랜덤 퍼지넘버의 대수의 법칙

Laws of large numbers for T-related L-R random fuzzy numbers

홍 덕 헌
경북 경산시 하양읍 금락1리330번지
대구효성가톨릭대학교 공과대학 자동차공학부

Dug Hun Hong
School of Mechanical and Automotive Engineering Catholic University of Taegu-Hyosung,Kyungbuk 712-702,South Korea.

ABSTRACT

In this paper, we introduce new results of law large numbers for mutually T-related fuzzy numbers, when T is an Archimedean t-norm and spreads are random variables, and generalize earlier result of Fullér[FSS 45(1992) 299-303].

I .Introduction.

Fullér[5] proved a law of large numbers for sequence of mutually T-related symmetric triangular fuzzy numbers with common spreads if $T(u,v) \le H_0(u,v) = \dfrac{uv}{(u+v-uv)}$ for all $0 \le u, v \le 1$.Badard [1] also proved a law of large numbers for fuzzy numbers with common spread when $T(u,v) = uv$. Recently,many different types of generalization have done by many authors, for example,Fullér[6], Triesch[13], Hong[8,9], Hong and Kim[10] and Williamson[15]. We note that Fullér and Triesch[7] show that the result of Williamson is not valid and Hong[11] also show that Theorem 2(ii)of Fullér is wrong. On the other hand, Näther et al.[14] considered a linear regression model for L-R random fuzzy numbers. In this paper, we introduce a new type of law of large numbers for fuzzy numbers. Indeed,we consider the case when spreads are random variables and generalize the result of Fullér[5].

II .Preliminaries.

As defined in [3],by a fuzzy number we mean a fuzzy subset ξ of the real line with a unimodal,upper semicontinuous membership function such that there exists a unique real number m satisfying $\xi(m) = SUP_x \xi(x) = 1$. The number $m = m(\xi)$is called the modal value of ξ.

Now suppose that a sequences of fuzzy numbers $\xi_1, \xi_2, \cdots, \xi_n, \cdots$and a t-norm T(see[17])are given. The T-sum $\xi_1 + \xi_2 + \cdots + \xi_n$and the T-arithmetic mea $(\xi_1 + \xi_2 + \cdots + \xi_n)/n$ are the fuzzy numbers defined by

$$\xi_1 + \xi_2 + \cdots + \xi_n(z) := \underset{x_1 + x_2 + \cdots + x_n = z}{SUP} T(\xi_1(x_1), \cdots, \xi_n(x_n))$$

and

$$\frac{1}{n}(\xi_1 + \xi_2 + \cdots + \xi_n)(z) := (\xi_1 + \xi_2 + \cdots + \xi_n)(nz),$$

respectively(sec[3]).It is easy to see that

$$m(\xi_1 + \xi_2 + \cdots + \xi_n) = m(\xi_1) + m(\xi_2) + \cdots + m(\xi_n) = nm(\frac{1}{n}(\xi_1 + \xi_2 + \cdots + \xi_n)).$$

For a fuzzy number ξ and any subset D of the real numbers, the quantity

$$Nes(\xi | D) := 1 - \underset{x \notin D}{SUP} \xi(x)$$

is considered to measure the necessity of ξ belonging to D(see[16]).If D is an interval (a,b) we also write $Nes(a < \xi < b)$ instead of $Nes(\xi \mid D)$.

Recall that a t-norm T is called Archimedean if and only if T is continuous and $T(x,x) < x$ for all $x \in (0,1)$. A t-norm T is called strict if it is continuous and all partial mappings $T(x,\cdot), x \in [0,1]$,are strictly increasing; it is called nilpotent if it is continuous, Archimedean and not strict. A well-known theorem (see[12]) asserts that for each Archimedean t-norm there exists a continuous, decreasing function $f : [0,1] \to [0, \infty]$ with $f(1) = 0$ such that

$$T(x_1, \cdots, x_n) = f^{[-1]}(f(x_1) + \cdots + f(x_n))$$

for all $x_i \in (0,1), 1 \le i \le n$. Here $f^{[-1]} : [0,\infty] \to [0,1]$is defined by

$$f^{[-1]}(y) = \begin{cases} f^{-1}(y) & \text{for } y \in [0, f(0)], \\ 0 & \text{if } y > f(0). \end{cases}$$

The function f is called the additive generator of T. Notice that if a continuous t-norm T has an additive generator f,then this additive generator is uniquely determined up to a non-zero positive multiplicative constant. A continuous Archmedean t-norm T with additive generator f is strict if and only if $f(0) = \infty$. Hence, for a nilpotent t-norm T,there always exists an additive generator f such that $f(0) = 1$,
called the normde additive generator of T.

Since f is continuous and decreasing, $f^{[-1]}$ is also continuous and non-increasing, we have

$$(\xi_1 + \cdots + \xi_n)(z) = \sup_{x_1 + \cdots + x_n = z} f^{[-1]}\Big(\sum_{i=1} f(\xi_i(x_i))\Big)$$

$$= f^{[-1]}\Big(\inf_{x_1 + \cdots + x_n = z} \Big(\sum_{i=1} f(\xi_i(x_i))\Big)\Big). \quad (1)$$

A triangular fuzzy number ã denoted by (a, α, β) is defined as

$$\tilde{a}(t) = \begin{cases} 1 - \frac{|a-t|}{\alpha} & \text{if } a - \alpha \le t \le a, \\ 1 - \frac{|a-t|}{\beta} & \text{if } a \le t \le a + \beta, \\ 0 & \text{otherwise,} \end{cases}$$

where $a \in R$ is the center and $\alpha > 0$ is the left spread, $\beta > 0$ is the right spread of ã.

If $\alpha = \beta$, then the triangular fuzzy number is called a symmetric triangular fuzzy number and denoted by (a, α).

An L-R fuzzy number $\tilde{a} = (a, \alpha, \beta)_{LR}$ is a function from the reals into the interval [0,1] satisfying

$$\tilde{a}(t) = \begin{cases} R(\frac{t-a}{\beta}) & \text{for } a \le t \le a + \beta, \\ L(\frac{t-a}{\alpha}) & \text{for } a - \alpha \le t \le a, \\ 0 & \text{else,} \end{cases}$$

Where L and R are decreasing and continuous functions from [0,1] to [0,1] satisfying $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$.

Let (Ω, T, P) be a probability space and $K(R)$ the set of all fuzzy sets A on R with upper semi-continuous normalized membership function and compact support $\{x \in R : A(x) > 0\}$.

Definition[14].A mapping $\tilde{Y} | \Omega \to K(R)$ is a random fuzzy set if every α-cut of $\tilde{Y}$, $\tilde{Y}_\alpha(\omega) = \{x \in R : (\tilde{Y}(\omega))(x) \ge \alpha\}$ is a compact random set.

For general discussions on fuzzy random sets, see Kwakernaak [13],Puri and Ralescu [14], and Zhang et al. [21]. Let $U, Y,$ and V be independent random variables on (Ω, T, P). Assume U and Y are positive. Let $T(R)$ be the set of all triangular fuzzy number T on R. Then clearly $T(R) \subset K(R)$.Let $g: R^3 \to T(R)$ be a function defined by $g(u, y, v)$. Then $g(U, Y, V) = (U, Y, V)$ is clearly a triangular random fuzzy set(number).

Ⅲ. Laws of large numbers for fuzzy numbers.

The function $H_\gamma : [0,1] \times [0,1] \to [0,1]$, where $\gamma \ge 0$, defined by $H_\gamma(u, v) = uv / (\gamma + (1-\gamma)(u + v - uv))$ is called the Hamacher norm with parameter γ. The following result is due to Fullér[5].

Theorem 1[5]. Let $T \leq H_0$ and let $\tilde{a}_i = (a_i, \alpha_i), i=1,2,\cdots$ be fuzzy numbers with $\alpha = \alpha_i$ for $i=1,2,\cdots$, then for any $\varepsilon > 0$

$$\lim_{n\to\infty} Nes(m_n - \varepsilon \leq \tilde{A}_n/n \leq m_n + \varepsilon) = 1$$

where $m_n = (a_1 + \cdots + a_n)/n$ and $\tilde{A}_n = \tilde{a}_1 + \cdots + \tilde{a}_n$.

In this theorem, if we assume $\{a_i\}_{i=1}^{\infty}$ is non-negative i.i.d.(independent and identically distributed) random variables, what conclusion can we have?

To answer this question we need the following lemmas.The first three lemmas are well known results in probability theory.

Lemma 1. Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d.random variables and $S_n = \sum_{i=1}^{n} X_i$. Then $S_n/n \to EX_1 a.s.$ iff $E|X_1| < \infty$ iff $\sum_{n=1}^{\infty} P\{|X_n| \geq n\} < \infty$.

Lemma 2. [2, Theorem 5.4.3] Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d.random variables with $EX_1 = \infty$ and $S_n = \sum_{i=1}^{n} X_i$.

Then $\lim_{n\to\infty} |S_n|/n_2 = 0$ $a.s.$ iff $\sum_{n=1}^{\infty} P\{|X_n| \geq n^2\} < \infty$.

Lemma 3. Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d.random variables.Then $E|X_1| < \infty$ iff $n^{-1} \max_{1\leq k\leq n} |X_k| \to 0$ $a.s.$

Lemma 4. Let T be an Archimedean t-norm with additive generator f,let $\tilde{a}_i = (a_i, \alpha_i, \beta_i)_{LR}, i=1,2,\cdots,n$ be L-R fuzzy numbers and let $\tilde{A}_n$ be the membership function of the T-sum. Suppose that $f\circ R$ and $f\circ L$ are convex function then

$$\tilde{A}_n(z) \leq \begin{cases} f^{[-1]}\left(nf\left(L\left(\dfrac{A_n - z}{n\alpha_n^*}\right)\right)\right) & \text{if } A_n - \alpha_n^* \leq z \leq A_n \\ f^{[-1]}\left(nf\left(R\left(\dfrac{z - A_n}{n\beta_n^*}\right)\right)\right) & \text{if } A_n \leq z \leq A_n + \beta_n^* \end{cases}$$

where

$A_n = a_1 + a_2 + \cdots + a_n$, $\alpha_n^* = \max\{\alpha_1, \cdots, \alpha_2\}$ and $\beta_n^* = \max\{\beta_1, \cdots, \beta_2\}$

.

Lemma 5. If T is an Archimedean t-norm with additive generator f and $\tilde{a}_i = (a_i, \alpha_i, \beta_i)_{LR}, i=1,2,\cdots,n$ are L-R fuzzy numbers. Then we have

$$\tilde{A}_n(z) \geq \begin{cases} f^{[-1]}\left(nf\circ\left(L\left(\dfrac{A_n - z}{\alpha_1 + \cdots + \alpha_n}\right)\right)\right) & \text{if } A_n - (\alpha_1 + \cdots + \alpha) \leq z \leq A_n \\ f^{[-1]}\left(nf\circ\left(R\left(\dfrac{z - A_n}{\beta_1 + \cdots + \beta_n}\right)\right)\right) & \text{if } A_n \leq z \leq A_n + (\beta_1 + \cdots + \beta_n) \end{cases}$$

Lemma 6. Let T be an Archimedean t-norm with additive generator f,let $\tilde{a}_i = (a_i, \alpha_i, \beta_i)_{LR}, i=1,2,\cdots,n$ be L-R fuzzy numbers and let $\tilde{A}_n$ be the membership function of their T-sum,then for any $\varepsilon > 0$ such that $\min(\alpha^*_n, \beta^*_n) > \varepsilon > 0$

I) $Nes(\tilde{A}_n/n \mid (m_n - \varepsilon, m_n + \varepsilon)) \le 1 - \max(f^{[-1]}(nf \circ L(\frac{\varepsilon}{\alpha^*_n})), f^{[-1]}(nf \circ R(\frac{\varepsilon}{\beta^*_n}))),$

ii) $Nes(\tilde{A}_n/n \mid (m_n - \varepsilon, m_n + \varepsilon)) \ge 1 - \max(f^{[-1]}(nf \circ L(\frac{n\varepsilon}{\alpha_1 + \cdots + \alpha_n})),$

$$f^{[-1]}(nf \circ R(\frac{n\varepsilon}{\beta_1 + \cdots + \beta_n}))).$$

Theorem 2. Let $\tilde{a}_i = (a_i, \alpha_i(\omega), \beta_i(\omega))_{LR}, i=1,2,\cdots,n$ denote a sequence of L-R random fuzzy numbers with non-negative real valued i.i.d random variables $\{\alpha_i(\omega)\}_{i=1}^{\infty}$ and $\{\beta_i(\omega)\}_{i=1}^{\infty}$. Suppose that an Archimedean t-norm with additive generator f is given and that $f \circ R$ and $f \circ L$ are convex function with $\lim_{x \to 0^+}(f \circ R)'(x) = \gamma_1 > 0$, $\lim_{x \to 0^+}(f \circ L)'(x) = \gamma_2 > 0$. If $E\alpha_1 < \infty$ and $E\beta_1 < \infty$, then

$$\lim_{n \to \infty} Nes(m_n - \varepsilon \le \tilde{A}_n/n \le m_n + \varepsilon) = 1 \quad a.s.$$

The following is the converse of Theorem 2, which can be proved under weaker conditions.

For a general strict t-norm the converse of Theorem 2 is also true.

Theorem 3. Let $\tilde{a}_i = (a_i, \alpha_i(\omega), \beta_i(\omega))_{LR}, i=1,2,\cdots,n$ denote a sequence of L-R random fuzzy numbers with non-negative real valued i.i.d random variables $\{\alpha_i(\omega)\}_{i=1}^{\infty}$ and $\{\beta_i(\omega)\}_{i=1}^{\infty}$. Suppose that an Archimedean t-norm with strict additive generator f is given and that $f \circ R$ and $f \circ L$ are convex function with $\lim_{x \to 0^+}(f \circ R)'(x) = \gamma_1 > 0$, $\lim_{x \to 0^+}(f \circ L)'(x) = \gamma_2 > 0$. Then $E\alpha_1 < \infty$ and $E\beta_1 < \infty$, iff

$$\lim_{n \to \infty} Nes(m_n - \varepsilon \le \tilde{A}_n/n \le m_n + \varepsilon) = 1 \quad a.s.$$

For the case of $E\alpha_1 = \infty$ and $E\beta_1 = \infty$,we have the following sufficient condition of law of large numbers for $L-R$ random fuzzy numbers.

Theorem 4. Let $\tilde{a}_i = (a_i, \alpha_i(\omega), \beta_i(\omega))_{LR}, i=1,2,\cdots,n$ denote a sequence of L-R random fuzzy numbers with non-negative real valued i.i.d random variables $\{\alpha_i(\omega)\}_{i=1}^{\infty}$ and $\{\beta_i(\omega)\}_{i=1}^{\infty}$ with $E\alpha_1 = \infty$ and $E\beta_1 = \infty$. Suppose that an Archimedean t-norm with additive generator f is given and that $f \circ R$ and $f \circ L$ are convex function with

$\lim_{x\to 0^+}(f\circ R)'(x)=\gamma_1>0$, $\lim_{x\to 0^+}(f\circ L)'(x)=\gamma_2>0$. If $\sum_{n=1}^{\infty}P\{\alpha_n\geq n^2\}<\infty$ and $\sum_{n=1}^{\infty}P\{\beta_n\geq n^2\}<\infty$, then

$$\lim_{n\to\infty} Nes(m_n-\varepsilon\leq \tilde{A}_n/n\leq m_n+\varepsilon)=1 \quad a.s.$$

Remark. Let a t-norm T be a Hamacher t-norm H_γ with $\gamma\geq 0$ and $L_x=R_x=1-x$.Then we have the following convex additive generator:

$$f_\gamma(x)=\begin{cases}\ln\dfrac{\gamma+(1-\gamma)x}{x} & \text{if } \gamma>0,\\ \dfrac{x}{1-x} & \text{if } \gamma=0.\end{cases}$$

We also have

$$\lim_{x\to 0^+}(f\circ R)'=\lim_{x\to 0^+}(f\circ L)'=\begin{cases}\gamma & \text{if } \gamma>0,\\ 1 & \text{if } \gamma=0.\end{cases}$$

Therefore.Theorem 1[5] is a special case of Theorem 2, since we can consider a sequence of constant real numbers as a sequence of i.i.d. random variables.

Lemma 7.Let h be a continuous and increasing function from [0,1] to $[0,\infty)$ with $h(0)=0$.

Then exists a increasing and convex function h^* such that

$$h^*(x)\leq h(x) \text{ for } x\in[0,1].$$

If we modify the proof of Theorem 2 using Lemma 7,we have the following general result.

Theorem 5. Let $(a_i, \alpha_i(\omega), \beta_i(\omega))_{LR}, i=1,2,\cdots,n$ denote a sequence of L-R fuzzy numbers with non-negative real valued i.i.d random variables $\{\alpha_i(\omega)\}_{i=1}^{\infty}$ and $\{\beta_i(\omega)\}_{i=1}^{\infty}$. Suppose that an Archimedean t-norm with additive generator f is given and that $\lim_{x\to 0^+}(f\circ R)^{*\prime}(x)=\gamma_1>0$, $\lim_{x\to 0^+}(f\circ L)^{*\prime}(x)=\gamma_2>0$.

If $E\alpha_1<\infty$ and $E\beta_1<\infty$, then

$$\lim_{n\to\infty} Nes(m_n-\varepsilon\leq \tilde{A}_n/n\leq m_n+\varepsilon)=1 \quad a.s.$$

Theorem 6. Let $(a_i(\omega), \alpha_i(\omega), \beta_i(\omega))_{LR}, i=1,2,\cdots,n$ denote a sequence of L-R fuzzy numbers with real valued i.i.d. random variables $\{a_i(\omega)\}_{i=1}^{\infty}$ and non-negative real valued i.i.d. random variables $\{\alpha_i(\omega)\}_{i=1}^{\infty}$ and $\{\beta_i(\omega)\}_{i=1}^{\infty}$. Suppose that an Archimedean t-norm with additive generator f is given and that $\lim_{x\to 0^+}(f\circ R)^{*\prime}(x)=\gamma_1>0$, $\lim_{x\to 0^+}(f\circ L)^{*\prime}(x)=\gamma_2>0$.

If $E\alpha_1<\infty$ and $E\beta_1<\infty$, then $\lim_{n\to\infty} Nes(m_n-\varepsilon\leq \tilde{A}_n/n\leq m_n+\varepsilon)=1 \quad a.s.$

REFERENCES

[1] R. Badard, The law of large numbers for fuzzy processes and the estimation problem, Information Science 28(1982) 161-178.

[2] K. L. Chung, A course in probability theory, 2nd edition (Academic Press, 1974).

[3] D. Dubois and H. Prade, Linear programming with fuzzy data,in : J. Bezdek, ed., Analysis of Fuzzy Information, vol. 3:
Application in Engineering and Science (CRC Press, Boca Raton, FL, 1987), 241-261.

[4] D. Dubois and H. Prade, Additions of interactive fuzzy numbers, IEEE Trans. Automat. Control 26(1981), 926-936.

[5] R. Fullér, A law of large numbers for fuzzy numbers, Fuzzy Sets and Systems 45(1992), 299-303.

[6] R. Fullér, it On law of large numbers for L-R fuzzy numbers, in: R. Lowen abd M. Roubens, eds., Proceedings of the IFSA Conference '91, Brussels,Mathematics.

[7] R. Fullér and E. Triesch, A note on the law of large numbers for fuzzy variables, Fuzzy Sets and Systems 45(1992) 299-303.

[8] D. H. Hong, A note on the law of large numbers for fuzzy numbers,Fuzzy Sets and Systems 64(1994) 59-61.

[9] D. H. Hong, A convergence theorem for arrays of L-R fuzzy numbers, Information Science 88(1996) 169-175.

[10] D. H. Hong and Y. M. Kim, A law of large numbers for fuzzy numbers in a Banach space, Fuzzy Sets and Systems 77(1996) 349-354.

[11] D. H. Hong, A note on the law of large numbers for fuzzy numbers, Fuzzy Sets and Systems 68(1994) 243.

[12] D. H. hong and C. Hwang, A T-sum bound of L-r fuzzy numbers, Fuzzy Sets and Systems 91(1997) 239-252.

[13] H. Kwakernaak, Fuzzy random variable:Definition and theorems, Information Sciences 15(1978) 1-29.

[14] W.Näther and M. Albrecht, Linear regression with random fuzzy obervations, Statistics 21(1190), No. 4,521-531.

[15] M. L. Puri and D. A. Ralescu, Fuzzy random variables, Journal of Mathematical Analysis and Applications 114(1986) 409-422.

[16] B. Schweizer and A. Sklar, Associative functions and abstract semigroups, Publ. Math. Debrecen 10(1963), 69-81.

[17] E. Triesch, Characterisation of Archimedian t-norms and a law of large numbers, Fuzzy Sets and Systems 58(1993), 339-342.

[18] R. L. Wheeden and A. Zygmund, Measure and integral, (Marcel Dekker,Inc., 1977).

[19] R. C. Willaimson, The law of large numbers for fuzzy variables under a general triangular norm extension principle, Fuzzy Sets and Systems 41(1991) 55-81.

[20] L. A. Zadeh, Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets and Systems 6(1978), 3-26.

[21] D. Zhang,H. Ouyang, E. S. Lee, and R. R. Yager, On fuzzy Sets and Systems 6(1978), 3-26.

[22] H. -J. Zimmermann, Fuzzy Set Theory-and Its Applications, 2nd edition(KluwerAcad. Publ., Dordrecht, 1991).