definition 3 *Let (I, dt) be a Pfaffian system on a manifold M. The* derived systems *of I are $I^{(0)} = I$ and, for each $k \geq 0$,*

$$I^{(k+1)} = \{\omega \in I^{(k)} \mid d\omega = 0 \mod I^{(k)}\}.$$

Cartan prolongations give rise to a more general notion of equivalence between systems that live in spaces of possibly different dimensions.

Definition 4 (Cartan Prolongation [6]) *Let (I, dt) be a Pfaffian system on a manifold M. Let B be a manifold such that $\pi : B \to M$ is a fiber bundle. A Pfaffian system $(J, \pi^* dt)$ on B is a* Cartan prolongation *of the system (I, dt) if the following conditions hold:*

1. *$\pi^*(I) \subset J$*
2. *For every integral curve of I, $c : (-\epsilon, \epsilon) \to M$, there is a* unique *lifted integral curve of J, $\tilde{c} : (-\epsilon, \epsilon) \to B$ with $\pi \circ \tilde{c} = c$.*

Definition 5 (Absolute Equivalence [2]) *Two systems I_1, I_2 are called* absolutely equivalent *if they have Cartan prolongations J_1, J_2 respectively that are equivalent in the usual sense, i.e., there exists a diffeomorphism ϕ such that $\phi^*(J_2) = J_1$. This is illustrated in the following diagram:*

$$\begin{array}{ccc} J_1 & \xleftrightarrow{\phi} & J_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ I_1 & & I_2 \end{array}$$

Theorem 1 ([2]) *A system (I, dt) on M is differentially flat if and only if it is absolutely equivalent to the trivial system $I_t = (\{0\}, dt)$ on a manifold N.*

If $(t, y_1, \ldots, y_p)$ are local coordinates on N then $(y_1, \ldots, y_p)$ are a set of *flat outputs*.

Remark Observe that the number of flat outputs is p where $p+1$ is the codimension of system (I, dt) on M. If the system is a control system then p is also the number of inputs. All Cartan prolongations are locally equivalent to total prolongations [2]. Starting with any system, taking derived systems enables one to "strip off" prolongations and reach the "core" system, which is not a total prolongation of any system. For differentially flat systems, the core is trivial. □

2.1 Flatness for Single Input Systems

For single input control systems, the corresponding differential system has codimension 2. There are a number of results available in codimension 2 systems which allow us to give a complete characterization of differentially flat single input control systems.

Theorem 2 ([2]) *A system (I, dt) of constant codimension 2 is differentially flat if and only if*

1. $\dim I^{(i)} = \dim I^{(i-1)} - 1$, *for* $i = 0, \ldots, n = \dim I$. *This implies* $I^{(n)} = \{0\}$
2. *The system* $I^{(i)} + \mathrm{span}\{dt\}$ *is integrable for each* $i = 0, \ldots, n$

Theorem 3 ([6]) *If a time invariant single input system is differentially flat we can always take the flat output as a function of the states only:* $y = h(x)$.

Remark These results can only be applied to codimension 2 systems. The characterization of flatness in systems of codimension higher than 2 remains open problem. □

3. Robust Stabilization

Consider a single input uncertain nonlinear system

$$\dot{x} = \tilde{f}(x, \alpha) + g(x)u \tag{1}$$

where $\tilde{f}(x, \alpha)$ and $g(x)$ are C^∞ vector fields defined on a dense submanifold $M \subset \mathbb{R}^n$, u is a scalar control input and the vector α is an unknown parameter vector. It is assumed that $\tilde{f}$ is a smooth vector field for every $\alpha \in B_r \subset \mathbb{R}^p$, where B_r is a compact set. Without loss of generality, we also assume that $\tilde{f}(0, \alpha) = 0$ and $g(x)$ does not vanish for every x. The nominal parameter vector α^n is assumed known and the perturbation about α^n are represented as $\alpha = \alpha^n + \delta\alpha$. And assume that the uncertain parameter vector α appears linearly in (1). Then the system (1) can be written as

$$\dot{x} = f(x) + \Delta f(x) + g(x)u. \tag{2}$$

Moreover, we also assume that the nominal system of (2) is differentially flat.

3.1 Coordinate Transformation

In this study, we consider an input-output linearization approach to the input-to-state linearization problem, i.e., we utilize the flat output as a given output function to linearize the system.

Lemma 1 *If a time invariant single input nonlinear system is differentially flat, there is no internal dynamics (or hence, zero dynamics) associated with the flat output.*

Proof: By Theorem 3, we can take the flat output as a function of the states only. In this case, the original system is a Cartan prolongation of the trivial system. And from the definition of Cartan prolongation, the flat output has an one-one correspondence with the states of the original system. This means that in the flat output space there exists no unobservable subspace, i.e., there is no internal dynamics (or hence, zero dynamics) associated with the flat output. ■

We shall now illustrate how to design coordinate transformation functions and robust stabilizing fictitious controls utilizing the flat output.

By Theorem 3, let the flat output of the nominal system of (2) be $y = h(x)$. And define a new state as

$$z_1 = h(x). \tag{3}$$

Differentiate (3) along the uncertain system (2) yields

$$\begin{aligned} \dot{z}_1 &= \frac{\partial h}{\partial x}\dot{x} \\ &= L_f h + L_{\Delta f} h \\ &\equiv \xi_1(x) + \Delta\xi_1(x). \end{aligned} \tag{4}$$

Note that the control input does not appear in (4). By Lemma 1 the system has relative degree n with the flat output [1, 5]. Define a bounding function for the uncertainty $\Delta\xi_1(x)$ in (4) as

$$\rho_1(x) \geq \|\Delta\xi_1(x)\|. \tag{5}$$

We then add a stabilizing robust fictitious function v_1, which will overcome the uncertainty at this stage,

$$v_1 = \mathrm{sgn}(z_1)(1 - e^{-\sigma_1|z_1|})\rho_1(x) \tag{6}$$

where $\mathrm{sgn}(\cdot)$ is the signum function and σ_1 is a positive design parameter that can be chosen to affect performance. We define the next state variable for the coordinate transformation, replacing the uncertainty in (4) by the robust fictitious control v_1 to give:

$$z_2 = \xi_1(x) + v_1. \tag{7}$$

Differentiating (7) yields,

$$\dot{z}_2 = \frac{\partial\xi_1}{\partial x}\dot{x} + \frac{\partial v_1}{\partial z_1}\dot{z}_1 + \frac{\partial v_1}{\partial x}\dot{x}. \tag{8}$$

Expanding by partials in (8), we obtain

$$\begin{aligned} \dot{z}_2 = {} & L_f\xi_1 + L_{\Delta f}\xi_1 + \frac{\partial v_1}{\partial z_1}[\xi_1(x) + \Delta\xi_1] \\ & + L_f v_1 + L_{\Delta f} v_1. \end{aligned} \tag{9}$$

Gathering terms

$$\dot{z}_2 = \xi_2(x) + \Delta\xi_2(x). \tag{10}$$

Define a bounding function for the uncertainty in (10)

$$\rho_2(x) \geq \|\Delta\xi_2(x)\|. \tag{11}$$

And we again add a stabilizing robust fictitious control v_2

$$v_2 = \mathrm{sgn}(z_2)(1 - e^{-\sigma_2|z_2|})\rho_2(x). \tag{12}$$

Continuing this procedure, we finally reach the equation of the form:

$$\dot{z}_n = \xi_n(x) + \Delta\xi_n(x) + \eta(x)u. \tag{13}$$

3.2 Robust Control Design

For the system (13), one can calculate the robust stabilizing controller by solving the following equation in terms of u:

$$\xi_n(x) + \Delta\xi_n(x) + \eta(x)u = \sum_{i=1}^{n} \beta_i\, z_i + v_n \qquad (14)$$

where β_i's are constant coefficients such that the associated polynomial $s^n - \beta_n s^{n-1} - \cdots - \beta_1$ is Hurwitz and v_n is a control variable to be determined. By (14) the nonlinear system (2) can be expressed as the following linear system

$$\dot{z} = Az + b\beta z + bv_n + b\Delta\xi_n$$

where the pair (A, b) is in the Brunovsky normal form and $\beta = [\beta_1, \ldots, \beta_n]^T$. Define $A_c = A + b\beta$, then we obtain

$$\dot{z} = A_c z + bv_n + b\Delta\xi_n. \qquad (15)$$

Define a bounding function for the uncertainty $\Delta\xi_n$ as

$$\rho_n(x) \geq \|\Delta\xi_n(x)\|. \qquad (16)$$

The theorem that follows gives us the control law v_n and the stability result for the full system.

Theorem 4 *Under assumptions, the equilibrium $z = 0$ of the system (15) is asymptotically stable if v_n satisfies*

$$v_n = \begin{cases} -\dfrac{\rho_n(x)}{\|b\|}, & z^T Pb \geq 0 \\ \dfrac{\rho_n(x)}{\|b\|}, & z^T Pb < 0 \end{cases} \qquad (17)$$

where $\|b\| = 1$ for the single input case, and P is the unique symmetric positive definite solution to the Lyapunov equation

$$PA_c + A_c^T P + Q = 0. \qquad (18)$$

with Q a given symmetric positive definite matrix.

Proof: Omitted. ■

4. An Illustrative Example

To demonstrate the approach, consider the following single input uncertain nonlinear system

$$\begin{aligned} \dot{x}_1 &= \alpha_1 x_1^2 + x_2 \\ \dot{x}_2 &= -2x_1^3 - 2\alpha_2 x_1 x_2 + u \end{aligned} \qquad (19)$$

where α_1 and α_2 are uncertain parameters. To check whether the nominal system is flat, we shall calculate the derived systems. The Pfaffian system of the nominal system of (19) is as follows:

$$I = \{dx_1 - (\alpha_1^n x_1^2 + x_2)dt, dx_2 - (-2x_1^3 - 2\alpha_2^n x_1 x_2 + u)dt\}.$$

Let $\omega = dx_1 - (\alpha_1^n x_1^2 + x_2)dt$, then

$$d\omega = -2\alpha_1^n x_1 dx_1 \wedge dt + dx_2 \wedge dt. \qquad (20)$$

After simple algebra, we get

$$d\omega = 0 \mod I.$$

Thus, we finally obtain the derived systems:

$$I^{(1)} = \{dx_1 - (\alpha_1^n x_1^2 + x_2)dt\}, \quad I^{(2)} = \{0\}.$$

Since the I, $I^{(1)}$, and $I^{(2)}$ have constant rank and each $I^{(i)} + \text{span}\{dt\}$ is integrable for $i = 0, \ldots, 2$, the nominal system of (19) is differentially flat by Theorem 2. And we obtain $y = x_1$ as a flat output. Follow the procedure explained in the previous section, we can design a robust stabilizing controller for the system (19).

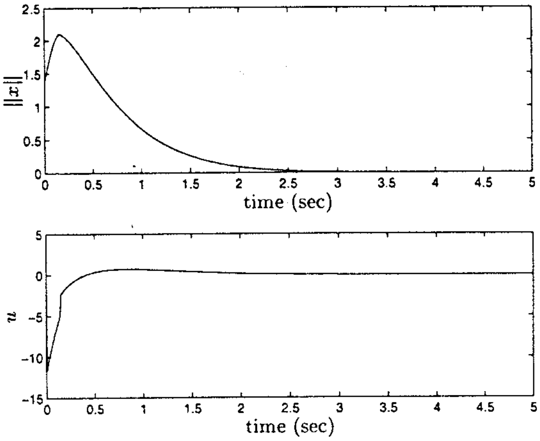

Figure 1: Norm of the states and control input

Figure 1 shows the norm of the states and the control input when the nominal values of the parameters are $\alpha_1^n = \alpha_2^n = 1$ and the relative parameter variation 30% is considered. We observe that the robust asymptotic stabilization is achieved in the presence of the unmatched uncertainty.

5. Conclusion

In this paper we addressed a robust stabilizing control of a class of uncertain nonlinear systems whose nominal system is differentially flat. In the design procedure, a flat output was utilized to obtain coordinate transformation functions recursively. At each stage, a robust stabilizing fictitious control was also designed for compensating the unmatched uncertainty. In the new coordinates, the original nonlinear system was transformed into a linear system with matched uncertainty and a robust control has been designed. The general case of higher codimension systems remains open problem.

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