TL-군에 대하여

On TL-subgroups

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ABSTRACT

We introduce the notion of TL-p-subgroups that is an extension of the notion of fuzzy p-subgroups and show that a torsion TL-subgroup of an Abelian group with $T = \wedge$ can be written as the intersection of its minimal TL-p-subgroups.

1. Introduction

Zadeh[14] introduced the concept of fuzzy subsets, and Rosenfeld [11] introduced the concept of fuzzy subgroups. Following these ideas, many authors are engaged in generalizing various notions of group theory in the fuzzy setting. In particular, the notion of fuzzy orders of the elements of a group relative to a fuzzy group and the notion of fuzzy orders of fuzzy subgroups have been introduced[6, 7, 10] and developed[1-8, 10] and conditions for a fuzzy subgroup to be written as the intersection of its minimal fuzzy p-subgroups have been investigated[2, 5, 7, 8]. Recently the concept of TL-subgroups that is an extension of the concept of fuzzy subgroups has been introduced and studied [13] and the notion of TL-orders of the elements of a group relative to a TL-subgroup and the notion of TL-orders of TL-subgroups that are extensions of the notion of fuzzy orders of the elements and the notion of fuzzy orders of fuzzy subgroups, respectively, were introduced [9]. In this paper, using the notion of TL-orders of the elements of a group relative to a TL-subgroup, we introduce the notion of TL-p-subgroups that is an extension of the notion of fuzzy p-subgroups and show that a torsion TL-subgroup of an Abelian group with $T = \wedge$ can be written as the intersection of its minimal TL-p-subgroups.

2. Orders of elements relative to TL-subgroups

Throughout this paper, we let L denote a complete lattice that contains at least two distinct elements. The meet, join, and partial ordering will be written as $\wedge$, $\vee$, and $\leq$, respectively. We also write 1 for the greatest element of L. We will write the identity element of a group G by e and the order of x in G by $O(x)$. And we let T denote a t-norm on L.

Definition 2.1[9]. Let μ be a TL-subgroup of a group G. For a given $x \in G$, the least positive integer n such that $\mu(x^n) = 1$ is said to be the (TL-)order of x with respect to μ (briefly, $O_u(x)$). If no such n exists, x is said to have infinite TL-order with respect to μ.

Note that the concept of a TL-order is an extension of the concept of a fuzzy order.

Proposition 2.2[9]. Let μ be a TL-subgroup of a group G. For $x \in G$, if $u(x^m) = 1$ for some integer m, then $O_u(x)$ divides m. In particular, if $O(x)$ is finite, then $O_u(x)$ divides $O(x)$.

Lemma 2.3. Let μ be a TL-subgroup of a group G. If $\mu(x) = 1$, **then $\mu(xy) = \mu(x) = \mu(yx)$ for all $y \in G$.**

Lemma 2.4. Let μ be a TL-subgroup of a group G with $T = \wedge$. Let $O_u(x) = m$ where $x \in G$. If n is an integer with $(m, n) = 1$, then $\mu(x^n) = \mu(x)$.

Lemma 2.5. **Let μ be a TL-subgroup of a group G. Let $(O_u(x), O_u(y)) = 1$ and $xy = yx$ where $x, y \in G$. If $\mu(xy) = 1$, then $\mu(x) = \mu(y) = 1$.**

Lemma 2.6. **Let μ be a TL-subgroup of a group G with $T = \wedge$. Let $(O_u(x), O_u(y)) = 1$ and $xy = yx$ where $x, y \in G$. Then $\mu(xy) = \mu(x) \wedge \mu(y)$.**

Note that the homomorphic images and the homomorphic preimages of TL-subgroups are TL-subgroups [13].

Proposition 2.7. Let f be a group homomorphism from G onto H. And let μ and ν be TL-subgroups of G and H, respectively. Then the following hold:

(1) $O_{f(u)}(f(x))$ **divides $O_u(x)$ for all $x \in G$.**

(2) $O_{f^{-1}(v)}(x)$ **divides $O_v(f(x))$ for all $x \in G$.**

Definition 2.8. Let μ be a TL-subgroup of a group G. For a prime p, μ is called a TL-p-subgroup of G if $O_u(x)$ is a power of p for every $x \in G$.

The following corollary is a direct consequence of Proposition 2.7.

Corollary 2.9. The homomorphic images and the homomorphic preimages of TL-p-subgroups are TL-p-subgroups.

3. Decompositions of TL-subgroups

Let μ be a TL-subgroup of a group G. If there exists a minimal TL-p-subgroup of G containing μ, then it is unique because the intersection of TL-p-subgroups of G is a TL-p-subgroup. We will denote it by $\mu_{(p)}$. Note that $\mu_{(p)}$ does not exist in general even if T = min and L = [0, 1] [7]. When $\mu_{(p)}$ exists for every prime p, μ is contained in $\bigcap_p \mu_{(p)}$. So we are interested in finding some conditions such that $\mu = \bigcap_p \mu_{(p)}$. If $\mu = \bigcap_p \mu_{(p)}$, then the structure of μ can be easily investigated because the structures of TL-p-subgroups are more simple than the structures of TL-subgroups. Now we will show that $\mu = \bigcap_p \mu_{(p)}$ where μ is a torsion TL-subgroup of an Abelian group and $T = \wedge$.

Definition 3.1. Let μ be a TL-subgroup of an Abelian group G. μ is said to be torsion if $O_u(x)$ is finite for all $x \in G$.

The following lemma can be easily verified by induction.

Lemma 3.2. Let μ be a TL-subgroup of a group G with $T = \wedge$. Then $\mu(x^n) \geq \mu(x)$ for all $x \in G$ and for all integers n.

Theorem 3.3. Let μ be a TL-subgroup of a group G with $T = \wedge$ such that $O_u(x)$ is finite for all $x \in G$ and $\mu_{(p)}$ exists for all primes p. If, for all primes p, $\mu(x) = \mu_{(p)}(x)$ where $O_u(x)$ is a power of p, then $\mu = \bigcap_p \mu_{(p)}$.

Theorem 3.4[9]. Let μ be a TL-subgroup of a group G. And let x and y be elements of G such that $xy = yx$ and $(O_u(x), O_u(y)) = 1$. Then $O_u(xy) = O_u(x) \times O_u(y)$.

Corollary 3.5[9]. Let μ be a TL-subgroup of a group G. And let x and y be elements of G such that $xy = yx$ and $(O(x), O(y)) = 1$. Then $O_u(xy) = O_u(x) \times O_u(y)$.

Theorem 3.6[9]. Let μ be a TL-subgroup of a group G. For $x \in G$, if $O_u(x) = mn$ with $(m, n) = 1$, then there exist x_1 and x_2 in G such that $x = x_1x_2 = x_2x_1$, $O_u(x_1) = m$, and $O_u(x_2) = n$. Furthermore such an expression for x is unique in the sense of TL-grades, i.e., if (x_1, x_2) and (y_1, y_2) are such pairs, then $u(x_1) = u(y_1)$ and $u(x_2) = u(y_2)$.

Lemma 3.7. Let μ be a TL-subgroup of an Abelian group G. Let $x = x_1x_2$ and $y = y_1y_2$ be expressions for x and y in G, respectively, as in Theorem 3.6 with $O_u(x) = m_1m_2$, $O_u(y) = n_1n_2$, $(m_1, m_2) = 1 = (n_1, n_2)$, $O_u(x_1) = m_1$, and $O_u(x_2) = m_2$, $O_u(y_1) = n_1$, and $O_u(y_2) = n_2$. If $(m_1n_1, m_2n_2) = 1$, then $xy = (x_1y_1)(x_2y_2)$ is an expression for xy as in Theorem 3.6 with $O_u(x_1y_1)$ **and** $O_u(x_2y_2)$ **divisors of** m_1n_1 and m_2n_2, respectively.

Let μ **be a TL-subgroup of a group** G **with** $T = \wedge$ such that $O_u(x)$ **is finite for all** $x \in G$. For every prime p, define a L-subset μ_p **of** G **by** $\mu_p(x) = \mu(x_2)$ where $x = x_1x_2$ is an expression for x with $O_\mu(x) = mp^t$, $(m, p) = 1$, $O_\mu(x_1) = m$, and $O_\mu(x_2) = p^t$. Then μ_p is well-defined.

Lemma 3.8. **Let** μ **be a TL-subgroup of a group** G **with** $T = \wedge$ such that $O_u(x)$ **is finite for all** $x \in G$. For a given prime p, if there exists $\mu_{(p)}$, **then** $\mu_p \subseteq \mu_{(p)}$.

Lemma 3.9. **Let** μ **be a TL-subgroup of a group G with** $T = \wedge$ such that $O_u(x)$ **is finite for all** $x \in G$. For a given prime p, μ_p **is a TL-subgroup of** G **if and only if** $\mu_p = \mu_{(p)}$.

Now we give a necessary and sufficient condition for a TL-subgroup to be written as the intersection of its all minimal TL-p-subgroups.

Theorem 3.10. **Let** μ **be a TL-subgroup of a group** G **with** $T = \wedge$ such that $O_u(x)$ **is finite for all** $x \in G$. Then μ_p **is a TL-subgroup of** G **for every prime p if and only if** $\mu = \bigcap_p \mu_{(p)}$.

Theorem 3.11. Let μ be a torsion TL-subgroup of an Abelian group G **with** $T = \wedge$. Then, for every prime p, $\mu_{(p)}$ **exists and** $\mu_{(p)} = \mu_p$.

Corollary 3.12 Let μ be a torsion TL-subgroup of an Abelian group G **with** $T = \wedge$. Then $\mu = \bigcap_p \mu_{(p)}$.

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