$TL-P^*$ 군의 몇가지 성질

Some properties of $TL-P^*$ subgroups

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ABSTRACT

We introduce the notion of $TL-p^*$-subgroups which is an extension of the notion of $TL-p$-subgroups and investigate basic properties of $TL-p^*$-subgroups. And we consider decompositions of TL-subgroups.

1. Introduction

Rosenfeld [8] introduced the concept of fuzzy subgroups of a group. Following these ideas, many authors are engaged in generalizing various notions of group theory in the fuzzy setting. In particular, the notion of fuzzy orders of the elements of a group relative to a fuzzy group and the notion of fuzzy orders of fuzzy subgroups have been introduced[4] and developed[3, 4, 6] and conditions for a fuzzy subgroup to be written as the intersection of its minimal fuzzy p-subgroups have been investigated[3, 4, 6]. Recently the concept of TL-subgroups that is an extension of the concept of fuzzy subgroups has been introduced and studied [10] and the notion of TL-orders of the elements of a group relative to a TL-subgroup and the notion of TL-orders of TL-subgroups that are extension of the notion of fuzzy orders of the elements and the notion of fuzzy orders of fuzzy subgroups, respectively, were introduced [7] and developed [1, 2, 5]. In this paper, we introduce the notion of $TL-p^*$-subgroups which is an extension of the notion of $TL-p$-subgroups and investigate basic properties of $TL-p^*$-subgroups. And we consider decompositions of TL-subgroups.

Throughout this paper, we let L denote a complete lattice that contains at least two distinct elements. The meet, join, and partial ordering will be written as $\wedge$, $\vee$, and $\leq$, respectively. We also write 1 for the greatest element of L.

2. Preliminaries

We recall basic definitions and some properties that are relevant for this paper. We will write the identity element of a group G by e and the order of x in G by $O(x)$. And we let T denote a t-norm on L.

Definition 2.1[7]. Let μ be a TL-subgroup of a group G. For a given $x \in G$, the least positive integer n such that $\mu(x^n) = 1$ is said to be the (TL-)order of x with respect to μ (briefly, $O_\mu(x)$). If no such n exists, x is said to have infinite TL-order with respect to μ.

Definition 2.2. Let μ be a TL-subgroup of a group G. For a prime p, μ is called a TL-p-subgroup of G if $O_\mu(x)$ is a power of p for every $x \in G$.

Let μ be a TL-subgroup of a group G. If there exists a minimal TL-p-subgroup of G containing μ, then it is unique because the intersection of TL-p-subgroups of G is obviously a TL-p-subgroup. We will call it by the least TL-p-subgroup of G containing μ and denote it by $\mu_{(p)}$. Note that for every prime p, $\mu_{(p)}$ does not exist in general even if $T = \wedge$ and $L = [0,1]$ [4].

Theorem 2.3[5]. Let μ be a TL-subgroup of a group G. For $x \in G$, if $O_\mu(x) = mn$ with $(m,n) = 1$, then there exist x_1 and x_2 in G such that $x = x_1x_2 = x_2x_1$, $O_\mu(x_1) = m$, and $O_\mu(x_2) = n$. Furthermore such an expression for x is unique in the sense of TL-grades, i.e., if (x_1, x_2) and (y_1, y_2) are such pairs, then $\mu(x_1) = \mu(y_1)$ and $\mu(x_2) = \mu(y_2)$.

Let μ be a TL-subgroup of a group G with $T = \wedge$ such that $O_\mu(x)$ is finite for all $x \in G$. For every prime p, define an L-subset μ_p of G by $\mu_p(x) = \mu(x_2)$ where $x = x_1x_2$ is an expression for x with $O_\mu(x) = mp^t$, $(p, m) = 1,$, $O_\mu(x_1) = m$ and $O_\mu(x_2) = p^t$. Then μ_p is well-defined by Theorem 2.3.

Proposition 2.4 [5]. Let μ be a TL-subgroup of an Abelian group G with $T = \wedge$ such that $O_\mu(x)$ is finite for all $x \in G$. For every prime p, μ_p is the least TL-p-subgroup of G containing μ i.e., $\mu_p = \mu_{(p)}$.

3. $TL-p^*$-subgroups

In this section, we introduce the notion of $TL-p^*$-subgroups and investigate basic properties of $TL-p^*$-subgroups.

Definition 3.1. Let μ be a TL-subgroup of a group G with $T=\wedge$ and p a prime. μ is said to be a $TL-p^*$-subgroup if, for every $x\in G$, $\min\{n\in \mathrm{N} \mid \mu(x) < \mu(x^n)\}$ is a power of p, whenever this minimum exists.

Let p be a prime. And let μ be a TL-subgroup of a group G satisfying the following condition : If $\mu(x)$ and $\mu(y)$ are comparable where $x, y\in G$, then $\mu(xy) = \mu(x)\wedge\mu(y)$. A TL-subgroup μ of a group G satisfying this condition is said to have the property (E). If μ has the property (E) and $T=\wedge$, then there exists the least $TL-p^*$-subgroup of G containing μ and we will denote it by $\mu_{(p)^*}$.

Note that if μ is a TL-subgroup of a group G with $T=\wedge$, then $\mu(x^n)\geq\mu(x)$ for all $x\in G$ and for all integers n.

Theorem 3.2. Every $TL-p$-subgroup of a group G with $T=\wedge$ is a $TL-p^*$ -subgroup of G.

Thus the notion of $TL-p^*$-subgroups is an extension of the notion of $TL-p$ -subgroups. Note that the homomorphic images and the homomorphic preimages of TL -subgroups are TL-subgroups[10].

Theorem 3.3. Let f be a group homomorphism from G onto H. And let μ and ν be TL-subgroups of G and H with $T=\wedge$, respectively. Then:

(1) If μ is a $TL-p^*$-subgroup of G, then $f(\mu)$ is a $TL-p^*$-subgroup of H, provided μ is f-invariant, i.e., if $f(x_1) = f(x_2)$ implies $\mu(x_1) = \mu(x_2)$.

(2) If ν is a $TL-p^*$-subgroup of H, then $f^{-1}(\nu)$ is a $TL-p^*$-subgroup of G.

4. Decompositions of TL-subgroups

The notion of μ_p in section 2 can be applied only to TL-subgroups of groups whose every element has a finite TL-order. To overcome such limit, we now introduce the notion of μ_{p^*}. While μ_p corresponds to $\mu_{(p)}$, μ_{p^*} corresponds to $\mu_{(p)^*}$. And the notion of μ_{p^*} is a generalization of the notion of μ_p.

Definition 4.1. Let μ be a TL-subgroup of a group G. For a given prime p, define an L-subset μ_{p^*} of G by $\mu_{p^*}(x) = \sup\{\mu(x^n) \mid n \in \mathrm{N}, (n, p) = 1\}$.

By Definition 4.1, it is clear that $\mu_{p^*} \supseteq \mu$. However $\mu_{p^*} \neq \mu$ and μ_{p^*} is not a TL-subgroup in general.

Proposition 4.2. Let μ be a TL-subgroup of a group G with $T = \wedge$. For every $x \in G$, $\min\{n \in \mathrm{N} \mid \mu_{p^*}(x) < \mu_{p^*}(x^n)\}$ is a power of p, whenever this minimum exists.

Theorem 4.3[7]. Let μ be a TL-subgroup of a group G. Let $O_u(x) = n$ where $x \in G$. Then $O_u(x^m) = n/(m, n)$ for all integers m.

Theorem 4.4. Let μ be a TL-subgroup of a group G with $T = \wedge$ such that $O_u(x)$ is finite for all $x \in G$. Then $\mu_{p^*} = \mu_p$ for every prime p.

Thus the notion of μ_{p^*} is a generalization of the notion of μ_p. Now we give a condition for a TL-subgroup μ to be written as the intersection of all μ_{p^*}. We will denote $\bigcap\{\mu_{p^*} \mid p \text{ is a prime}\}$ by $\bigcap \mu_{p^*}$ for the sake of convenience.

Proposition 4.5. Let μ be a TL-subgroup of a group G. If there exists a prime q such that $\mu(x^n) = \mu(x)$ for all $x \in G$ where $(n, q) = 1$, then $\mu = \bigcap \mu_{p^*}$.

Proposition 4.6. Let μ be a TL-subgroup of a group G with $T = \wedge$. For all $x \in G$, let $\mu(x^l)$ and $\mu(x^{m_x q})$ be comparable where $l = m_x q + r$ with $0 \leq r < m_x$ and where $m_x = \min\{n \in \mathrm{N} \mid \mu(x) < \mu(x^n)\}$. Then $\mu = \bigcap \mu_{p^*}$.

Theorem 4.7. Let μ be a TL-subgroup of a group G with $T = \wedge$. And let μ have the property (E). Then μ_{p^*} is a TL-subgroup of G with $T = \wedge$ if and only if $\mu_{p^*} = \mu_{(p)^*}$.

Proposition 4.8. Let μ be a TL-subgroup of a group G with $T = \wedge$. And let μ have the property (E). For all $x \in G$, let $\mu(x^l)$ and $\mu(x^{m_x q})$ are comparable where $l = m_x q + r$ with $0 \leq r < m_x$ and where $m_x = \min\{n \in \mathrm{N} \mid \mu(x) < \mu(x^n)\}$. If μ_{p^*} is a TL-subgroup of G with $T = \wedge$ for every prime p, then $\mu = \bigcap \mu_{(p)^*}$.

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